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ON (2, 3, 7)-GENERATION OF MAXIMAL PARABOLIC SUBGROUPS

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To Laci Kovács on his 65th birthday

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Abstract

Let R be a ring with 1 and $E_n(R)$ be the subgroup of $\mathrm{GL}_n(R)$ generated by the matrices $I + re_{ij}$, $r \in R$, $i \neq j$. We prove that the subgroup $P_{n,\bar{n}}$ of $E_{n+\bar{n}}(R)$ consisting of the matrices of shape $\left(\begin{smallmatrix} A & B \\ 0 & \bar{A} \end{smallmatrix}\right)$, where $A \in E_n(R)$, $\bar{A} \in E_{\bar{n}}(R)$ and $B \in \mathrm{Mat}_{n,\bar{n}}(R)$, is (2, 3, 7)-generated whenever R is finitely generated and n, $\bar{n}$ are large enough.

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1. Introduction

We recall that a non-trivial group is said to be (2, 3, 7)-generated if it is an epimorphic image of the infinite triangle group $\Delta(2, 3, 7)$, defined by the presentation $\langle X, Y \mid X^2 = Y^3 = (XY)^7 = 1\rangle$. The relevance of (2, 3, 7)-generated groups stems from their relation with the normal structure of the classical modular group and the theory of Riemann surfaces. In particular, the finite (2, 3, 7)-generated groups (the so-called Hurwitz groups) are realizable as automorphism groups of maximal order of compact Riemann surfaces of genus at least 2. Over the recent years, it has been shown that the class of (2, 3, 7)-generated groups is indeed quite large (see [LTW, LT], and the references quoted there). Naturally, since (2, 3, 7)-generated groups are perfect, much attention has been devoted to simple or close to simple groups. In particular, while the seminal paper [Co] had already shown that the finite alternating groups A_n are

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Hurwitz provided $n > 167$, [LTW] and [LT] show that most finite classical groups are Hurwitz, provided their Lie rank is large enough. The prototype key-result in this context is Theorem A in [LTW]. Let R be an arbitrary ring with 1 and, for each $n \in \mathbb{N}$, define $E_n(R)$ to be the group generated by the set of $n \times n$ matrices $\{I + re_{ij} \mid r \in R,\ 1 \leq i \neq j \leq n\}$. It is well known that, for $n \geq 3$, $E_n(R) = \mathrm{SL}_n(R)$ if R is commutative and either a semi-local or a Euclidean domain (see [HO'M]). In particular, $E_n(R)$ contains $E_n(R_0) = \mathrm{SL}_n(R_0)$, where R_0 denotes the subring of R generated by 1. Theorem A in [LTW] asserts that, if R is finitely generated, then $E_n(R)$ is $(2, 3, 7)$-generated for all sufficiently large n. However, it was also noticed that Theorem A could be applied in order to prove $(2, 3, 7)$-generation for certain semi-simple groups and even for groups that are far apart from the semi-simple ones. Namely, it was proven in [LTW] that if $n \geq 287$, the direct product of t copies of $\mathrm{SL}_n(F_q)$, where F_q denotes the finite field of order $q = p^a$, is Hurwitz provided $t \leq q^{[(n-287)/84]}$; and moreover, that there exist Hurwitz groups which are extensions of p-groups of arbitrarily large derived length by the group $\mathrm{SL}_n(F_q)$. Pushing further in this direction, but still in connection with groups of Lie type, in this paper we prove that the subgroup

$$P_{n,\bar{n}}(R) = \left\{ \begin{pmatrix} A & B \\ 0 & \bar{A} \end{pmatrix} \middle| \, A \in E_n(R),\ \bar{A} \in E_{\bar{n}}(R),\ B \in \mathrm{Mat}_{n,\bar{n}}(R) \right\}$$

of $E_{n+\bar{n}}(R)$ is $(2, 3, 7)$-generated, whenever R is finitely generated and $n, \bar{n}$ are large enough. Note that, for $n, \bar{n} \geq 3$, $P_{n,\bar{n}}(R)$ coincides with the commutator subgroup of a maximal parabolic of $E_{n+\bar{n}}(R)$. Clearly, $P_{n,\bar{n}}(R)$ is the semidirect product of the 'unipotent radical' N, consisting of the matrices of shape $\left(\begin{smallmatrix} I_n & B \\ 0 & I_{\bar{n}} \end{smallmatrix}\right)$, and the 'standard Levi subgroup' L, isomorphic to $E_n(R) \times E_{\bar{n}}(R)$, consisting of the matrices of shape $\left(\begin{smallmatrix} A & 0 \\ 0 & \bar{A} \end{smallmatrix}\right)$. Then, via suitable variations of the techniques developed in [LTW], the following is proven:

THEOREM. *Let R be generated by elements $t_1, \ldots, t_m$, where $2t_1 - t_1^2$ is a unit of R of finite multiplicative order. Then $P_{n,\bar{n}}(R)$ (and therefore also the Levi subgroup L) is $(2, 3, 7)$-generated for all $n, \bar{n} \geq 84(m + 1) + 180 + 216$.*

2. Joining representations via handles

Let Σ be the canonical basis of the free R-module $\langle \Sigma \rangle$ consisting of all row vectors of size $|\Sigma|$. We assume $|\Sigma| < \infty$ and let the group $\mathrm{GL}_{|\Sigma|}(R)$ act on the right on $\langle \Sigma \rangle$. We also identify the symmetric group $\mathrm{Sym}(\Sigma)$ with the group of permutation matrices. As above, we let $\Delta(2, 3, 7) = \langle X, Y \mid X^2 = Y^3 = (XY)^7 = 1 \rangle$.

DEFINITION. If $\psi : \Delta(2,3,7) \to \mathrm{GL}_{|\Sigma|}(R)$ is a representation, a 2-subset $\{a_2, a_3\}$ of Σ is called a *handle for* ψ if the following conditions are satisfied:

(1) $\psi(X)$ fixes a_2, a_3 and the submodule $\langle \Sigma \setminus \{a_2, a_3\}\rangle$;
(2) $\psi(Y)$ acts as one of the cycles $(a_1, a_2, a_3)^{\pm 1}$ for some $a_1 \in \Sigma$ and fixes $\langle \Sigma \setminus \{a_1, a_2, a_3\}\rangle$.

We note that, whenever ψ is a permutation representation, our definition of a handle coincides with that given by Conder in [Co]. In the sequel, we will always assume that $\psi(Y)$ acts as (a_1, a_2, a_3).

LEMMA 1. *Let $\{a_2, a_3\}$, $\{b_2, b_3\}$ be disjoint handles for a representation ψ of $\Delta(2,3,7)$ and suppose that $Z \in \mathrm{GL}_{|\Sigma|}(R)$ induces the identity on $\langle \Sigma \setminus \{a_2, a_3, b_2, b_3\}\rangle$ and acts on $\langle a_2, a_3, b_2, b_3\rangle$ in one of the following ways:*

$$(1)\quad \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad or \quad (2)\quad \begin{pmatrix} -I & \alpha I \\ 0 & I \end{pmatrix},$$

where $\alpha \in R$.

