Glasgow Math. J. **58** (2016) 717–726. © Glasgow Mathematical Journal Trust 2015.
doi:10.1017/S0017089515000427.

ON THE EXISTENCE OF A GLOBAL NEIGHBOURHOOD

TOM COATES
Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom
e-mail: t.coates@imperial.ac.uk

and HIROSHI IRITANI
Department of Mathematics, Graduate School of Science, Kyoto University, Kitashirakawa-Oiwake-cho, Sakyo-ku, Kyoto, 606-8502, Japan
e-mail: iritani@math.kyoto-u.ac.jp

(Received 3 December 2014; accepted 17 April 2015; first published online 21 July 2015)

Abstract. Suppose that a complex manifold M is locally embedded into a higher-dimensional neighbourhood as a submanifold. We show that, if the local neighbourhood germs are compatible in a suitable sense, then they glue together to give a global neighbourhood of M. As an application, we prove a global version of Hertling–Manin's unfolding theorem for germs of TEP structures; this has applications in the study of quantum cohomology.

2010 *Mathematics Subject Classification.* 32Q99 (Primary), 32Q40, 14N35, 53D45.

1. Introduction.

We prove

THEOREM 1. *Let M be a complex manifold of dimension m and let $\mathcal{A}$ be a sheaf of $\mathbb{C}$-algebras over M. Suppose that there exist a natural number n and a morphism of sheaves of $\mathbb{C}$-algebras $\pi\colon \mathcal{A} \to \mathcal{O}_M$ such that for each $x \in M$ there exists an open neighbourhood U of x and an isomorphism $\mathcal{A}|_U \cong \iota^{-1}\mathcal{O}_{U\times\mathbb{C}^n}$, where $\iota\colon U \hookrightarrow U \times \mathbb{C}^n$ is the embedding $x \mapsto (x, 0)$, such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A}|_U & \xrightarrow{\ \sim\ } & \iota^{-1}\mathcal{O}_{U\times\mathbb{C}^n} \\ & {\scriptstyle \pi|_U}\searrow \quad \swarrow{\scriptstyle \iota^{\#}} & \\ & \mathcal{O}_U & \end{array} \tag{1}$$

Here $\iota^{\#}$ is the canonical morphism induced by ι. Then

(*i*) *There exist a complex manifold M' of dimension $m + n$ and a closed embedding $\iota\colon M \to M'$ such that we have an isomorphism $\mathcal{A} \cong \iota^{-1}\mathcal{O}_{M'}$ of sheaves of $\mathbb{C}$-algebras which commutes with the surjections to $\mathcal{O}_M$.*

(*ii*) *The manifold-germ M' is unique up to unique isomorphism in the following sense. If we have two complex manifolds M'_1, M'_2, two closed embeddings $\iota_1\colon M \to M'_1$, $\iota_2\colon M \to M'_2$, and an isomorphism of sheaves of $\mathbb{C}$-algebras $\phi\colon \iota_1^{-1}\mathcal{O}_{M'_1} \to \iota_2^{-1}\mathcal{O}_{M'_2}$ commuting with the surjections to $\mathcal{O}_M$, then ϕ is induced by a biholomorphic map $\varphi\colon N_2 \to N_1$ between open neighbourhoods N_i of M in M'_i, $i \in \{1, 2\}$, such that*

φ is the identity map on M. Such a map φ is unique as a germ of maps on a neighbourhood of M.

We also discuss the extension of sheaves, proving

THEOREM 2. *Suppose that $\iota\colon M \hookrightarrow M'$ is a closed embedding of complex manifolds. Let $\mathcal{A}$ be the sheaf of $\mathbb{C}$-algebras $\mathcal{A} = \iota^{-1}\mathcal{O}_{M'}$ over M and let $\mathcal{B}$ be a coherent $\mathcal{A}$-module. Then*

(*i*) *There exist an open neighbourhood N of M in M' and a coherent $\mathcal{O}_N$-module $\mathcal{B}'$ on N such that $\iota^{-1}\mathcal{B}' \cong \mathcal{B}$ as sheaves of $\mathcal{A}$-modules.*

(*ii*) *The sheaf-germ $\mathcal{B}'$ is unique up to unique isomorphism in the following sense. If we have two coherent $\mathcal{O}_N$-modules $\mathcal{B}'_1$ and $\mathcal{B}'_2$ on a neighbourhood N of M in M' and isomorphisms $\iota^{-1}\mathcal{B}'_1 \cong \mathcal{B} \cong \iota^{-1}\mathcal{B}'_2$ of $\mathcal{A}$-modules, with $\phi\colon \iota^{-1}\mathcal{B}'_1 \to \iota^{-1}\mathcal{B}'_2$ denoting the composite isomorphism, then ϕ is induced by an isomorphism $\Phi\colon \mathcal{B}'_1|_P \to \mathcal{B}'_2|_P$ of $\mathcal{O}_P$-modules on an open neighbourhood P of M in N. Such a morphism Φ is unique as a germ of homomorphisms over a neighbourhood of M.*

REMARK 3. By Oka's coherence theorem, $\mathcal{O}_{M'}$ is coherent and hence $\mathcal{A} = \iota^{-1}\mathcal{O}_{M'}$ is also coherent as a sheaf of algebras. Therefore, being a coherent $\mathcal{A}$-module is equivalent to being *locally finitely presented* as an $\mathcal{A}$-module, i.e. for each $x \in M$, there exists an open neighbourhood U of x in M and an exact sequence of $\mathcal{A}$-modules

$$\mathcal{A}_U^{\oplus k} \longrightarrow \mathcal{A}_U^{\oplus l} \longrightarrow \mathcal{B}|_U \longrightarrow 0 \tag{2}$$

for some $k, l \in \mathbb{N}$. See e.g. [**6**, Appendix].

REMARK 4. If in addition $\mathcal{B}$ is locally free as an $\mathcal{A}$-module in Theorem 2, then $\mathcal{B}'$ becomes locally free as an $\mathcal{O}_N$-module in a neighbourhood N of M, because the stalk $\mathcal{B}'_x$ at each $x \in M$ is a free $\mathcal{O}_{M',x}$-module.

