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Schwartz Functions on Real Algebraic Varieties

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Abstract. We define Schwartz functions, tempered functions, and tempered distributions on (possibly singular) real algebraic varieties. We prove that all classical properties of these spaces, defined previously on affine spaces and on Nash manifolds, also hold in the case of affine real algebraic varieties, and give partial results for the non-affine case.

1 Introduction

Schwartz functions are classically defined as smooth functions such that they, and all their (partial) derivatives, decay at infinity faster than the inverse of any polynomial. On $\mathbb{R}$, for instance, a smooth function f is called *Schwartz* if for any $n, k \in \mathbb{N} \cup \{0\}$, $x^n f^{(k)}$ is bounded (where $f^{(k)}$ is the k-th derivative of f). This was formulated on $\mathbb{R}^n$ by Laurent Schwartz, and later on Nash manifolds (smooth semi-algebraic varieties); see [dC, AG]. As Schwartz functions are defined using algebraic notions, it is natural to define Schwartz spaces of real algebraic varieties; this is the main goal of this paper.

The basic idea is to define the set of *Schwartz functions* on a real algebraic set in $\mathbb{R}^n$ as the quotient of the space of Schwartz functions on $\mathbb{R}^n$ by the ideal of Schwartz functions that vanish identically on the set. We define *tempered functions* similarly. Starting with these definitions we develop a theory of Schwartz spaces for arbitrary real algebraic varieties, and in particular prove that many properties of Schwartz spaces shown in [dC, AG] also hold in the (singular) algebraic case.

The main results for affine varieties appearing in this paper are as follows:

(1) Let $X \subset \mathbb{R}^n$ be an algebraic set; then $\mathcal{S}(X)$ is a Fréchet space (Lemma 3.3).

(2) Let $\varphi: X_1 \to X_2$ be a biregular isomorphism between two algebraic sets $X_1 \subset \mathbb{R}^{n_1}, X_2 \subset \mathbb{R}^{n_2}$. Then $\varphi^*|_{\mathcal{S}(X_2)}: \mathcal{S}(X_2) \to \mathcal{S}(X_1)$ is an isomorphism of Fréchet spaces. This implies that the definition of Schwartz functions on an affine algebraic variety does not depend on the embedding (Lemma 3.6(i)).

For (3)–(6) below, let X be some affine algebraic variety, and let $Z \subset X$ be a Zariski closed subset.

(3) (Tempered partition of unity) Let $\{V_i\}_{i=1}^m$ be a Zariski open cover of X. Then there exist tempered functions $\{\beta_i\}_{i=1}^m$ on X, such that $\operatorname{supp}(\beta_i) \subset V_i$ and $\sum_{i=1}^m \beta_i = 1$. Furthermore, for any m-tuple $(\beta_1, \dots, \beta_m)$ of tempered functions on X satisfying

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these conditions, and for any $\varphi \in \mathcal{S}(X)$, we have $(\beta_i \cdot \varphi)|_{V_i} \in \mathcal{S}(V_i)$ (Proposition 3.14 and Corollary 3.26).

(4) The restriction $\phi \mapsto \phi|_Z$ maps $\mathcal{S}(X)$ onto $\mathcal{S}(Z)$ (Theorem 3.9).

(5) Define $U := X \setminus Z$ and $W_Z := \{\phi \in \mathcal{S}(X) | \phi \text{ is flat on } Z\}$. Then W_Z is a closed subspace of $\mathcal{S}(X)$ (and so is a Fréchet space), and extension by zero $\mathcal{S}(U) \to W_Z$ is an isomorphism of Fréchet spaces whose inverse is the restriction of functions (Theorem 3.23). As a consequence, the restriction morphism of tempered distributions $\mathcal{S}^*(X) \to \mathcal{S}^*(U)$ is onto (Theorem 3.29).