Then $\psi(X)Z$ is an involution and $\psi(X)Z\psi(Y)$ is conjugate to $\psi(XY)$. In particular, one can define a representation $\hat{\psi} : \Delta(2,3,7) \to \mathrm{GL}_{|\Sigma|}(R)$ by setting $\hat{\psi}(X) = \psi(X)Z$, $\hat{\psi}(Y) = \psi(Y)$.

The easy proof of the above Lemma is given in [LT]. Clearly, the handles of ψ different from $\{a_2, a_3\}$ and $\{b_2, b_3\}$, if any, are still handles of $\hat{\psi}$. This fact allows repeated application of Lemma 1. In the following we are interested in commutators. In order to use in a more direct way the results of Conder, we find it convenient to write $[a, b]$ for the commutator $aba^{-1}b^{-1}$. (We warn the reader that this notation differs from that used in [LTW] and [LT], where $[a, b]$ stands for $a^{-1}b^{-1}ab$).

LEMMA 2. *Under the assumptions of Lemma 1, set $\gamma = [\psi(X), \psi(Y)]$ and $\hat{\gamma} = [\hat{\psi}(X), \hat{\psi}(Y)]$.*

(1) *Both a_2 and b_2 are fixed by γ and $\hat{\gamma}$.*
(2) *Suppose that Δ is a subset of Σ such that each of the submodules $\langle \Delta\rangle$ and $\langle \Sigma \setminus \Delta\rangle$ is invariant under γ. Setting $a_1 = a_2\psi(Y^{-1})$, $b_1 = b_2\psi(Y^{-1})$, assume further that $\Delta_1 = \{a_1\psi(X),\ a_3,\ b_1\psi(X),\ b_3\}$ is a subset of Δ. Then*

  (i) *$\langle \Sigma \setminus \Delta\rangle$ is $\hat{\gamma}$-invariant and $\hat{\gamma}$ acts on $\langle \Sigma \setminus \Delta\rangle$ in the same way as γ;*
  (ii) *$\langle \Delta\rangle$ is $\hat{\gamma}$-invariant.*

(3) *Suppose that γ acts on Δ as a permutation consisting of two cycles of respective lengths $\mu = \mu_1 + \mu_2$, $\nu = \nu_1 + \nu_2$ of the following shape:*

$$\Big(\underbrace{a_1\psi(XY^{-1}), \dots, a_1\psi(X)}_{\mu_1}, \underbrace{a_1, \dots, a_3}_{\mu_2}\Big)\quad \Big(\underbrace{b_1\psi(XY^{-1}), \dots, b_1\psi(X)}_{\nu_1}, \underbrace{b_1, \dots, b_3}_{\nu_2}\Big).$$

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If Z is of type (2) assume further that $\mu_2 = \nu_2$. Then $\hat{\gamma}_{|\langle\Delta\rangle}$ has order l.c.m.$(\mu_1 + \nu_2, \mu_2 + \nu_1)$.

PROOF. (1) Direct calculation.

(2) Using (1), we see that, for each $v \in \Sigma \setminus \Delta_1$, $v\hat{\gamma} = v\gamma$. Thus (i) follows at once from the assumptions that $\Delta_1 \subseteq \Delta$ and $\langle \Sigma \setminus \Delta \rangle$ is γ-invariant. As for (ii), for each $v \in \Delta \setminus \Delta_1$, we get $v\hat{\gamma} = v\gamma \in \langle\Delta\rangle$. Thus, we need to show that, if $v \in \Delta_1$, then $v\hat{\gamma} \in \langle\Delta\rangle$. For this purpose we assume first that Z is of type (1). In this case:

$$a_1\psi(X)\hat{\gamma} = b_1 = b_1\psi(X)\gamma, \quad a_3\hat{\gamma} = b_1\psi(XY^{-1}) = b_3\gamma,$$
$$b_1\psi(X)\hat{\gamma} = a_1 = a_1\psi(X)\gamma, \quad b_3\hat{\gamma} = a_1\psi(XY^{-1}) = a_3\gamma.$$

Hence $\Delta_1\hat{\gamma} = \Delta_1\gamma \subseteq \langle\Delta\rangle$, by our assumptions. Next, assume that Z is of type (2). Then

$$a_1\psi(X)\hat{\gamma} = -a_1 + \alpha b_1 = -a_1\psi(X)\gamma + \alpha b_1\psi(X)\gamma \in \langle\Delta\rangle$$
$$a_3\hat{\gamma} = -a_1\psi(XY^{-1}) + \alpha b_1\psi(XY^{-1}) = -a_3\gamma + \alpha b_3\gamma \in \langle\Delta\rangle.$$

Finally, $b_1\psi(X)\hat{\gamma} = b_1 = b_1\psi(X)\gamma \in \langle\Delta\rangle$ and $b_3\hat{\gamma} = b_1\psi(XY^{-1}) = b_3\gamma \in \langle\Delta\rangle$.

(3) Suppose first that Z is of type (1). By the above considerations, $\hat{\gamma}$ acts on Δ as

$$\big(\underbrace{a_1\psi(XY^{-1}), \dots, a_1\psi(X)}_{\mu_1}, \underbrace{b_1, \dots, b_3}_{\nu_2}\big) \quad \big(\underbrace{b_1\psi(XY^{-1}), \dots, b_1\psi(X)}_{\nu_1}, \underbrace{a_1, \dots, a_3}_{\mu_2}\big).$$

Next, suppose that Z is of type (2), hence by assumption $\mu_2 = \nu_2$. Then $\hat{\gamma}$ acts on Δ as

$$\big(\underbrace{a_1\psi(XY^{-1}), \dots, a_1\psi(X)}_{\mu_1}, \underbrace{-a_1 + \alpha b_1, \dots, -a_3 + \alpha b_3}_{\mu_2}\big)$$
$$\big(\underbrace{b_1\psi(XY^{-1}), \dots, b_1\psi(X)}_{\nu-\mu_2}, \underbrace{b_1, \dots, b_3}_{\mu_2}\big).$$