REMARK 5. We have stated Theorems 1 and 2 in the category of holomorphic manifolds, but the same statements hold true, with the same proofs, in the real analytic category.

We can reformulate our results as an *equivalence of categories*. Namely, for Theorem 1, the category of sheaves $\mathcal{A}$ of $\mathbb{C}$-algebras on M equipped with surjections $\pi : \mathcal{A} \to \mathcal{O}_M$ satisfying the local condition (1) is equivalent to the category of germs of neighbourhoods $\iota\colon M \hookrightarrow M'$ of M. For Theorem 2, the category of coherent $\mathcal{A}$-modules is equivalent to the category of germs of coherent sheaves on a neighbourhood of M in M'. It is not difficult to modify the discussion below to prove these categorical equivalences.

In the real analytic category (Remark 5), Theorem 1 may be viewed as a generalization of the existence theorem for the complexification of a real-analytic manifold, see e.g. [**9**]. In the C^∞ category, Lemma 8 below is not valid (see e.g. [**8**]) and our results do not hold. However, most of the arguments for Theorem 1 work if we can take representatives of neighbourhood germs and C^∞ gluing maps between them which satisfy cocycle conditions as germs; similar arguments appear in the context of Kuranishi structures, see K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono "Technical details on Kuranishi structure and virtual fundamental chain", arXiv:1209.4410 and [**5**]. Our original motivation was to globalize the unfolding of Frobenius-type structures (or meromorphic connections, or TEP structures) which has been studied by Hertling–Manin [**4**] and Reichelt [**7**] on the level of germs. We give a global version of Hertling–Manin's unfolding theorem for TEP structures in Section 4 below.

NOTATION 6. We require that manifolds be paracompact.

NOTATION 7. We write $A \Subset B$ if and only if A is a relatively compact subset of B.

2. The proof of Theorem 1.

Our assumptions on $\mathcal{A}$ imply that we can find a locally-finite open covering $\{W_i : i \in I\}$ of M with index set I such that $\mathcal{A}|_{W_i} \cong \iota^{-1}\mathcal{O}_{W_i \times Z_i}$, where Z_i is a copy of $\mathbb{C}^n$ and $\iota\colon W_i \to W_i \times Z_i$ is the embedding $x \mapsto (x, 0)$. Without loss of generality we can assume both that W_i is a co-ordinate neighbourhood on M (i.e. is identified with an open subset of $\mathbb{C}^m$) and that W_i is relatively compact in M. We take locally finite coverings $\{U_i : i \in I\}$, $\{V_i : i \in I\}$ with the same index set I such that $U_i \Subset V_i \Subset W_i$. We write $W_{ij} := W_i \cap W_j$, $V_{ij} := V_i \cap V_j$, $V_{ijk} := V_i \cap V_j \cap V_k$. The basic fact we use for the gluing is the following Lemma.

LEMMA 8. *Let $U \subset \mathbb{C}^m$ be an open set and let $\iota\colon U \to U \times \mathbb{C}^n$ be the embedding $x \mapsto (x, 0)$. Let $\phi\colon \iota^{-1}\mathcal{O}_{U\times\mathbb{C}^n} \to \iota^{-1}\mathcal{O}_{U\times\mathbb{C}^n}$ be a homomorphism of sheaves of $\mathbb{C}$-algebras which commutes with the natural surjections to $\mathcal{O}_U$. Then*

(a) there exists an open neighbourhood U' of $U \times \{0\}$ in $U \times \mathbb{C}^n$ and a holomorphic map $\varphi\colon U' \to U \times \mathbb{C}^n$ which is the identity on $U \times \{0\}$ such that ϕ coincides with the pull-back by φ;

(b) if ϕ is an isomorphism then the map φ is a biholomorphic isomorphism onto its image.

Proof of Lemma 8. Statement (a) implies statement (b), by the inverse function theorem, so we prove (a). Consider first the case where $m = 0$ and U is a point. Then ϕ is a $\mathbb{C}$-algebra endomorphism of the ring $\iota^{-1}\mathcal{O}_{\mathbb{C}^n} = \mathbb{C}\{z_1, \ldots, z_n\}$ of convergent power series which preserves the maximal ideal $\mathfrak{m} = (z_1, \ldots, z_n)$. Because such a ϕ is continuous with respect to the $\mathfrak{m}$-adic topology, ϕ is determined by the images of the generators $z_1, \ldots, z_n$. The images determine a holomorphic map $\varphi\colon (z_1, \ldots, z_n) \mapsto (\phi(z_1), \ldots, \phi(z_n))$ which is defined on a neighbourhood U' of 0 in $\mathbb{C}^n$, and ϕ coincides with the pull-back by φ.

Consider now the general case. Let $t_1, \ldots, t_m$ denote the standard co-ordinates on $U \subset \mathbb{C}^m$ and let $z_1, \ldots, z_n$ denote the standard co-ordinates on $\mathbb{C}^n$. Then the images of $t_1, \ldots, t_m$ and $z_1, \ldots, z_n$ under ϕ give global sections of $\iota^{-1}\mathcal{O}_{U\times\mathbb{C}^m}$, and thus they define a holomorphic map

$$\varphi\colon (t_1, \ldots, t_m, z_1, \ldots, z_n) \mapsto (\phi(t_1), \ldots, \phi(t_m), \phi(z_1), \ldots, \phi(z_n)),$$

on a neighbourhood U' of $U \times \{0\}$ in $U \times \mathbb{C}^n$. Since, ϕ commutes with the surjections to $\mathcal{O}_U$, φ restricts to the identity map on $U \times \{0\}$. The pull-back $\varphi^\star\colon \iota^{-1}\mathcal{O}_{U\times\mathbb{C}^n} \to \iota^{-1}\mathcal{O}_{U\times\mathbb{C}^n}$ defines a homomorphism of sheaves of $\mathbb{C}$-algebras commuting with the surjections to $\mathcal{O}_U$. The $m = 0$, $U =$ point case implies that $(\varphi^\star)_t = \phi_t$ on the stalk at every point $t \in U$. Thus, $\varphi^\star = \phi$. □