(6) The assignment of the space of Schwartz functions (resp. tempered functions, tempered distributions) to any open $U \subset X$, together with the extension by zero, Ext_U^V, from U to any other open $V \supset U$ (restriction of functions, restrictions of functionals from $\mathcal{S}^*(V)$ to $\mathcal{S}^*(U)$), form a flabby cosheaf (sheaf, flabby sheaf) on X (Propositions 4.5, 4.3, and 4.4).

The most difficult result above is (5). A smooth function defined on a smooth algebraic set that vanishes identically with all its derivatives at some point is called *flat* at this point. In order to define W_Z, we have to make sense of the notion "flat" at a singular point. We do this by the following (a-priori naive) definition: a function f on an algebraic set $X \subset \mathbb{R}^n$ is flat at $y \in X$ if it is the restriction of some $C^\infty(\mathbb{R}^n)$ function that is flat at y. Then the proof of (5) is quite quickly reduced to a Whitney type extension problem. This point of view suggests the characterization of Schwartz functions by local means only: the global conditions of "rapid decaying at infinity" are translated to local conditions of flatness at "all points added in infinity" in some compactification process. We make this claim precise in Theorem 3.23 and Remark 3.27.

For general (not necessarily affine) varieties, we define Schwartz functions as sums of extensions by zero of Schwartz functions on affine open subvarieties and prove some generalizations of (1)–(6) above. The main obstacle for generalizing the rest is the absence of non-affine partition of unity, *i.e.,* our inability to generalize (3) to the non-affine case.

Structure of the paper In Section 2 we present the preliminary definitions and results used in this paper, mainly from real algebraic geometry and Schwartz spaces on affine algebraic manifolds.

In Section 3 we define the space of Schwartz functions on an affine algebraic variety and study its properties. We start by showing that it is a Fréchet space and proving that a useful partition of unity exists. Afterwards we define the notion of flat functions at a point on an affine algebraic variety and characterize the spaces of Schwartz functions on Zariski open subsets of an affine algebraic variety. We also define tempered distributions and prove that the restriction morphism from the space of tempered distributions on an affine algebraic variety to the space of tempered distributions on an open subset of it is onto. The proofs of two key lemmas in Section 3 require some tools from subanalytic geometry; Appendix A is dedicated to presenting these tools and completing the two proofs. Mainly, a Whitney type extension theorem is proved (Lemma 3.16): that a function on a compact algebraic set that can be extended to a flat function "pointwise" in some algebraic subset, can be "uniformly" extended to a single function that is flat everywhere in the same subset.

As proving that tempered functions and tempered distributions form sheaves and that Schwartz functions form a cosheaf are quite technical, Section 4 is dedicated to these proofs.

In Section 5 we define the spaces of Scwhartz functions and of tempered functions on an arbitrary (not necessarily affine) real algebraic variety, and repeat some of the results we proved in the affine case. We also briefly discuss the difficulty of generalizing the rest of these results to the non-affine case and suggest an idea that might enable overcoming this difficulty.

Conventions Throughout this paper the base field is always $\mathbb{R}$. We always consider the Zariski topology, unless otherwise stated. If X is a set, $Y \subset X$ is some subset and f is a real valued function on Y, we denote by $\mathrm{Ext}_Y^X(f)$ the real valued function on X defined by

$$\mathrm{Ext}_Y^X(f)(x) := \begin{cases} f(x) & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y; \end{cases}$$

i.e., Ext_Y^X is the "extension by zero" operator. If g is a real valued function on X we denote its restriction to Y by $\mathrm{Res}_X^Y(g)$.

2 Preliminaries

In this section we present the basic definitions and results used in this paper from real algebraic geometry (2.1), Schwartz functions on affine algebraic manifolds (2.2), and Fréchet spaces (2.3).