□

COROLLARY 3. *Let $\psi : \Delta(2,3,7) \to \mathrm{GL}_{|\Sigma|}(R)$ be a permutation representation. For $i = 1, \dots, k$, assume that $\{a_2^i, a_3^i\}$ and $\{b_2^i, b_3^i\}$ are disjoint handles for ψ and that Z_i acts on the submodule $\langle a_2^i, a_3^i, b_2^i, b_3^i\rangle$ in one of the ways described in Lemma 1, fixing pointwise $\langle\Sigma \setminus \{a_2^i, a_3^i, b_2^i, b_3^i\}\rangle$. Let $\hat{\psi} : \Delta(2,3,7) \to \mathrm{GL}_{|\Sigma|}(R)$ be defined by*

$$\hat{\psi}(X) = \psi(X)Z_1 \dots Z_k, \quad \hat{\psi}(Y) = \psi(Y).$$

As above, set $\gamma = [\psi(X), \psi(Y]$ and $\hat{\gamma} = [\hat{\psi}(X), \hat{\psi}(Y]$. For each $i \leq k$, let A^i be a $\langle\gamma\rangle$-invariant subset of Σ containing $\{a_1^i\psi(X), a_3^i\}$, where $a_1^i = a_2^i\psi(Y^{-1})$. Similarly, let B^i be a $\langle\gamma\rangle$-invariant subset of Σ containing $\{b_1^i\psi(X), b_3^i\}$, where $b_1^i = b_2^i\psi(Y^{-1})$. Assume further that $A^i \cap B^i = \emptyset$. Then, setting $U_k = \cup_{i=1}^k (A^i \cup B^i)$, each of the submodules $\langle A^1, B^1\rangle, \dots, \langle A^k, B^k\rangle$ and $\langle\Sigma \setminus U_k\rangle$ is $\hat{\gamma}$-invariant, and $\hat{\gamma}$ acts on $\langle\Sigma \setminus U_k\rangle$ in the same way as γ.

PROOF. Assume $k = 1$. Then $A^1 \cup B^1 = U_1$ is γ-invariant by assumption. It follows that $\Sigma \setminus U_1$ is also γ-invariant, since γ acts as a permutation on Σ. Thus our claim follows from Lemma 2, with $\Delta = U_1$. Now assume $k > 1$ and consider the representation φ defined by $\varphi(X) = \psi(X)Z_1 \dots Z_{k-1}$, $\varphi(Y) = \psi(Y)$. Set $U_{k-1} = \bigcup_{i=1}^{k-1}(A^i \cup B^i)$. By induction, each of the submodules $\langle A^1, B^1\rangle, \dots, \langle A^{k-1}, B^{k-1}\rangle$ and $\langle \Sigma \setminus U_{k-1}\rangle$ is invariant under $[\varphi(X), \varphi(Y)] = \gamma_\varphi$, and γ_φ acts on $\langle \Sigma \setminus U_{k-1}\rangle$ in the same way as γ. Thus γ_φ acts as a permutation on $\Sigma \setminus U_{k-1}$, fixing each of the sets A^k, B^k and $\Sigma \setminus U_k$. In particular, γ_φ fixes $\langle \Sigma \setminus U_k\rangle \oplus \langle A^1, B^1\rangle \oplus \cdots \oplus \langle A^{k-1}, B^{k-1}\rangle = \langle \Sigma \setminus (A^k \cup B^k)\rangle$. Now we have $\hat{\psi}(X) = \varphi(X)Z_k$, $\hat{\psi}(Y) = \varphi(Y)$. Application of Lemma 2 to the representation φ, with $\Delta = A^k \cup B^k$, shows that the submodules $\langle A^k, B^k\rangle$ and $\langle \Sigma \setminus (A^k \cup B^k)\rangle$ are both invariant under $\hat{\gamma}$, and that $\hat{\gamma}$ acts on $\langle \Sigma \setminus (A^k \cup B^k)\rangle$ in the same way as γ_φ. It follows that $\langle A^1, B^1\rangle, \dots, \langle A^{k-1}, B^{k-1}\rangle$ are also $\hat{\gamma}$-invariant. Since $\langle \Sigma \setminus U_k\rangle \le \langle \Sigma \setminus (A^k \cup B^k)\rangle \cap \langle \Sigma \setminus U_{k-1}\rangle$, we conclude that $\hat{\gamma}$ acts on $\langle \Sigma \setminus U_k\rangle$ in the same way as γ. □

3. The (2, 3, 7)-generators for $P_{n,\bar{n}}(R)$

Let $\Omega = \{v_i \mid 1 \le i \le n\}$ and $\bar{\Omega} = \{\bar{v}_i \mid 1 \le i \le \bar{n}\}$ be the canonical bases for the free R-modules consisting of row vectors of sizes n and $\bar{n}$ respectively. Thus $P_{n,\bar{n}}(R)$ acts naturally, on the right, on the direct sum $\langle \Omega\rangle \oplus \langle \bar{\Omega}\rangle$. In order to describe a pair of (2, 3, 7)-generators $\hat{x}$, $\hat{y}$ for the group $P_{n,\bar{n}}(R)$, we make use of 17 diagrams introduced by Conder in [Co]: namely the diagrams G, E, A (pages 78 and 79) with 42, 28, 14 vertices respectively, and 14 diagrams H_v (page 84) with v vertices, $v \in D = \{36, 42, 57, 77, 115, 135, 136, 142, 144, 165, 180, 187, 195, 216\}$.

We assume $n \not\equiv 10 \pmod{14}$ unless $n \equiv \bar{n} \equiv 10 \pmod{14}$, and set:

$$\begin{cases} l = 42, \quad \bar{l} = 0 & \text{if } n \equiv \bar{n} \equiv 10 \pmod{14}; \\ l = 180, \quad \bar{l} = 0 & \text{if } n \not\equiv 10 \pmod{14},\ \bar{n} \not\equiv 12 \pmod{14}; \\ l = 180, \quad \bar{l} = 187 & \text{if } n \not\equiv 10 \pmod{14},\ \bar{n} \equiv 12 \pmod{14}. \end{cases}$$

As the elements of D give all residues modulo 14, if n and $\bar{n}$ are large enough, then there exist uniquely determined $a, \bar{a} \ge 2$, $b, \bar{b} \in \{0, 1, 2\}$ and $d, \bar{d} \in D$ such that we can write:

$$n - l = 42a + 14b + d, \quad \bar{n} - \bar{l} = 42\bar{a} + 14\bar{b} + \bar{d}.$$

We think of Ω and $\bar{\Omega}$ as a union of Conder diagrams. Namely, let

$$\Omega_0 = G_1 \cup \cdots \cup G_a \cup H_d \cup H_l, \quad \bar{\Omega}_0 = \bar{G}_1 \cup \cdots \cup \bar{G}_{\bar{a}} \cup \bar{H}_{\bar{d}} \cup \bar{H}_{\bar{l}},$$

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where each G_i, $\bar{G}_i$ is a copy of G, whereas $\bar{A}$, $\bar{E}$, $\bar{H}_{\bar{d}}$, $\bar{H}_{\bar{l}}$ are copies of A, E, $H_{\bar{d}}$, $H_{\bar{l}}$ respectively, and $H_l = \emptyset$ if $l = 0$, $\bar{H}_{\bar{l}} = \emptyset$ if $\bar{l} = 0$. Then we set:

$$\Omega = \Omega_0 \text{ if } b = 0, \qquad \Omega = \Omega_0 \cup A \text{ if } b = 1, \qquad \Omega = \Omega_0 \cup E \text{ if } b = 2,$$
$$\bar{\Omega} = \bar{\Omega}_0 \text{ if } \bar{b} = 0, \qquad \bar{\Omega} = \bar{\Omega}_0 \cup \bar{A} \text{ if } \bar{b} = 1, \qquad \bar{\Omega} = \bar{\Omega}_0 \cup \bar{E} \text{ if } \bar{b} = 2.$$