The composite isomorphism $\iota^{-1}\mathcal{O}_{W_{ij}\times Z_i} \cong \mathcal{A}|_{W_{ij}} \cong \iota^{-1}\mathcal{O}_{W_{ij}\times Z_j}$ induces a biholomorphic isomorphism $\varphi_{ij}\colon N_{ij} \to N_{ji}$ for each $i, j \in I$, where N_{ij} is an open neighbourhood of $W_{ij} \times \{0\}$ in $W_{ij} \times Z_i$ and φ_{ij} is the identity on $W_{ij} \times \{0\}$. Note that N_{ij} and N_{ji} are subsets of different spaces. Without loss of generality we may

https://doi.org/10.1017/S0017089515000427 Published online by Cambridge University Press

assume that $N_{ii} = W_i \times Z_i$ and that φ_{ii} is the identity map. Define:

$$O_i := V_i \times Z_i$$
$$O_{ij} := (V_{ij} \times Z_i) \cap N_{ij} \cap \varphi_{ij}^{-1}(V_{ij} \times Z_j).$$

Then, O_{ij} is an open subset of O_i which contains $V_{ij} \times \{0\}$. By restricting φ_{ij} to O_{ij}, we obtain a biholomorphic isomorphism $\varphi_{ij} \colon O_{ij} \to O_{ji}$ such that $\varphi_{ij}|_{V_{ij} \times \{0\}}$ is the identity map.

LEMMA 9. *There exist open subsets Q_i of O_i, $i \in I$, such that for each $i, j \in I$ we have*
(a) $Q_i \Subset O_i$;
(b) $U_i \times \{0\} \subset Q_i \subset U_i \times Z_i$;
(c) $Q_{ij} \Subset O_{ij}$, *where* $Q_{ij} := Q_i \cap O_{ij} \cap \varphi_{ij}^{-1}(Q_j)$.

Proof of Lemma 9. Denote by $\overline{U}_i$ the closure of U_i in V_i; by assumption $\overline{U}_i$ is compact. We have that $\overline{U}_i \cap \overline{U}_j$ is contained in V_{ij}, and hence that $(\overline{U}_i \cap \overline{U}_j) \times \{0\} \subset O_{ij}$. Fix $i, j \in I$, and fix a relatively compact open subset P of O_{ij} such that P contains $(\overline{U}_i \cap \overline{U}_j) \times \{0\}$; such a subset exists because the set $\overline{U}_i \cap \overline{U}_j$ is compact. Set

$$Q_i(n) := U_i \times \left\{x \in Z_i : |x| < \tfrac{1}{n}\right\},$$

noting that $Q_i(n)$ satisfies conditions (a) and (b) of the Lemma.

We claim that there exists n such that

$$Q_i(n) \cap O_{ij} \cap \varphi_{ij}^{-1}\big(Q_j(n)\big) \subset P. \tag{3}$$

Suppose, on the contrary, that for each n there exists an element $x_n \in Q_i(n) \cap O_{ij} \cap \varphi_{ij}^{-1}\big(Q_j(n)\big)$ such that $x_n \notin P$. After passing to a subsequence, we have that (x_n) converges to a limit $x \in \overline{U}_i \times \{0\}$. Thus, (x_n) converges in O_i. On the other hand, each x_n lies in the closed subset $O_i \setminus P$ of O_i, and so $x \in O_i \setminus P$. Thus, x lies in $(\overline{U}_i \setminus \overline{U}_j) \times \{0\}$. Now x_n lies in O_{ij} for each n, hence x_n lies in $V_{ij} \times Z_i$ and the limit x lies in the closure of V_{ij} in M. However the closure of V_{ij} is contained in W_{ij}. Recall that φ_{ij} is defined and continuous on the open neighbourhood N_{ij} of $W_{ij} \times \{0\}$ in W_i'. Thus

$$\varphi_{ij}(x) = \lim_{n \to \infty} \varphi_{ij}(x_n).$$

The right-hand side here converges to an element in $\overline{U}_j \times \{0\}$, since $\varphi_{ij}(x_n) \in Q_j(n)$. This is a contradiction: we have shown that $x \in \overline{U}_i \setminus \overline{U}_j$, and $\varphi_{ij}|_{W_{ij} \times \{0\}}$ is the identity map. Thus, for each i and $j \in I$, there exists an integer $n = n(i, j)$ such that (3) holds.

Since, V_i is relatively compact in M and since the covering $\{V_i : i \in I\}$ is locally finite, only finitely many V_j have nonempty intersection with a fixed V_i. Thus we can define

$$n(i) := \max\{n(i, j) : j \in I \text{ such that } V_i \cap V_j \neq \varnothing\}$$
$$Q_i := Q_i\big(n(i)\big),$$

to obtain open sets $\{Q_i : i \in I\}$ with the properties claimed. □

LEMMA 10. *Let $\{Q_i : i \in I\}$ be such that Q_i is an open subset of O_i and that properties (a–c) in Lemma 9 hold. Then the image of the map $Q_{ij} \to Q_i \times Q_j$ given by* (inclusion, φ_{ij}) *is closed in $Q_i \times Q_j$.*

https://doi.org/10.1017/S0017089515000427 Published online by Cambridge University Press

Proof of Lemma 10. Let (x_n) be a sequence in Q_{ij} such that (x_n) converges in Q_i and $\big(\varphi_{ij}(x_n)\big)$ converges in Q_j. Let x denote the limit of (x_n) in Q_i. Since Q_{ij} is relatively compact in O_{ij}, the limit x lies in O_{ij}. But $\big(\varphi_{ij}(x_n)\big)$ converges in Q_j and thus $\varphi_{ij}(x) \in Q_j$, or in other words $x \in \varphi_{ij}^{-1}(Q_j)$. Thus, $x \in Q_{ij}$. $\square$

LEMMA 11. *There exist open subsets Q_i of O_i, $i \in I$, such that properties (a–c) in Lemma 9 hold and further, for each $i, j, k \in I$, we have*

(d) $Q_{ij} \cap Q_{ik} \subset \varphi_{ij}^{-1}(O_{jk})$;
(e) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $Q_{ij} \cap Q_{ik}$.