2.1 Real Algebraic Geometry

We start by recalling the basic definitions:

Definition 2.1 (following [BCR]) Let $X \subset \mathbb{R}^n$ be an algebraic set (*i.e.,* the zero locus of a family of polynomials in $\mathbb{R}[x_1, \dots, x_n]$). Let

$$I_{\mathrm{Alg}}(X) := \left\{ p \in \mathbb{R}[x_1, \dots, x_n] : p|_X = 0 \right\}.$$

Define *the coordinate ring of* X by $\mathbb{R}[X] := \mathbb{R}[x_1, \dots, x_n]/I_{\mathrm{Alg}}(X)$. Let V be an open subset of X. A function $f\colon V \to \mathbb{R}$ is called *a regular function* if $f = \frac{g}{h}$, where $g, h \in \mathbb{R}[X]$ and $h^{-1}(0) \cap V = \varnothing$. Note that the space of regular functions on V forms a ring. Moreover, the assignment of such a ring to any open subset of X defines a sheaf on X. We call this sheaf *the sheaf of regular functions on* X and denote it by $\mathcal{R}_X$. A map $F\colon V \to \mathbb{R}^m$ $(F(x) = (F_1(x), \dots, F_m(x)))$ is called *a regular map* if for any $1 \le i \le m$: F_i is a regular function. Let $Y \subset \mathbb{R}^m$ be an algebraic set, and let U be an open subset of Y. A map from V to U is called *a biregular isomorphism* if it is a bijective regular map whose inverse map is also regular. In that case we say that V is biregular isomorphic to U. An *affine algebraic variety* is a topological space X' equipped with a sheaf of real valued functions $\mathcal{R}_{X'}$ and isomorphic (as a ringed space) to an algebraic set $X \subset \mathbb{R}^n$ with its Zariski topology equipped with its sheaf of regular functions $\mathcal{R}_X$. The sheaf $\mathcal{R}_{X'}$ is called *the sheaf of regular functions on* X', and the topology of X' is called *the Zariski topology*. An *algebraic variety* is a topological space X', equipped with a sheaf of real valued functions $\mathcal{R}_{X'}$ such that there exists a finite open cover $\{U_i\}_{i=1}^n$ of X', with each U_i equipped with the sheaf $\mathcal{R}_{X'}|_{U_i}$ being an affine algebraic variety. The sheaf $\mathcal{R}_{X'}$ is called *the sheaf of regular functions on* X', and the topology of X' is called *the Zariski topology*. Note that unlike in the complex case, the ring of regular functions on $\mathbb{R}^n$ is not $\mathbb{R}[x_1,\dots,x_n]$; *e.g.*, $\frac{1}{x^2+1}$ on $\mathbb{R}$.

The following two propositions discuss the nature of algebraic sets and of their open subsets.

Proposition 2.2 ([BCR, Proposition 2.1.3]) *Let $X \subset \mathbb{R}^n$ be an algebraic set. There exists $f \in \mathbb{R}[x_1,x_2,\dots,x_n]$ such that X is the zero locus of f,* i.e., $X = \{x \in \mathbb{R}^n | f(x) = 0\}$.

Proposition 2.3 ([BCR, Proposition 3.2.10]) *Let $X \subset \mathbb{R}^n$ be an algebraic set, and U an open subset of X. Then $(U, \mathcal{R}_X|_U)$ is an affine algebraic variety (when we define for any open $U' \subset U \subset X$: $\mathcal{R}_X|_U(U') := \mathcal{R}_X(U')$).*

Proposition 2.4 is implicitly used in [BCR] (see for instance [BCR, Corollary 3.2.4]). For the reader's convenience we give its detailed proof in Appendix B.

Proposition 2.4 (the Zariski topology is Noetherian) *Let X be a real algebraic variety, $U \subset X$ an open subset, and let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of U. Then there exists a finite subcover $\{U_{\alpha_i}\}_{i=1}^k$.*

Definition 2.5 An affine algebraic variety is *complete* if any regular function on it is bounded.