We label the points of Ω (and likewise those of $\bar{\Omega}$, putting bars everywhere) as follows:

$$G_i = \{v_j \mid 42(i-1)+1 \le j \le 42i\}, \quad 1 \le i \le a;$$
$$A = \{w_i \mid 1 \le i \le 14\}, \quad E = \{w_i \mid 1 \le i \le 28\};$$
$$H_d = \{v_j \mid 42a+1 \le j \le n-l\}; \quad H_l = \{v_j \mid n-l+1 \le j \le n\}.$$

In order to construct $\hat{x}$, $\hat{y}$, we start with a permutation representation of $\Delta(2,3,7) = \langle X, Y \mid X^2 = Y^3 = (XY)^7 = 1\rangle$ on $\langle\Omega\rangle \oplus \langle\bar{\Omega}\rangle$, defined by

$$X \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \bar{\xi} \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} y & 0 \\ 0 & \bar{y} \end{pmatrix},$$

where ξ, y, $\bar{\xi}$, $\bar{y}$ are given below. For this purpose, let U be one of Conder diagrams mentioned above. U depicts a transitive permutation representation ψ_U of $\Delta(2,3,7)$ on the set of vertices of U, hence of degree $v = |U|$. For each $i \le a$, we write $\xi_i = \psi_G(X)$, $y_i = \psi_G(Y)$. Then we define:

$$\xi_0 = \left(\prod_{i=1}^{a} \xi_i\right) \psi_{H_d}(X)\psi_{H_l}(X), \quad y_0 = \left(\prod_{i=1}^{a} y_i\right) \psi_{H_d}(Y)\psi_{H_l}(Y),$$

where $\psi_{H_l}(X) = \psi_{H_l}(Y) = \emptyset$ if $l = 0$, and set

$$\begin{cases} \xi = \xi_0, \; y = y_0, & \text{if } b = 0; \\ \xi = \xi_0\psi_A(X), \; y = y_0\psi_A(Y), & \text{if } b = 1; \\ \xi = \xi_0\psi_E(X), \; y = y_0\psi_E(Y), & \text{if } b = 2. \end{cases}$$

We fix our labelling so that

$$\begin{aligned} \xi_1 = \xi_{|G_1} = {} & (v_1, v_4)(v_5, v_7)(v_6, v_{10})(v_8, v_{12})(v_9, v_{24})(v_{11}, v_{29})(v_{13}, v_{16})(v_{17}, v_{19}) \\ & (v_{18}, v_{25})(v_{20}, v_{27})(v_{21}, v_{23})(v_{22}, v_{39})(v_{26}, v_{30})(v_{28}, v_{41})(v_{31}, v_{34}) \\ & (v_{35}, v_{37})(v_{36}, v_{40})(v_{38}, v_{42}), \\ y_1 = y_{|G_1} = {} & \prod_{0 \le j \le 13} (v_{3j+1}, v_{3j+2}, v_{3j+3}); \end{aligned}$$

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for $i > 1$, the actions of $\xi_i = \xi_{|G_i}$ and $y_i = y_{|G_i}$ are obtained translating the indices by $42(i-1)$;

$$\psi_A(X) = (w_1, w_4)(w_5, w_9)(w_6, w_{11})(w_7, w_{10})(w_8, w_{13})(w_{12}, w_{14}),$$
$$\psi_A(Y) = \prod_{0 \le j \le 3} (w_{3j+1}, w_{3j+2}, w_{3j+3});$$
$$\psi_E(X) = (w_1, w_4)(w_5, w_9)(w_6, w_{11})(w_7, w_{10})(w_8, w_{13})(w_{12}, w_{24})$$
$$(w_{14}, w_{26})(w_{15}, w_{16})(w_{18}, w_{19})(w_{21}, w_{22})(w_{23}, w_{25})(w_{27}, w_{28}),$$
$$\psi_E(Y) = \prod_{0 \le j \le 8} (w_{3j+1}, w_{3j+2}, w_{3j+3}).$$

According to the chosen labelling, the three handles in each G_i are denoted respectively by

$$\{v_{2+42(i-1)}, v_{3+42(i-1)}\}, \quad \{v_{14+42(i-1)}, v_{15+42(i-1)}\}, \quad \{v_{32+42(i-1)}, v_{33+42(i-1)}\}$$

and the handle in A or E by $\{w_2, w_3\}$. We denote by $\{b_2^v, b_3^v\}$ the handle in H_v, $v \in D$. $\bar{\xi}$ and $\bar{y}$ are defined in a similar way with respect to $\bar{\Omega}$.

DEFINITION OF $\hat{x}$ AND $\hat{y}$. The (2, 3, 7)-generators of $P_{n,\bar{n}}(R)$ are obtained by extending the above permutation representation of $\Delta(2, 3, 7)$ to the linear representation on $\langle\Omega\rangle \oplus \langle\bar{\Omega}\rangle$:

$$X \mapsto \hat{x} = \begin{pmatrix} \xi\zeta x_1 & 0 \\ 0 & \bar{\xi}\bar{\zeta}\bar{x}_1 \end{pmatrix} T, \qquad Y \mapsto \hat{y} = \begin{pmatrix} y & 0 \\ 0 & \bar{y} \end{pmatrix},$$

where $\zeta, \bar{\zeta}, x_1, \bar{x}_1$ and T are defined below. Note that, in order to have enough handles for repeated application of Lemma 1, we need to impose the condition $a, \bar{a} \ge 2m+2$, where m is the number of generators $t_1, \ldots, t_m$ of the ring R.

ζ is a permutation of order 2 which, by repeated application of Lemma 1 to ξ, with $\Sigma = \Omega$ and Z of type (1): joins each diagram G_i to G_{i+1} $(1 \le i < a-1)$ via the last handle of G_i and the first handle of G_{i+1}; joins G_{a-1} either to A or to E via the central handle of G_{a-1}; joins G_a to H_d via the central handle and, if $l \ne 0$, joins G_a to H_l via the last handle. Namely:

$$v_{32+42(i-1)}\zeta = v_{2+42i}, \qquad v_{33+42(i-1)}\zeta = v_{3+42i} \quad (1 \le i < a-1)$$
$$v_{14+42(a-2)}\zeta = w_2, \qquad v_{15+42(a-2)}\zeta = w_3$$
$$v_{32+42(a-1)}\zeta = b_2^d, \qquad v_{33+42(a-1)}\zeta = b_3^d \quad \text{and if } l \ne 0,$$
$$v_{14+42(a-1)}\zeta = b_2^l, \qquad v_{15+42(a-1)} = b_3^l.$$

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The permutation $\bar{\zeta}$ is defined similarly with respect to $\bar{\Omega}$. It follows that $X \mapsto \begin{pmatrix} \xi\zeta & 0 \\ 0 & \bar{\xi}\bar{\zeta} \end{pmatrix}$, $Y \mapsto \hat{y}$ defines a permutation representation of $\Delta(2,3,7)$, which is transitive on each of Ω and $\bar{\Omega}$. Actually, it will turn out that $\langle \xi\zeta, y \rangle = \mathrm{Alt}(\Omega)$ and $\langle \bar{\xi}\bar{\zeta}, \bar{y} \rangle = \mathrm{Alt}(\bar{\Omega})$.