Note that condition (d) guarantees that the composition in (e) is well-defined.

Proof of Lemma 11. Let $\{Q_i : i \in I\}$ be such that, for each $i \in I$, Q_i is an open subset of O_i and that properties (a–c) hold. Such subsets exist by Lemma 9. Define

$$Q_{ijk} := Q_{ij} \cap Q_{ik}.$$

Let $\overline{Q}_{ij}$ denote the closure of Q_{ij} in O_{ij}. This is compact, and hence $\overline{Q}_{ij}$ is at the same time the closure of Q_{ij} in O_i. Let $\overline{Q}_{ijk}$ denote the closure of Q_{ijk} in O_i. This is contained in the compact set $\overline{Q}_{ij} \cap \overline{Q}_{ik}$, hence in particular is contained in $O_{ij} \cap O_{ik}$.

We claim that there exists an open neighbourhood N_{ijk} of $\overline{Q}_{ijk} \cap (U_i \times \{0\})$ in $O_{ij} \cap O_{ik}$ such that

- $N_{ijk} \subset \varphi_{ij}^{-1}(O_{jk})$; and
- $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ holds on N_{ijk}.

It suffices to show that each x in $\overline{Q}_{ijk} \cap (U_i \times \{0\})$ has an open neighbourhood N_x in $O_{ij} \cap O_{ik}$ such that $N_x \subset \varphi_{ij}^{-1}(O_{jk})$ and that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ holds on N_x. Let $x \in \overline{Q}_{ijk} \cap (U_i \times \{0\})$. Then x lies in $O_{ij} \cap O_{ik} \cap (U_i \times \{0\}) = (V_{ijk} \cap U_i) \times \{0\}$ and hence lies in $O_{ij} \cap O_{ik} \cap \varphi_{ij}^{-1}(O_{jk})$. But $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ holds as germs at x, and so choosing N_x to be a sufficiently small open neighbourhood of x in $O_{ij} \cap O_{ik} \cap \varphi_{ij}^{-1}(O_{jk})$ proves the claim.

Recall that properties (a–c) in Lemma 9 hold for $\{Q_i : i \in I\}$ and note that, with the exception of the assertion that $U_i \times \{0\} \subset Q_i$, these properties are preserved under shrinking the sets Q_i. For a fixed $i \in I$, there are only finitely many pairs $(j, k) \in I \times I$ such that the triple intersection V_{ijk} is nonempty. Thus, as $Q_{ijk} \subset V_{ijk} \times Z_i$, there are only finitely many pairs $(j, k) \in I \times I$ such that Q_{ijk} is nonempty. Let $(j_1, k_1), \ldots, (j_f, k_f)$ be all such pairs. We shrink Q_i inductively as follows: set $Q_i^{(0)} := Q_i$, set

$$Q_i^{(a)} := (Q_i^{(a-1)} \setminus \overline{Q}_{ij_ak_a}) \cup (N_{ij_ak_a} \cap Q_i^{(a-1)}) \subset Q_i^{(a-1)},$$

for $a = 1, \ldots, f$, and set $Q_i^{\mathrm{new}} := Q_i^{(f)}$. Then, Q_i^{new} is an open subset of the original set Q_i, and it contains $U_i \times \{0\}$. Thus, properties (a–c) in Lemma 9 hold for the new sets $\{Q_i^{\mathrm{new}} : i \in I\}$. Furthermore, for each $i, j, k \in I$, $Q_{ij}^{\mathrm{new}} := Q_i^{\mathrm{new}} \cap O_{ij} \cap \varphi_{ij}^{-1}(Q_j^{\mathrm{new}})$ is contained in Q_{ij} and $Q_{ijk}^{\mathrm{new}} := Q_{ij}^{\mathrm{new}} \cap Q_{ik}^{\mathrm{new}}$ is contained in Q_{ijk}. Also Q_{ijk}^{new} is contained in N_{ijk}. Therefore, properties (d) and (e) hold for $\{Q_i^{\mathrm{new}} : i \in I\}$, and the Lemma is proved. $\square$

REMARK. Let $Q_i \subset O_i$, $i \in I$, be open subsets such that properties (a–e) in Lemma 11 hold. In particular, then, $Q_{ij} \cap Q_{ik} \subset \varphi_{ij}^{-1}(O_{jk})$. But slightly more is true: in fact

$Q_{ij} \cap Q_{ik} \subset \varphi_{ij}^{-1}(Q_{jk})$. For if $x \in Q_{ij} \cap Q_{ik}$ then $\varphi_{ij}(x) \in Q_{ji} \subset Q_j$ and $\varphi_{ik}(x) \in Q_{ki} \subset Q_k$. Also $\varphi_{jk} \circ \varphi_{ij}(x) = \varphi_{ik}(x)$, which lies in Q_k. Thus $\varphi_{ij}(x)$ lies in $Q_j \cap O_{jk} \cap \varphi_{jk}^{-1}(Q_k) =: Q_{jk}$.

We now complete the proof of Theorem 1. Choose open subsets $Q_i \subset O_i$, $i \in I$, such that properties (a–e) in Lemma 11 hold. (This is possible by Lemma 11.) Set M' equal to the quotient space:

$$\left(\coprod_{i\in I} Q_i\right)\Big/\sim,$$

by the equivalence relation $\sim$ generated by $x \sim \varphi_{ij}(x)$ where $x \in Q_{ij}$. We claim that M' is a complex manifold.