Remark 2.6 Definition 2.5 is a special case of [BCR, Definition 3.4.10]. Note that if X is a complete affine algebraic variety, then for any closed embedding $i: X \hookrightarrow \mathbb{R}^n$, $i(X)$ is compact in the Euclidean topology on $\mathbb{R}^n$.

Proposition 2.7 (Algebraic Alexandrov compactification [BCR, Proposition 3.5.3]) *Let X be an affine algebraic variety that is not complete; then there exists a pair $(\dot{X}, i)$ such that*

(i) *$\dot{X}$ is a complete affine algebraic variety;*
(ii) *$i: X \to \dot{X}$ is a biregular isomorphism from X onto $i(X)$;*
(iii) *$\dot{X} \smallsetminus i(X)$ consists of a single point.*

2.2 Schwartz Functions and Tempered Functions on Affine Algebraic Manifolds

An affine algebraic variety that has a structure of a smooth differential manifold when being closely embedded in $\mathbb{R}^n$ is called *an affine algebraic manifold* (this property is independent of the embedding). We now present the basic theory of Schwartz functions and tempered functions on affine algebraic manifolds, as developed in [AG]. In [AG] the theory was developed for a much richer category, that is the category of Nash manifolds, and all the results we present here are special cases. In particular our very basic definitions of Schwartz and tempered functions are not the original definitions used in [AG]; however, they are equivalent.

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Definition 2.8 (*cf.* [AG, Definition 4.1.1, Theorem 4.6.1]) Let $M \subset \mathbb{R}^n$ be an algebraic subset that is also a smooth differential submanifold of $\mathbb{R}^n$. Let $\mathcal{S}(\mathbb{R}^n)$ be the Fréchet space of classical real valued Schwartz functions on $\mathbb{R}^n$, and let $I_{\mathrm{Sch}}(M) \subset \mathcal{S}(\mathbb{R}^n)$ be the ideal of all Schwartz functions that vanish identically on M. Define *the space of Schwartz functions on M* by $\mathcal{S}(M) := \mathcal{S}(\mathbb{R}^n)/I_{\mathrm{Sch}}(M)$ equipped with the quotient topology (equivalently we can define $\mathcal{S}(M)$ by restrictions of functions from $\mathcal{S}(\mathbb{R}^n)$, but then the definition of the topology is a bit more complicated). Let M be an affine algebraic manifold, and let $i: M \hookrightarrow \mathbb{R}^n$ be a closed embedding. A function $f: M \to \mathbb{R}$ is called *a Schwartz function on M* if $i_* f := f \circ i^{-1} \in \mathcal{S}(i(M))$. Denote the space of all Schwartz functions on M by $\mathcal{S}(M)$, and define a topology on $\mathcal{S}(M)$ by declaring a subset $U \subset \mathcal{S}(M)$ to be open if $i_*(U) \subset \mathcal{S}(i(M))$ is an open subset. $\mathcal{S}(M)$ is a well-defined Fréchet space (independent of the chosen embedding).

Definition 2.9 (*cf.* [AG, Definition 4.2.1, Theorem 4.6.2]) A function $t: \mathbb{R}^n \to \mathbb{R}$ is called *tempered* if it is a smooth function such that for any $\alpha \in (\mathbb{N} \cup \{0\})^n$ there exists a polynomial $p_\alpha \in \mathbb{R}[x_1, \dots, x_n]$ such that $\left|\frac{\partial^{|\alpha|} t}{\partial^\alpha x}(x)\right| < p_\alpha(x)$ for any $x \in \mathbb{R}^n$. Let M be an affine algebraic manifold, and let $i: M \hookrightarrow \mathbb{R}^n$ be a closed embedding. A function $t: M \to \mathbb{R}$ is called *a tempered function on M* if $i_* f := f \circ i^{-1}$ is the restriction to $i(M)$ of a tempered function from $\mathbb{R}^n$. Denote the space of all tempered functions on M by $\mathcal{T}(M)$. $\mathcal{T}(M)$ is a well defined space (independent of the chosen embedding).