As in [LTW], x_1 is an involution of $E_n(R)$ which, by repeated application of Lemma 1 to $\xi\zeta$ with $\Sigma = \Omega$ and Z of type (2), joins the first two handles of G_1 and then joins in pairs the central handles of $\underbrace{G_2, G_3}, \dots, \underbrace{G_{2m-2}, G_{2m-1}}$. Namely:

$$v_2 x_1 = -v_2 + t_1 v_{14}, \quad v_3 x_1 = -v_3 + t_1 v_{15};$$
$$v_{14+42(2j-3)} x_1 = -v_{14+42(2j-3)} + t_j v_{14+42(2j-2)}, \quad 2 \le j \le m;$$
$$v_{15+42(2j-3)} x_1 = -v_{15+42(2j-3)} + t_j v_{15+42(2j-2)}, \quad 2 \le j \le m.$$

$\bar{x}_1$ is defined in a similar way with respect to $\bar{\Omega}$. Thus $X \mapsto \begin{pmatrix} \xi\zeta x_1 & 0 \\ 0 & \bar{\xi}\bar{\zeta}\bar{x}_1 \end{pmatrix}$, $Y \mapsto \hat{y}$ defines a linear representation of $\Delta(2,3,7)$. It will be shown that the image of $\Delta(2,3,7)$ under this representation is the full Levi subgroup L of $P_{n,\bar{n}}(R)$.

Finally, applying Lemma 1 to $\begin{pmatrix} \xi\zeta x_1 & 0 \\ 0 & \bar{\xi}\bar{\zeta}\bar{x}_1 \end{pmatrix}$, with $\Sigma = \Omega \cup \bar{\Omega}$ and Z of type (2), we define an element T of the unipotent radical N, which joins the central handles of G_{a-2} and $\bar{G}_{\bar{a}-2}$ as follows:

$$v_{14+42(a-3)} T = -v_{14+42(a-3)} + \bar{v}_{14+42(a-3)},$$
$$v_{15+42(a-3)} T = -v_{15+42(a-3)} + \bar{v}_{15+42(a-3)}.$$

4. Action of the commutator

For each subset Δ of Ω we identify $\mathrm{Alt}(\Delta)$ with the group of even permutation matrices of $E_n(R)$ which fix every point of $\Omega \setminus \Delta$. We make a similar identification for $\mathrm{Alt}(\bar{\Delta})$, where $\bar{\Delta} \subseteq \bar{\Omega}$. Moreover, we set:

$$\widehat{\mathrm{Alt}}(\Delta) = \left\{ \begin{pmatrix} s & 0 \\ 0 & I_{\bar{n}} \end{pmatrix} \middle| s \in \mathrm{Alt}(\Delta) \right\}, \quad \widehat{\mathrm{Alt}}(\bar{\Delta}) = \left\{ \begin{pmatrix} I_n & 0 \\ 0 & \bar{s} \end{pmatrix} \mid \bar{s} \in \mathrm{Alt}(\bar{\Delta}) \right\}.$$

Let ψ be a permutation representation of $\Delta(2,3,7)$. A cycle c of the commutator $[\psi(X), \psi(Y)]$ is called *useful* in [Co], if c has prime odd length and contains an orbit of $\psi(X)$ and two points from an orbit of $\psi(Y)$.

LEMMA 4. *Let d, l, $\hat{x}$, $\hat{y}$ be defined as in Section* 3. *The following properties hold*:

(i) $\hat{x}^2 = \hat{y}^3 = (\hat{x}\hat{y})^7 = 1$;

(ii) $\langle \hat{x}, \hat{y} \rangle$ *contains* $\widehat{\mathrm{Alt}}(\Gamma \cup \Gamma y)$, *where* $\Gamma \cup \Gamma y \subseteq H_v$ *and* Γ *is the support of a useful cycle c of* $[\psi_{H_v}(X), \psi_{H_v}(Y)]$, $v \in \{d, l\}$.

PROOF. (i) Repeated application of Lemma 1, starting with the representation ψ : $\Delta(2,3,7) \to \mathrm{GL}_{n+\bar{n}}(R)$ such that $\psi(X) = \begin{pmatrix} \xi & 0 \\ 0 & \bar{\xi} \end{pmatrix} = \hat{\xi}$, $\psi(Y) = \begin{pmatrix} y & 0 \\ 0 & \bar{y} \end{pmatrix} = \hat{y}$.

(ii) We need to determine the decomposition of $\langle \Omega, \bar{\Omega} \rangle$ into $[\hat{x}, \hat{y}]$-invariant submodules and the orders of the corresponding restrictions of $[\hat{x}, \hat{y}]$. Following the notation of Section 3, we first note that:

$$\begin{aligned}
[\psi_G(X), \psi_G(Y)] &= [\xi_1, y_1] \\
&= (\underbrace{v_6, v_{28}, v_{37}, v_{42}, v_{24}, v_4}, \underbrace{v_1, v_9, v_{38}, v_{35}, v_{41}, v_{10}, v_3}) \\
&\quad (\underbrace{v_{18}, v_{29}, v_7, v_{12}, v_{23}, v_{16}}, \underbrace{v_{13}, v_{21}, v_8, v_5, v_{11}, v_{25}, v_{15}}) \\
&\quad (\underbrace{v_{36}, v_{30}, v_{19}, v_{27}, v_{22}, v_{34}}, \underbrace{v_{31}, v_{39}, v_{20}, v_{17}, v_{26}, v_{40}, v_{33}}) \text{ of order } 13; \\
[\psi_A(X), \psi_A(Y)] & \\
&= (\underbrace{w_6, w_{14}, w_9, w_{10}, w_{13}, w_4}, \underbrace{w_1, w_8, w_7, w_5, w_{12}, w_{11}, w_3}) \text{ of order } 13; \\
[\psi_E(X), \psi_E(Y)] &= (\underbrace{w_6, w_{23}, w_{13}, w_4}, \underbrace{w_1, w_8, w_{25}, w_{11}, w_3}) \\
&\quad (w_5, w_{12}, w_{20}, w_{24}, w_9, w_{10}, w_{15}, w_{16}, w_7) \\
&\quad (w_{14}, w_{28}, w_{22}, w_{17}, w_{21}, w_{27}, w_{26}, w_{18}, w_{19}) \text{ of order } 9.
\end{aligned}$$