Let $X = \coprod_{i\in I} Q_i$ and consider the binary relation R on $X \times X$ given by

$$R := \coprod_{i,j\in I} Q_{ij} \subset \coprod_{i,j\in I} Q_i \times Q_j = X \times X,$$

where the map $Q_{ij} \to Q_i \times Q_j$ is given by (inclusion, φ_{ij}). Then M' is the quotient space of X by the equivalence relation generated by R. To show that M' is a complex manifold it suffices to prove that M' is Hausdorff; hence it suffices to prove that R is closed and that R is an equivalence relation. We have shown that the image of the map $Q_{ij} \to Q_i \times Q_j$ is closed (Lemma 10), so R is closed. It remains to show that R is an equivalence relation. Reflexivity ($x \sim x$) is obvious, since $Q_{ii} = Q_i$ and φ_{ii} is the identity map. Symmetry ($x \sim y \implies y \sim x$) is also obvious, since φ_{ij} and φ_{ji} are inverse to each other. For transitivity ($x \sim y \wedge y \sim z \implies x \sim z$) assume that $x \in Q_j$, $y \in Q_i$, $z \in Q_k$, $x \sim y$, and $y \sim z$. Then, $y \in Q_{ij}$ (since $y \sim x$) and $y \in Q_{ik}$ (since $y \sim z$), thus $y \in Q_{ij} \cap Q_{ik}$. The Remark after Lemma 11 implies that $y \in \varphi_{ij}^{-1}(Q_{jk})$, and Lemma 11 implies that $\varphi_{jk} \circ \varphi_{ij}(y) = \varphi_{ik}(y)$. But $x = \varphi_{ij}(y)$ and $z = \varphi_{ik}(y)$, so $z = \varphi_{jk}(x)$. Thus, $x \sim z$. It follows that R is an equivalence relation, and that M' is a complex manifold. It is clear that M is a closed submanifold of M'. This completes the proof of part (i) of Theorem 1.

Let us prove part (ii) of Theorem 1. Suppose we have two closed embeddings $\iota_1 \colon M \to M_1'$, $\iota_2 \colon M \to M_2'$ and an isomorphism $\phi \colon \iota_1^{-1}\mathcal{O}_{M_1'} \cong \iota_2^{-1}\mathcal{O}_{M_2'}$ of sheaves of $\mathbb{C}$-algebras commuting with natural surjections to $\mathcal{O}_M$. By Lemma 8, the isomorphism $\phi \colon \iota_1^{-1}\mathcal{O}_{M_1'} \to \iota_2^{-1}\mathcal{O}_{M_2'}$ is locally induced by a biholomorphic map which is the identity on M. Therefore, we have a locally finite open covering $\{S_i : i \in I\}$ of M, open neighbourhoods T_i of S_i in M_2', and holomorphic maps $\varphi \colon T_i \to M_1'$ such that φ_i is the identity map on $T_i \cap M$ and $\phi|_{S_i} = \varphi_i^\star$. Without loss of generality we may assume that S_i is relatively compact in M. Choose an open covering $\{R_i : i \in I\}$ of M such that $R_i \Subset S_i$, and choose an open tubular neighbourhood of M in M_2'. The tubular neighbourhood here is identified with a neighbourhood of the zero section of the normal bundle of M in M_2'; we choose a (fibrewise) Riemannian metric on it. We write $U(\epsilon) \subset M_2'$ for the open tube of length $\epsilon > 0$ over an open subset $U \subset M$. The maps φ_i and φ_j on the overlap $T_{ij} := T_i \cap T_j$ coincide on an open neighbourhood $T_{ij}^\circ \subset T_{ij}$ of $S_{ij} := S_i \cap S_j$. Since, for fixed $i \in I$, there are only finitely many $j \in I$ such that S_{ij} is nonempty, there exists $\epsilon_i > 0$ such that

- $R_i(\epsilon_i) \subset T_i$,
- $R_{ij}(\epsilon_i) \subset T_{ij}^\circ$ for all $j \in I$, where $R_{ij} := R_i \cap R_j \Subset S_{ij}$.

https://doi.org/10.1017/S0017089515000427 Published online by Cambridge University Press

Then the maps $\{\varphi_i|_{R_i(\epsilon_i)} : i \in I\}$ coincide over each overlap $R_i(\epsilon_i) \cap R_j(\epsilon_j) \subset T^\circ_{ij}$, $i, j \in I$, and thus determine a global holomorphic map φ on $N = \bigcup_{i\in I} R_i(\epsilon_i) \subset M'_2$. The uniqueness of φ as a germ is obvious. This completes the proof of Theorem 1.

3. The proof of Theorem 2. LEMMA 12. *Let $\iota\colon M \to M'$ be a closed embedding of complex manifolds, let $\mathcal{A}$ be the sheaf of $\mathbb{C}$-algebras $\mathcal{A} = \iota^{-1}\mathcal{O}_{M'}$ over M, and let $\mathcal{B}$ be a locally finitely presented $\mathcal{A} = \iota^{-1}\mathcal{O}_{M'}$-module. There exist*

- *an open covering $\{V_i : i \in I\}$ of M such that V_i is relatively compact in M;*
- *for each $i \in I$, an open subset W'_i of M' such that $V_i \subset W'_i$;*
- *for each $i, j \in I$, an open subset A_{ij} of M' such that $V_{ij} \subset A_{ij} \subset W'_{ij}$, where $V_{ij} := V_i \cap V_j$ and $W'_{ij} := W'_i \cap W'_j$;*
- *for each $i, j, k \in I$, an open subset B_{ijk} of M' such that $V_{ijk} \subset B_{ijk} \subset A_{ijk}$, where $V_{ijk} := V_i \cap V_j \cap V_k$ and $A_{ijk} := A_{ij} \cap A_{jk} \cap A_{ik}$;*
- *for each $i \in I$, a coherent $\mathcal{O}_{W'_i}$-module $\mathcal{B}'_i$ on W'_i;*
- *for each $i \in I$, an isomorphism $\theta_i\colon \iota^{-1}\mathcal{B}'_i \cong \mathcal{B}|_{V_i}$ of $\mathcal{A}_{V_i}$-modules;*
- *for each $i, j \in I$, an isomorphism $\phi_{ij}\colon \mathcal{B}'_i|_{A_{ij}} \cong \mathcal{B}'_j|_{A_{ij}}$ of $\mathcal{O}_{A_{ij}}$-modules;*

such that A_{ij}, B_{ijk} are symmetric in their indices and that the diagrams:

$$\begin{array}{ccc} \iota^{-1}\mathcal{B}'_i|_{A_{ij}} & \xrightarrow{\iota^{-1}\phi_{ij}} & \iota^{-1}\mathcal{B}'_j|_{A_{ij}} \\ & {\scriptstyle\theta_i}\searrow \quad \swarrow{\scriptstyle\theta_j} & \\ & \mathcal{B}|_{V_{ij}} & \end{array} \qquad \begin{array}{ccc} \mathcal{B}'_i|_{B_{ijk}} & \xrightarrow{\phi_{ik}} & \mathcal{B}'_k|_{B_{ijk}} \\ & {\scriptstyle\phi_{ij}}\searrow \quad \nearrow{\scriptstyle\phi_{jk}} & \\ & \mathcal{B}'_j|_{B_{ijk}} & \end{array} \tag{4}$$

commute for each $i, j, k \in I$.