The following results are of special importance for us.

Proposition 2.10 (*cf.* [AG, Proposition 4.2.1]) *Let M be an affine algebraic manifold and α a tempered function on M. Then $\alpha\mathcal{S}(M) \subset \mathcal{S}(M)$.*

Theorem 2.11 (Partition of unity *cf.* [AG, Theorem 4.4.1]) *Let M be an affine algebraic manifold, and let $\{U_i\}_{i=1}^n$ be a finite open cover of M by affine algebraic manifolds.*

(i) *There exist tempered functions $\alpha_1, \alpha_2, \dots, \alpha_n$ on M such that* $\operatorname{supp}(\alpha_i) \subset U_i$ *and* $\sum_{i=1}^n \alpha_i = 1$.

(ii) *Moreover, α_i can be chosen in such a way that for any $\phi \in \mathcal{S}(M)$, $(\alpha_i \cdot \phi)|_{U_i} \in \mathcal{S}(U_i)$.*

Proposition 2.12 (*cf.* [AG, Proposition 4.5.3] and Proposition 2.4) *Let M be an affine algebraic manifold. The assignment of the space of tempered functions on U, to any open $U \subset M$, together with the usual restriction maps, define a sheaf of algebras on M.*

Theorem 2.13 (*cf.* [AG, Theorem 4.6.1]) *Let M be an affine algebraic manifold, and $Z \hookrightarrow M$ be a closed algebraic submanifold. The restriction $\mathcal{S}(M) \to \mathcal{S}(Z)$ is defined, continuous, and onto. Moreover, it has a section $s: \mathcal{S}(Z) \to \mathcal{S}(M)$ such that if $\phi \in \mathcal{S}(Z)$ is zero at some point p with all its derivatives, then $s(\phi)$ is also zero at p with all its derivatives.*

Theorem 2.14 (*cf.* [AG, Theorem 4.6.2]) *Let M be an affine algebraic manifold, and $Z \hookrightarrow M$ be a closed algebraic submanifold. The restriction $\mathcal{T}(M) \to \mathcal{T}(Z)$ is defined, continuous and onto. Moreover, it has a section $s: \mathcal{T}(Z) \to \mathcal{T}(M)$ such that if $\alpha \in \mathcal{T}(Z)$*

is zero at some point p with all its derivatives, then $s(\alpha)$ is also zero at p with all its derivatives.

Theorem 2.15 (Characterization of Schwartz functions on open subsets: *cf.* [AG, Theorem 5.4.1]) *Let M be an affine algebraic manifold, $Z \hookrightarrow M$ be a closed algebraic submanifold, and $U = M \setminus Z$. Let W_Z be the closed subspace of $\mathcal{S}(M)$ defined by*

$$W_Z := \{\phi \in \mathcal{S}(M) \mid \phi \text{ vanishes with all its derivatives on } Z\}.$$

Then restriction and extension by zero give an isomorphism $\mathcal{S}(U) \cong W_Z$.

2.3 Fréchet Spaces

A Fréchet space is a metrizable, complete locally convex topological vector space. It can be shown that the topology of a Fréchet space can always be defined by a countable family of semi-norms. We use the following results.