We intend to apply Corollary 3 to the permutation representation $X \mapsto \hat{\xi}$, $Y \mapsto \hat{y}$, where $\hat{\xi}$ is defined as in (i). Thus, for each pair of handles $\{a_2^i, a_3^i\}$, $\{b_2^i, b_3^i\}$ in $\Omega \cup \bar{\Omega}$ ($i \leq k$, say) which have been used in the definition of ζ, $\bar{\zeta}$, x_1, $\bar{x}_1$ and T in order to join Conder diagrams, we assume that $\{a_2^i, a_3^i\}$ is contained in a diagram of type G, and denote by A^i the orbit of length 13 of $[\hat{\xi}, \hat{y}]$ which contains $\{a_1^i\hat{\xi}, a_3^i\}$, where $a_1^i = a_2^i\hat{y}^{-1}$. Similarly, if $\{b_2^i, b_3^i\}$ is contained in a diagram of type G, A or E, we denote by B^i the orbit of length 13 or 9 of $[\hat{\xi}, \hat{y}]$ which contains $\{b_1^i\hat{\xi}, b_3^i\}$, where $b_1^i = b_2^i\hat{y}^{-1}$. Otherwise, we denote by B^i the diagram H_v or $\bar{H}_{\bar{v}}$, $v, \bar{v} \in D$, which contains $\{b_2^i, b_3^i\}$. By Corollary 3, setting $U = \bigcup_{i=1}^k (A^i \cup B^i)$, each of the submodules $\langle A^i, B^i \rangle$ and $\langle (\Omega \cup \bar{\Omega}) \setminus U \rangle$ is invariant under $[\hat{x}, \hat{y}]$. Moreover, $[\hat{x}, \hat{y}]$ acts on the last submodule in the same way as $[\hat{\xi}, \hat{y}]$, that is, as a permutation matrix of order dividing $9 \cdot 13$. To compute the orders of the restrictions of $[\hat{x}, \hat{y}]$ to each submodule $\langle A^i, B^i \rangle$ we distinguish the following cases.

(a) Suppose $\{b_2^i, b_3^i\}$ belongs to a diagram of type G, A or E. Noting that, in the last two cases, the join is of type (1), we are in the situation of Lemma 2 (3), with $\Delta = A^i \cup B^i$, $\mu_1 = 6$, $\mu_2 = 7$ and either $\nu_1 = 6$, $\nu_2 = 7$ or $\nu_1 = 4$, $\nu_2 = 5$. It follows that $[\hat{x}, \hat{y}]_{|\langle A^i, B^i \rangle}$ has order 13 or 11.

(b) Suppose $\{b_2^i, b_3^i\}$ belongs to a diagram of type H. Table 1 below gives the cycle structure of the restriction of $[\hat{x}, \hat{y}]$ to $A^i \cup H_v$, $v \in D$. This table is deducible from the table in [Co, page 87], noting that $[\hat{x}, \hat{y}]_{|\langle A^i, H_v \rangle}$ coincides in Conder's notation with $(xyt)^2$, where t is the symmetry in the vertical axis of $G \cup H_v$. Every useful cycle

TABLE 1.

$\langle A^i, H_{42}\rangle$	1 23	3 11 **17**
$\langle A^i, H_{57}\rangle$	1 23	3 **5** 7^2 12^2
$\langle A^i, H_{142}\rangle$	1 13^2	1^4 3 11^3 **17** 12^4 23
$\langle A^i, H_{115}\rangle$	1 11^2	2^2 5^2 11^4 15^2 **17**
$\langle A^i, H_{144}\rangle$	1 12^2	1^2 5^3 8^2 11^2 **17** 30^2
$\langle A^i, H_{187}\rangle$	1 15^2	1^2 4^2 9^2 10^2 12^4 15^2 **43**
$\langle A^i, H_{216}\rangle$	1 13^2	1^4 4^2 **5** 11 6^2 7^2 12^2 13^2 17^2 32^2
$\langle A^i, H_{77}\rangle$	1 11^2	1^2 2^2 4^2 9^4 **17**
$\langle A^i, H_{36}\rangle$	1 12^2	4^2 **5** 11
$\langle A^i, H_{135}\rangle$	1 12^2	1^3 3 4^2 5^2 8^2 11^2 **19** 21^2
$\langle A^i, H_{136}\rangle$	1 13^2	1^4 4^2 **5** 11^3 12^6
$\langle A^i, H_{165}\rangle$	1 12^2	1^4 2^2 4^2 5^4 8^4 11^6 **19**
$\langle A^i, H_{180}\rangle$	1 12^2	1^2 5^2 6^2 7^2 8^2 11^2 13^2 19 **47**
$\langle A^i, H_{195}\rangle$	1 32^2	2^2 5^2 6^2 7^2 13^4 14^2 **23**

is denoted in bold and has length belonging to the set $\Pi = \{\mathbf{5}, \mathbf{17}, \mathbf{19}, \mathbf{23}, \mathbf{43}, \mathbf{47}\}$. Note that b_2^i is fixed, whereas b_3^i belongs to a non-trivial orbit whose length is listed in column 2. In particular, the support of the useful cycle intersects trivially the handle $\{b_2^i, b_3^i\}$.

Using all the above data and setting $\hat{\gamma} = [\hat{x}, \hat{y}]$, $m = 32 \cdot 9 \cdot 7 \cdot 11 \cdot 13$, we see that

$$\hat{\gamma}^m = \begin{pmatrix} [\xi\zeta, y]^m & 0 \\ 0 & [\bar{\xi}\bar{\zeta}, \bar{y}]^m \end{pmatrix}$$

acts as the identity on both $\Omega \setminus (G_a \cup H_d \cup H_l)$ and $\bar{\Omega} \setminus (\bar{G}_{\bar{a}} \cup \bar{H}_{\bar{d}} \cup \bar{H}_{\bar{l}})$. Moreover, denoting by h the product of the primes in Π, it is easy to check the following facts:

(1) Suppose $n \equiv \bar{n} \equiv 10 \pmod{14}$. Then, by assumption, $l = 42$ and $\bar{l} = 0$. It follows $n - l \equiv \bar{n} - \bar{l} \equiv 10 \pmod{14}$. Hence $d = \bar{d} = 136 \equiv 10 \pmod{14}$. Thus $[\xi\zeta, y]^m$ consists of cycles of length $k \in \{5, 23\}$ and a single cycle c_{17} of length 17, which is useful and whose support is contained in H_{42}. On the other hand, $[\bar{\xi}\bar{\zeta}, \bar{y}]^m$ consists of a single cycle $\bar{c}_5$ of length 5, which is useful and whose support is contained in $\bar{H}_{136}$. It follows that

$$(\hat{\gamma}^m)^{h/17} = \begin{pmatrix} c_{17} & 0 \\ 0 & I_{\bar{n}} \end{pmatrix}, \quad (\hat{\gamma}^m)^{h/5} = \begin{pmatrix} c_5 & 0 \\ 0 & \bar{c}_5 \end{pmatrix}.$$

(2) Suppose $n \not\equiv 10 \pmod{14}$. Then $l = 180 \equiv 12 \pmod{14}$. It follows that $n - l \not\equiv -2 \pmod{14}$, hence $d \neq 180 \equiv -2 \pmod{14}$. We claim that also $\bar{d} \neq 180$. To check this, consider first the case $\bar{n} \not\equiv 12 \pmod{14}$. Then $\bar{l} = 0$ implies $\bar{d} \neq 180$. Finally, if $\bar{n} \equiv 12 \pmod{14}$, then $\bar{l} = 187 \equiv 5 \pmod{14}$ implies $\bar{d} = 77 \equiv 7$

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(mod 14). Thus $[\xi\zeta, y]^m$ consists of cycles of length $k \in \Pi \setminus \{47\}$ and a single cycle c_{47} of length 47, which is useful with support contained in H_{180}.