Proof of Lemma 12. We take open coverings $\{V_i : i \in I\}$, $\{W_i : i \in I\}$ of M by Stein open subsets V_i, W_i such that $V_i \Subset W_i \Subset M$ and that $\mathcal{B}|_{W_i}$ has a finite presentation

$$\mathcal{A}^{\oplus k_i}_{W_i} \xrightarrow{\xi_i} \mathcal{A}^{\oplus l_i}_{W_i} \xrightarrow{\eta_i} \mathcal{B}|_{W_i} \longrightarrow 0$$

as in (2). For example, we can take V_i, W_i to be small open balls centred at the same point of a co-ordinate chart. The $\mathcal{A}$-module homomorphism $\xi_i\colon \mathcal{A}^{\oplus k_i}_{W_i} \to \mathcal{A}^{\oplus l_i}_{W_i}$ extends to an $\mathcal{O}$-module homomorphism $\xi'_i\colon \mathcal{O}^{\oplus k_i}_{W'_i} \to \mathcal{O}^{\oplus l_i}_{W'_i}$ on a neighbourhood W'_i of W_i in M' and defines a coherent $\mathcal{O}_{W'_i}$-module $\mathcal{B}'_i$ by the exact sequence

$$\mathcal{O}^{\oplus k_i}_{W'_i} \xrightarrow{\xi'_i} \mathcal{O}^{\oplus l_i}_{W'_i} \xrightarrow{\eta'_i} \mathcal{B}'_i \longrightarrow 0$$

By construction there is an $\mathcal{A}$-module isomorphism $\theta_i\colon \iota^{-1}\mathcal{B}'_i \to \mathcal{B}|_{W_i}$. For each pair (i, j) such that $V_{ij} := V_i \cap V_j$ is nonempty, we construct a homomorphism ϕ_{ij} from $\mathcal{B}'_i$ to $\mathcal{B}'_j$. Let $e_1, \ldots, e_{l_i}$ denote the standard basis of $\mathcal{O}^{\oplus l_i}_{W_i}$. For each $1 \le a \le l_i$, the image $\eta_i(e_a)$ is a section of $\mathcal{B}'_i$ and induces a section s_a of $\iota^{-1}\mathcal{B}'_i \cong \mathcal{B}|_{W_i}$. Via the isomorphism $\iota^{-1}\mathcal{B}'_j|_{W_{ij}} \cong \mathcal{B}|_{W_{ij}}$, the restriction $s_a|_{W_{ij}}$ can be lifted to a section t_a of $\mathcal{B}'_j$ over an open neighbourhood C_{ij} of W_{ij} in W'_{ij}. Because the intersection V_{ij} of Stein open sets V_i, V_j is Stein and because $V_{ij} \Subset W_{ij}$, we can find a Stein open neighbourhood A_{ij} of V_{ij} in C_{ij}. Because A_{ij} is Stein, we can find a lift $u_a \in \Gamma(A_{ij}, \mathcal{O}^{\oplus l_j})$ of $t_a|_{A_{ij}}$ such that $\eta_j(u_a) = t_a|_{A_{ij}}$.

https://doi.org/10.1017/S0017089515000427 Published online by Cambridge University Press

The sections $u_1, \ldots, u_{l_i}$ define a homomorphism $\psi_{ij} \colon \mathcal{O}_{A_{ij}}^{\oplus l_i} \to \mathcal{O}_{A_{ij}}^{\oplus l_j}$ sending e_a to u_a.

$$
\begin{array}{ccccccc}
\mathcal{O}_{A_{ij}}^{\oplus k_i} & \xrightarrow{\xi_i} & \mathcal{O}_{A_{ij}}^{\oplus l_i} & \xrightarrow{\eta_i} & \mathcal{B}'_i|_{A_{ij}} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \psi_{ij}} & & & & \\
\mathcal{O}_{A_{ij}}^{\oplus k_j} & \xrightarrow{\xi_j} & \mathcal{O}_{A_{ij}}^{\oplus l_j} & \xrightarrow{\eta_j} & \mathcal{B}'_j|_{A_{ij}} & \longrightarrow & 0
\end{array}
$$

We claim that, after shrinking A_{ij} if necessary, ψ_{ij} induces a homomorphism $\phi_{ij} \colon \mathcal{B}'_i|_{A_{ij}} \to \mathcal{B}'_j|_{A_{ij}}$. It suffices to show that the composition $\eta_j \circ \psi_{ij} \circ \xi_i$ is zero in a neighbourhood of V_{ij}. By construction, $\eta_i(v)$ and $\eta_j \circ \psi_{ij}(v)$ define the same section of $\mathcal{B}$ over V_{ij} for every $v \in \Gamma(A_{ij}, \mathcal{O}_{A_{ij}}^{\oplus l_i})$. Hence, for $w \in \Gamma(A_{ij}, \mathcal{O}_{A_{ij}}^{\oplus k_i})$, $\eta_j \circ \psi_{ij} \circ \xi_i(w)$ and $\eta_i \circ \xi_i(w) = 0$ define the same section of $\mathcal{B}$ over V_{ij}. This means that $\eta_j \circ \psi_{ij} \circ \xi_i(w)$ vanishes in a neighbourhood of V_{ij}, and the claim follows. It is clear that the first diagram in (4) commutes. We can also assume that $A_{ij} = A_{ji}$ by replacing A_{ij} with $A_{ij} \cap A_{ji}$ and restricting ϕ_{ij} to it if necessary.