Proposition 2.16 (*cf.* [T, Chapter 10]) *A closed subspace of a Fréchet space is a Fréchet space (for the induced topology).*

Proposition 2.17 (*cf.* [T, Proposition 7.9 and Chapter 10]) *A quotient of a Fréchet space by a closed subspace is a Fréchet space (for the quotient topology). Moreover, let F be a Fréchet space whose topology is defined by a basis of continuous semi-norms $\mathcal{P}$, let $K \subset F$ be a closed subspace, and let $\phi\colon F \to F/K$ be the canonical mapping of F onto F/K. Then the topology on F/K is defined by the basis of continuous semi-norms $\dot{p}(\dot{x}) = \inf_{\phi(x)=\dot{x}} p(x)$, where $p \in \mathcal{P}$.*

Theorem 2.18 (Banach open mapping [T, Chapter 17, Corollary 1]) *A bijective continuous linear map from a Fréchet space to another Fréchet space is an isomorphism.*

Theorem 2.19 (Hahn-Banach *cf.* [T, Chapter 18]) *Let F be a Fréchet space, and let $K \subset F$ be a closed subspace. By Proposition 2.16, K is a Fréchet space (with the induced topology). Define F^* (resp. K^*) to be the space of continuous linear functionals on F (on K). Then the restriction map $F^* \to K^*$ is onto.*

3 The Affine Case

Definition 3.1 Let $X \subset \mathbb{R}^n$ be an algebraic subset. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of classical real valued Schwartz functions on $\mathbb{R}^n$, and let $I_{\mathrm{Sch}}(X) \subset \mathcal{S}(\mathbb{R}^n)$ be the ideal of all Schwartz functions that vanish identically on X. Define *the space of Schwartz functions on X* by $\mathcal{S}(X) := \mathcal{S}(\mathbb{R}^n)/I_{\mathrm{Sch}}(X)$ equipped with the quotient topology.

Remark 3.2 An equivalent definition is

$$\mathcal{S}(X) := \{f\colon X \to \mathbb{R} : \exists \widetilde{f} \in \mathcal{S}(\mathbb{R}^n) \text{ such that } \widetilde{f}|_X = f\},$$

but then the definition of the topology is a bit more complicated. Recall that the topology of $\mathcal{S}(\mathbb{R}^n)$ is given by a system of semi-norms $|f|_D := \sup_{x\in\mathbb{R}^n} |Df(x)|$, where D is an algebraic differential operator on $\mathbb{R}^n$. This enables us to introduce topology on $\mathcal{S}(X)$ by the system of semi-norms $|f|_D := \inf\{|\widetilde{f}|_D : \widetilde{f} \in \mathcal{S}(\mathbb{R}^n), \widetilde{f}|_X = f\}$, where D

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is an algebraic differential operator on $\mathbb{R}^n$. By Proposition 2.17, the two definitions coincide.

Lemma 3.3 *$\mathcal{S}(X)$ is a Fréchet space.*

Proof We have that $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space; $I_{\mathrm{Sch}}(X) = \bigcap_{x\in X}\{f \in \mathcal{S}(\mathbb{R}^n) | f(x) = 0\}$ is an intersection of closed subsets and so the quotient is a Fréchet space (see Proposition 2.17). ∎

Lemma 3.4 *Let $X \subset \mathbb{R}^n$ be an algebraic set, and $U \subset \mathbb{R}^n$ be some open set containing X. Consider $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(U)$ as defined in Definition 2.8 (U is an open algebraic set, hence it can be considered as an affine algebraic manifold). Let $I^U_{\mathrm{Sch}}(X) \subset \mathcal{S}(U)$ be the ideal of all Schwartz functions on U that vanish identically on X. Then $\mathcal{S}(X) \cong \mathcal{S}(U)/I^U_{\mathrm{Sch}}(X)$ (isomorphism of Fréchet spaces).*