(2.1) Assume $\bar{n} \not\equiv 12$. Then $[\bar{\xi}\bar{\zeta}, \bar{y}]^m$ consists of cycles of length $k \in \Pi \setminus \{47\}$ and a single cycle $\bar{c}_p$ of length $p \in \Pi \setminus \{47\}$, which is useful with support contained in $\bar{H}_{\bar{d}}$. It follows that

$$(\hat{y}^m)^{h/47} = \begin{pmatrix} c_{47} & 0 \\ 0 & I_{\bar{n}} \end{pmatrix} \quad \text{and} \quad (\hat{y}^m)^{h/p} = \begin{pmatrix} \sigma & 0 \\ 0 & \bar{c}_p \end{pmatrix}, \quad \text{where } \sigma \in \mathrm{Alt}(\Omega).$$

(2.2) Assume $\bar{n} \equiv 12$. Then $[\bar{\xi}\bar{\zeta}, \bar{y}]^m$ consists of cycles of length $k \in \Pi \setminus \{47\}$ and a single cycle $\bar{c}_{43}$ of length 43, which is useful with support contained in $\bar{H}_{187}$. It follows that

$$(\hat{y}^m)^{h/47} = \begin{pmatrix} c_{47} & 0 \\ 0 & I_{\bar{n}} \end{pmatrix}, \quad (\hat{y}^m)^{h/43} = \begin{pmatrix} I_n & 0 \\ 0 & \bar{c}_{43} \end{pmatrix}.$$

If c is one of the useful cycles occurring in Table 1 and Γ denotes its support, then $\Gamma \cap \Gamma y \neq \emptyset$ by definition, and a direct computation via Conder diagrams (see [Mo]) shows that $|\Gamma \cup \Gamma y| \geq |\Gamma| + 3$. Thus, by Lemma 3 in [LT], based on a classical theorem of Jordan, $\langle c, c^y \rangle = \mathrm{Alt}(\Gamma \cup \Gamma y)$. Setting $c = c_{17}$ in case (1) and $c = c_{47}$ in case (2), we conclude that $\left\langle \left(\begin{smallmatrix} c & 0 \\ 0 & I_{\bar{n}} \end{smallmatrix}\right), \left(\begin{smallmatrix} c^y & 0 \\ 0 & I_{\bar{n}} \end{smallmatrix}\right) \right\rangle = \widehat{\mathrm{Alt}}(\Gamma \cup \Gamma y)$, where $\Gamma \subseteq H_v$, $v \in \{d, l\}$. □

5. Proof of the theorem

LEMMA 5. $\widehat{\mathrm{Alt}}(\Omega) \times \widehat{\mathrm{Alt}}(\bar{\Omega}) \leq \langle \hat{x}, \hat{y} \rangle$.

PROOF. Let $\Gamma \subseteq H_v$ and c be as in Lemma 4. Observe that, by the remarks made just before Table 1, $\hat{x}_{|\Gamma} = (\xi\zeta)_{|\Gamma} = \xi_{|\Gamma} = (\psi_{H_v}(X))_{|\Gamma}$. It follows that c is also a cycle of each of the commutators $[\hat{x}, \hat{y}]$, $[\xi\zeta, y]$ and $[\xi, y]$. Suppose that S is a maximal subset of Ω with respect to the following properties: $\Gamma \cup \Gamma y \subseteq S$ and $\widehat{\mathrm{Alt}}(S) \leq \langle \hat{x}, \hat{y} \rangle$. Since Γ contains two points from an orbit of y, we have $S \cap Sy \neq \emptyset$. It follows that $\langle \mathrm{Alt}(S), \mathrm{Alt}(S)^y \rangle = \mathrm{Alt}(S \cup Sy)$, hence $\langle \widehat{\mathrm{Alt}}(S), \widehat{\mathrm{Alt}}(S)^{\hat{y}} \rangle = \widehat{\mathrm{Alt}}(S \cup Sy)$. Thus $S = Sy$ by the maximality of S. Suppose that $S \neq \Omega$. By the transitivity of $\langle \xi\zeta, y \rangle$ on Ω, there exists $v \in S$ such that $v\xi\zeta \in \Omega \setminus S$. Since Γ contains an orbit of ξ and $|\Gamma| \geq 5$, there exists $w \neq w' \in \Gamma$ such that $w\xi \in \{w, w'\}$ and $v \notin \{w, w'\}$. Note that $s = (v, w, w') \in \mathrm{Alt}(S)$ and either $s^{\xi\zeta} = (v\xi\zeta, w', w)$ or $s^{\xi\zeta} = (v\xi\zeta, w, w'\xi)$. Thus, in each case, $w \in S \cap \mathrm{supp}(s^{\xi\zeta})$. It follows that $\langle \mathrm{Alt}(S), s^{\xi\zeta} \rangle = \mathrm{Alt}(S \cup \{v\xi\zeta\} \cup \{w'\xi\})$. Let $\Delta \cup \bar{\Delta}$ be the subset of $\Omega \cup \bar{\Omega}$ consisting of the handles used to define $x_1, \bar{x}_1$ and T. Then $\langle \Omega, \bar{\Omega} \rangle = \langle \Delta, \bar{\Delta} \rangle \oplus \langle \Omega \setminus \Delta \rangle \oplus \langle \bar{\Omega} \setminus \bar{\Delta} \rangle$, where each direct summand is $\hat{x}$-invariant and $\hat{x}$ acts on $\langle \Omega \setminus \Delta \rangle$ as $\xi\zeta$ does. Since $v \neq v\xi\zeta$ and $\{w, w'\} \subseteq \Gamma$, it is

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clear that $\langle v, w, w' \rangle \leq \langle \Omega \setminus \Delta \rangle$. Hence $\left(\begin{smallmatrix} s & 0 \\ 0 & I_{\bar{n}} \end{smallmatrix}\right) \in \widehat{\mathrm{Alt}}(S)$ and

$$\left\langle \widehat{\mathrm{Alt}}(S), \begin{pmatrix} s & 0 \\ 0 & I_{\bar{n}} \end{pmatrix}^{\hat{x}} \right\rangle = \left\langle \widehat{\mathrm{Alt}}(S), \begin{pmatrix} s^{\xi\zeta} & 0 \\ 0 & I_{\bar{n}} \end{pmatrix} \right\rangle = \widehat{\mathrm{Alt}}\big(S \cup \{v\xi\zeta\} \cup \{w'\xi\}\big).$$