Finally, we find an open subset B_{ijk} of $A_{ijk} = A_{ij} \cap A_{jk} \cap A_{ij}$ containing $V_{ijk} = V_i \cap V_j \cap V_k$ on which the second diagram in (4) commutes (i.e. the cocycle condition holds). But this is straightforward, because $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ holds at the stalk of each $x \in V_{ijk}$. □

Let $\iota \colon M \hookrightarrow M'$, $\mathcal{A}$, and $\mathcal{B}$ be as in the statement of Theorem 2. Take the data constructed in Lemma 12. By taking a refinement if necessary, we may assume that the open covering $\{V_i : i \in I\}$ is locally finite. Choose an open covering $\{U_i : i \in I\}$ of M such that $U_i \Subset V_i$. Take an open tubular neighbourhood of M in M' and, as in the proof of Theorem 1, fix a fibrewise Riemannian metric on it. For a open subset U of M and $\epsilon > 0$, we denote by $U(\epsilon) \subset M'$ the open tube of length ϵ over U. For each $i \in I$, there are only finitely many $j \in I$ such that the intersection V_{ij} is nonempty. Therefore, we can find $\epsilon_i > 0$ such that:

- $U_i(\epsilon_i) \subset W'_i$;
- $U_{ij}(\epsilon_i) \subset A_{ij}$ for all $j \in I$, where $U_{ij} := U_i \cap U_j$;
- $U_{ijk}(\epsilon_i) \subset B_{ijk}$ for all $j, k \in I$, where $U_{ijk} := U_i \cap U_j \cap U_k$.

Then the coherent sheaves $\mathcal{B}'_i|_{U_i(\epsilon_i)}$ glue together via the homomorphisms $\phi_{ij}|_{U_i(\epsilon_i) \cap U_j(\epsilon_j)}$ to give a global coherent $\mathcal{O}_N$-module $\mathcal{B}'$ on $N = \bigcup_{i \in I} U_i(\epsilon_i)$. It is clear that there is an isomorphism $\iota^{-1}\mathcal{B}' \cong \mathcal{B}$ of $\mathcal{A}$-modules. This completes the proof of part (i) of Theorem 2.

Let us prove part (ii) of Theorem 2. Suppose that we have coherent $\mathcal{O}_N$-modules $\mathcal{B}'_1$ and $\mathcal{B}'_2$ and a isomorphisms $\iota^{-1}\mathcal{B}'_1 \cong \mathcal{B} \cong \iota^{-1}\mathcal{B}'_2$ of $\mathcal{A}$-modules. Let $\phi \colon \iota^{-1}\mathcal{B}'_1 \to \iota^{-1}\mathcal{B}'_2$ denote the composite isomorphism. We can find a locally finite open covering $\{S_i : i \in I\}$ of M together with a family $\{T_i : i \in I\}$ of open subsets of M' such that $S_i \Subset T_i$ and $\mathcal{B}'_1|_{T_i}$ and $\mathcal{B}'_2|_{T_i}$ have finite presentations as $\mathcal{O}_{T_i}$-modules. Without loss of generality we may assume that S_i is Stein. The argument in the proof of Lemma 12 shows that we can find an open neighbourhood T_i° of S_i in T_i and a homomorphism $\Phi_i \colon \mathcal{B}'_1|_{T_i^\circ} \to \mathcal{B}'_2|_{T_i^\circ}$ such that $\iota^{-1}\Phi_i = \phi|_{S_i}$. For each $i, j \in I$, there exists an open subset P_{ij} of $T_i^\circ \cap T_j^\circ$ containing $S_{ij} := S_i \cap S_j$ such that $\Phi_i|_{P_{ij}} = \Phi_j|_{P_{ij}}$. Take an open covering $\{R_i : i \in I\}$ of M such that $R_i \Subset S_i$. As before, we choose a tubular neighbourhood of M in M' and a (fibrewise) Riemannian metric on it, denoting by $U(\epsilon)$ the open ϵ-tube over the open subset $U \subset M$. Then for each $i \in I$, there exists $\epsilon_i > 0$ such that

https://doi.org/10.1017/S0017089515000427 Published online by Cambridge University Press

- $R_i(\epsilon_i) \subset T_i^\circ$;
- $R_{ij}(\epsilon_i) \subset P_{ij}$;

because there are only finitely many $j \in I$ such that S_{ij} is nonempty. Now the homomorphisms $\Phi_i|_{R_i(\epsilon_i)}$ glue together to define a global homomorphism $\Phi\colon \mathcal{B}'_1|_P \to \mathcal{B}'_2|_P$ over $P = \bigcup_{i\in I} R_i(\epsilon_i)$ such that $\phi = \iota^{-1}\Phi$. The uniqueness of Φ as a germ is obvious. This completes the proof of Theorem 2.

4. Global unfolding of TEP structures. As an application of our results we now prove a global unfolding theorem for TEP structures, by globalizing the reconstruction theorem for germs of TEP structures due to Hertling–Manin [**4**]. This global unfolding theorem has applications in mirror symmetry and the study of quantum cohomology [see, T. Coates and H. Iritani "A Fock sheaf for Givental quantization", arXiv:1411.7039; Section 8.1.6]. TEP structures were introduced by Hertling [**3**]; they are closely related to Dubrovin's notion of Frobenius manifold [**2**, **1**, **4**].