Proof By the same reasoning as in Lemma 3.3, $\mathcal{S}(U)/I^U_{\mathrm{Sch}}(X)$ is a Fréchet space. By Theorem 2.15, $\mathcal{S}(U)$ is isomorphic to a closed subspace of $\mathcal{S}(\mathbb{R}^n)$, and so by Proposition 2.17 it is enough to check that $\mathcal{S}(X) := \mathcal{S}(\mathbb{R}^n)/I_{\mathrm{Sch}}(X)$ and $\mathcal{S}(U)/I^U_{\mathrm{Sch}}(X)$ are equal as sets, *i.e.,* that a function on X is a restriction of a Schwartz function on $\mathbb{R}^n$ if and only if it is a restriction of a Schwartz function on U. Let $f \in \mathcal{S}(U)|_X$. There exists $F \in \mathcal{S}(U)$ such that $F|_X = f$. By Theorem 2.15, extending F by zero to a function on $\mathbb{R}^n$ (denote it by $\widetilde{F}$) is a function in $\mathcal{S}(\mathbb{R}^n)$. Then $f = \widetilde{F}|_X$ and so $f \in \mathcal{S}(\mathbb{R}^n)|_X$. Let $f \in \mathcal{S}(\mathbb{R}^n)|_X$. There exists $F \in \mathcal{S}(\mathbb{R}^n)$ such that $F|_X = f$. Denote $U' := \mathbb{R}^n \setminus X$. Then $\{U, U'\}$ form an open cover of $\mathbb{R}^n$ and so, by Theorem 2.11, there exist tempered functions α_1, α_2 such that $\operatorname{supp}(\alpha_1) \subset U$, $\operatorname{supp}(\alpha_2) \subset U'$, and $\alpha_1 + \alpha_2 = 1$ as a real valued function on $\mathbb{R}^n$. Moreover, α_1 and α_2 can be chosen such that $(\alpha_1 \cdot F)|_U \in \mathcal{S}(U)$. As $\alpha_1|_X = 1$, it follows that $((\alpha_1 \cdot F)|_U)|_X = (\alpha_1 \cdot F)|_X = F|_X = f$, and so $f \in \mathcal{S}(U)|_X$. ∎

Lemma 3.5 *Let $X \subset \mathbb{R}^n$ be an algebraic set, and let $U \subset \mathbb{R}^n$ be some open set containing X. Consider $\mathcal{T}(\mathbb{R}^n)$ and $\mathcal{T}(U)$ (the spaces of tempered functions on $\mathbb{R}^n$ and on U, respectively) as defined in Definition 2.9 (U is an open algebraic set, hence it can be considered as an affine algebraic manifold). Then a function $f: X \to \mathbb{R}$ is a restriction of a function $F \in \mathcal{T}(\mathbb{R}^n)$ if and only if it is a restriction of a function $\widetilde{F} \in \mathcal{T}(U)$.*

Proof Let $f: X \to \mathbb{R}$ be a restriction of some function $F \in \mathcal{T}(\mathbb{R}^n)$. By Proposition 2.12, $F|_U \in \mathcal{T}(U)$, and clearly $f = (F|_U)|_X$; *i.e.,* f is a restriction of a tempered function on U. Let $f: X \to \mathbb{R}$ be a restriction of some function $F \in \mathcal{T}(U)$. Let $U' := \mathbb{R}^n \setminus X$. Then $\{U, U'\}$ form an open cover of $\mathbb{R}^n$, and so, by Theorem 2.11, there exist tempered functions $\alpha_1, \alpha_2 \in \mathcal{T}(\mathbb{R}^n)$ such that $\operatorname{supp}(\alpha_1) \subset U$, $\operatorname{supp}(\alpha_2) \subset U'$, and $\alpha_1 + \alpha_2 = 1$ as a real valued function on $\mathbb{R}^n$. As tempered functions on affine algebraic manifolds form a sheaf (see Proposition 2.12), $\alpha_1|_U \in \mathcal{T}(U)$ and as $\mathcal{T}(U)$ is an algebra, $\alpha_1|_U \cdot F \in \mathcal{T}(U)$. Moreover, defining $F': \mathbb{R}^n \to \mathbb{R}$ by $F'|_U := F$ and $F'|_{\mathbb{R}^n \setminus U} := 0$; then as $\operatorname{supp}(\alpha_1) \subset U$, we get that $\alpha_1 \cdot F' \in \mathcal{T}(\mathbb{R}^n)$. Since $\alpha_1|_X = 1$, we have $(\alpha_1 \cdot F')|_X = f$; *i.e.,* f is a restriction of a tempered function on $\mathbb{R}^n$. ∎