But this contradicts the maximality of S, as $v\xi\zeta \notin S$. We conclude that $S = \Omega$, that is $\widehat{\mathrm{Alt}}(\Omega) \leq \langle \hat{x}, \hat{y} \rangle$. This, together with (1), (2.1) and (2.2) in the proof of Lemma 4, implies that $\left(\begin{smallmatrix} I_n & 0 \\ 0 & \bar{c} \end{smallmatrix}\right) \in \langle \hat{x}, \hat{y} \rangle$, where $\bar{c}$ is a useful cycle of prime length. Applying to $\bar{\Omega}$ the same arguments as above, we finally get $\widehat{\mathrm{Alt}}(\bar{\Omega}) \leq \langle \hat{x}, \hat{y} \rangle$. □

For any subset Δ of Ω we denote by $E_\Delta(R)$ the subgroup of $E_\Omega(R) = E_n(R)$ which fixes every point of $\Omega \setminus \Delta$. Similarly for $\bar{\Omega}$. Moreover, we set

$$\widehat{E}_\Delta(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & I_{\bar{n}} \end{pmatrix} \middle| \, a \in E_\Delta(R) \right\}, \quad \widehat{E}_{\bar{\Delta}}(R) = \left\{ \begin{pmatrix} I_n & 0 \\ 0 & \bar{a} \end{pmatrix} \middle| \, \bar{a} \in E_{\bar{\Delta}}(R) \right\}.$$

LEMMA 6. *$\langle \hat{x}, \hat{y} \rangle$ contains $\langle L, T \rangle$, where $L = \widehat{E}_\Omega(R) \times \widehat{E}_{\bar{\Omega}}(R)$ is the Levi subgroup of $P_{n,\bar{n}}(R)$.*

PROOF. We recall that $\hat{x} = \left(\begin{smallmatrix} \xi\zeta x_1 & 0 \\ 0 & \bar{\xi}\bar{\zeta}\bar{x}_1 \end{smallmatrix}\right) T$. As $\left(\begin{smallmatrix} \xi\zeta & 0 \\ 0 & \bar{\xi}\bar{\zeta} \end{smallmatrix}\right) \in \widehat{\mathrm{Alt}}(\Omega) \times \widehat{\mathrm{Alt}}(\bar{\Omega})$, it follows from Lemma 5 that $\left(\begin{smallmatrix} x_1 & 0 \\ 0 & \bar{x}_1 \end{smallmatrix}\right) T \in \langle \hat{x}, \hat{y} \rangle$. Note that the subgroup E generated by the latter element and $\widehat{\mathrm{Alt}}(\Omega \setminus G_{a-2})$ preserves the decomposition $\langle \Omega, \bar{\Omega} \rangle = \langle \Omega \setminus G_{a-2} \rangle \oplus \langle \bar{\Omega} \setminus G_{\bar{a}-2} \rangle \oplus \langle G_{a-2}, \bar{G}_{\bar{a}-2} \rangle$; as a matter of fact, E is a subgroup of the direct product:

$$\left\langle \begin{pmatrix} x_1 & 0 \\ 0 & I_{\bar{n}} \end{pmatrix}, \widehat{\mathrm{Alt}}(\Omega \setminus G_{a-2}) \right\rangle \times \left\langle \begin{pmatrix} I_n & 0 \\ 0 & \bar{x}_1 \end{pmatrix} \right\rangle \times \langle T \rangle.$$

Since, under our assumptions, $n = |\Omega| \geq 84(m+1)+180+216$, it follows from [LTW] that $E_{\Omega \setminus G_{a-2}}(R)$ is generated by elements xx_1 and y, where $x, y \in \mathrm{Alt}(\Omega \setminus G_{a-2})$. A fortiori, $\langle x_1, \mathrm{Alt}(\Omega \setminus G_{a-2}) \rangle = E_{\langle \Omega \setminus G_{a-2} \rangle}(R)$. Since the latter group is perfect, $\langle \hat{x}, \hat{y} \rangle$ contains the derived subgroup $E' = \widehat{E}_{\Omega \setminus G_{a-2}}(R)$ of E. In particular, $\left(\begin{smallmatrix} x_1 & 0 \\ 0 & I_{\bar{n}} \end{smallmatrix}\right) \in \langle \hat{x}, \hat{y} \rangle$ and, again by [LTW], $\left\langle \left(\begin{smallmatrix} x_1 & 0 \\ 0 & I_{\bar{n}} \end{smallmatrix}\right), \mathrm{Alt}(\Omega) \right\rangle = \widehat{E}_\Omega(R)$ is contained in $\langle \hat{x}, \hat{y} \rangle$. In a similar way one shows that $\langle \hat{x}, \hat{y} \rangle$ contains $\widehat{E}_{\bar{\Omega}}(R)$. It follows that $T \in \langle \hat{x}, \hat{y} \rangle$. □

LEMMA 7. *Assume $n, \bar{n} \geq 3$. Then $P_{n,\bar{n}}(R) = \langle L, T_1 \rangle$, where $T_1 = \left(\begin{smallmatrix} I_n & e_{11}+e_{22} \\ 0 & I_{\bar{n}} \end{smallmatrix}\right)$.*

PROOF. Let $A = \left(\begin{smallmatrix} I_n - e_{11} - e_{12} + e_{21} & 0 \\ 0 & I_{\bar{n}} \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} I_n - e_{11} - e_{22} + e_{12} - e_{21} & 0 \\ 0 & I_{\bar{n}} \end{smallmatrix}\right)$. As mentioned in the introduction, $\mathrm{SL}(n, R_0) \leq E_n(R)$, where $R_0 = \mathbb{Z}1_R$. It follows that $A, B \in L$ and $Z = T_1^A T_1^B = \left(\begin{smallmatrix} I_n & e_{11} \\ 0 & I_{\bar{n}} \end{smallmatrix}\right) \in \langle L, T_1 \rangle$. For any $r \in R$, let $S_r = \left(\begin{smallmatrix} I_n - re_{21} & 0 \\ 0 & I_{\bar{n}} \end{smallmatrix}\right)$. Then $\{Z^{S_r} Z^{-1} \mid r \in R\} = \left\{ \left(\begin{smallmatrix} I_n & re_{21} \\ 0 & I_{\bar{n}} \end{smallmatrix}\right) \middle| \, r \in R \right\} \leq \langle L, T_1 \rangle$. The normal closure of this root subgroup under the matrices of shape $\left(\begin{smallmatrix} \sigma & 0 \\ 0 & \tau \end{smallmatrix}\right)$, where σ and τ run over the set of even permutation matrices, is the whole of N. □

Observe that T_1 differs from T by an element of L. Thus, by Lemma 6 and Lemma 7, $\langle \hat{x}, \hat{y} \rangle = P_{n,\bar{n}}(R)$, and our theorem is proven.

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