DEFINITION 13 (TEP structure). Let M be a complex manifold. A *TEP structure* $(\mathcal{F}, \nabla, (\cdot,\cdot)_{\mathcal{F}})$ with base M consists of a locally free $\mathcal{O}_{M\times\mathbb{C}}$-module $\mathcal{F}$ of rank $N+1$, and a meromorphic flat connection

$$\nabla\colon \mathcal{F} \to (\pi^*\Omega^1_M \oplus \mathcal{O}_{M\times\mathbb{C}} z^{-1}dz) \otimes_{\mathcal{O}_{M\times\mathbb{C}}} \mathcal{F}(M\times\{0\}),$$

so that for $f \in \mathcal{O}_{M\times\mathbb{C}}$, $s \in \mathcal{F}$, and tangent vector fields $v_1, v_2 \in \Theta_{M\times\mathbb{C}}$:

$$\nabla(fs) = df \otimes s + f\nabla s, \qquad [\nabla_{v_1}, \nabla_{v_2}] = \nabla_{[v_1,v_2]},$$

together with a non-degenerate pairing

$$(\cdot,\cdot)_{\mathcal{F}}\colon (-)^*\mathcal{F} \otimes_{\mathcal{O}_{M\times\mathbb{C}}} \mathcal{F} \to \mathcal{O}_{M\times\mathbb{C}},$$

which satisfies

$$\begin{gathered}((-)^*s_1, s_2)_{\mathcal{F}} = (-)^*((-)^*s_2, s_1)_{\mathcal{F}} \\ d((-)^*s_1, s_2)_{\mathcal{F}} = ((-)^*\nabla s_1, s_2)_{\mathcal{F}} + ((-)^*s_1, \nabla s_2)_{\mathcal{F}},\end{gathered}$$

for $s_1, s_2 \in \mathcal{F}$. Here $\mathcal{F}(M\times\{0\})$ denotes the sheaf of sections of $\mathcal{F}$ with poles of order at most 1 along the divisor $M\times\{0\} \subset M\times\mathbb{C}$ and $(-)\colon M\times\mathbb{C} \to M\times\mathbb{C}$ is the map sending (y,z) to $(y,-z)$.

DEFINITION 14 (Miniversality). Let M be a complex manifold. A *TEP structure* $(\mathcal{F}, \nabla, (\cdot,\cdot)_{\mathcal{F}})$ with base M is called *miniversal* if for each $y \in M$, the set

$$\left\{x \in \mathcal{F}|_{(y,0)} : \text{the map } T_yM \to \mathcal{F}|_{(y,0)},\ v \mapsto (z\nabla_v x)|_{(y,0)} \text{ is an isomorphism}\right\}$$

is nonempty in the fibre $\mathcal{F}|_{(y,0)}$.

THEOREM 15 (Global unfolding for TEP structures). *Let M be a complex manifold and $(\mathcal{F}, \nabla, (\cdot,\cdot)_{\mathcal{F}})$ a TEP structure with base M. Suppose that for each $y \in M$, there exists a section ζ of $\mathcal{F}$ over a neighbourhood of $(y,0) \in M\times\mathbb{C}$ such that*

https://doi.org/10.1017/S0017089515000427 Published online by Cambridge University Press

(IC) *the map* $T_yM \to \mathcal{F}|_{(y,0)}$ *defined by* $v \mapsto z\nabla_v\zeta|_{(y,0)}$ *is injective;*
(GC) *the fibre* $\mathcal{F}|_{(y,0)}$ *is generated by iterated derivatives*

$$(z^2\nabla_{\partial_z})^l z\nabla_{v_1}\cdots z\nabla_{v_k}\zeta\big|_{(y,0)} \qquad l \geq 0$$

with respect to local vector fields $v_1, \ldots, v_k$ *on* M *near* y *and* $z^2\partial_z$.

Then there exist a complex manifold M', *a miniversal TEP structure* $(\mathcal{F}', \nabla', (\cdot,\cdot)_{\mathcal{F}'})$ *with base* M', *and a closed embedding* $\iota\colon M \to M'$ *such that*

$$\iota^\star\Big(\mathcal{F}', \nabla', (\cdot,\cdot)_{\mathcal{F}'}\Big) = \Big(\mathcal{F}, \nabla, (\cdot,\cdot)_{\mathcal{F}}\Big).$$

Furthermore, the manifold-germ M' *and the TEP structure* $(\mathcal{F}', \nabla', (\cdot,\cdot)_{\mathcal{F}'})$ *are unique up to unique isomorphism in the sense of Theorem 1 and 2.*

Proof. Combine Theorems 1 and 2 with the universal unfolding theorem for germs of TEP structures proved by Hertling–Manin [**4**, Theorem 2.5, Lemma 3.2]. □

Analogous global unfolding theorems for TE structures [**4**], log-trTLEP structures [**7**], and so on can be proved in exactly the same way. Global unfoldings of log-trTLEP structures have interesting applications in Gromov–Witten theory see, T. Coates and H. Iritani “A Fock sheaf for Givental quantization”, arXiv:1411.7039.

Acknowledgements. T.C. thanks Eugenia Cheng for useful correspondence. H.I. thanks Ono Kaoru and Ken-ichi Yoshikawa for useful discussions.

REFERENCES

1. B. Dubrovin, Geometry of 2D topological field theories, in *Integrable systems and quantum groups (Montecatini Terme, 1993)*, Lecture Notes in Mathematics, vol. 1620 (Springer, Berlin, 1996), 120–348.
2. B. Dubrovin, Geometry and analytic theory of Frobenius manifolds, in *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, number Extra vol. II (1998), 315–326.
3. C. Hertling, *tt** geometry, Frobenius manifolds, their connections, and the construction for singularities, *J. Reine Angew. Math.* **555** (2003), 77–161.
4. C. Hertling and Y. Manin, Unfoldings of meromorphic connections and a construction of Frobenius manifolds, in *Frobenius manifolds*, Aspects of Mathematics, vol. E36 (Vieweg, Wiesbaden, 2004), 113–144.
5. D. Joyce, D-manifolds and d-orbifolds: A theory of derived differential geometry, Available at http://people.maths.ox.ac.uk/joyce/dmbook.pdf, 2012.
6. M. Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217 (American Mathematical Society, Providence, RI, 2003). (Translated from the 2000 Japanese original by Mutsumi Saito, Iwanami Series in Modern Mathematics.)
7. T. Reichelt, A construction of Frobenius manifolds with logarithmic poles and applications, *Comm. Math. Phys.* **287**(3) (2009), 1145–1187.
8. M. Shiota, Some results on formal power series and differentiable functions, *Publ. Res. Inst. Math. Sci.* **12**(1) (1967/77), 49–53.
9. H. Whitney and F. Bruhat, Quelques propriétés fondamentales des ensembles analytiques-réels, *Comment. Math. Helv.* **33** (1959), 132–160.

https://doi.org/10.1017/S0017089515000427 Published online by Cambridge University Press