Lemma 3.6 *Let $\varphi: X_1 \to X_2$ be a biregular isomorphism between two algebraic sets $X_1 \subset \mathbb{R}^{n_1}$ and $X_2 \subset \mathbb{R}^{n_2}$.*

(i) *$\varphi^*|_{\mathcal{S}(X_2)}: \mathcal{S}(X_2) \to \mathcal{S}(X_1)$ is an isomorphism of Fréchet spaces.*
(ii) *If $f: X_2 \to \mathbb{R}$ is a restriction of a tempered function on $\mathbb{R}^{n_2}$ (see Definition 2.9), then $\varphi^* f := f \circ \varphi$ is a restriction of a tempered function on $\mathbb{R}^{n_1}$.*

Proof By definition, for any $x \in X_1$, we have

$$\varphi(x) = \Big(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_{n_2}(x)}{g_{n_2}(x)} \Big),$$

where $f_1, \dots, f_{n_2}, g_1, \dots, g_{n_2} \in \mathbb{R}[X_1]$ and $g_i^{-1}(0) \cap X_1 = \varnothing$ for any $1 \leq i \leq n_2$. By abuse of notation we choose some representatives in $\mathbb{R}[x_1, \dots, x_{n_1}]$ and consider $f_1, \dots, f_{n_2}, g_1, \dots, g_{n_2}$ as functions in $\mathbb{R}[x_1, \dots, x_{n_1}]$. Define

$$U := \Big\{ x \in \mathbb{R}^{n_1} \,|\, \prod_{i=1}^{n_2} g_i(x) \neq 0 \Big\}.$$

Then U is open in $\mathbb{R}^{n_1}$ (also in the Euclidean topology), X_1 is a closed subset of U, and φ can be naturally extended to a regular map $\widetilde{\varphi}: U \to \mathbb{R}^{n_2}$ (by the same formula as φ). Note that U is an affine algebraic manifold.

Similarly to the construction of U and $\widetilde{\varphi}$ above, we can construct an open $V \subset \mathbb{R}^{n_2}$ and a function $\phi: V \to \mathbb{R}^{n_1}$ such that $\phi|_{X_2} = \varphi^{-1}$. Note that $\phi \neq \widetilde{\varphi}^{-1}$: in general $\widetilde{\varphi}$ is not a bijection and $U \ncong V$. Consider the following diagram, where α is defined by $\alpha(x, y) := (x, y + \widetilde{\varphi}(x))$:

$$\begin{tikzcd} X_1 \arrow[r, hook] & U \arrow[r, bend left, "\mathrm{Id} \times 0"] \arrow[r, bend right, "\mathrm{Id} \times \widetilde{\varphi}"'] & U \times \mathbb{R}^{n_2} \arrow[loop above, "\alpha"] \end{tikzcd}.$$

Clearly, $U \times \{0\}$ is an affine algebraic manifold isomorphic to U. Denote $\widetilde{U} := \alpha(U \times \{0\})$; then α restricted to $U \times \{0\}$ is an isomorphism of the affine algebraic manifolds $U \times \{0\}$ and $\widetilde{U}$ – the inverse map is given by $\alpha^{-1}(x, y) := (x, y - \widetilde{\varphi}(x))$. Thus, we have

$$\mathcal{S}(X_1) \cong \mathcal{S}(U)/I^U_{\mathrm{Sch}}(X_1) \cong \mathcal{S}(\widetilde{U})/I^{\widetilde{U}}_{\mathrm{Sch}}\big(\alpha(X_1 \times \{0\})\big) = \mathcal{S}(\widetilde{U})/I^{\widetilde{U}}_{\mathrm{Sch}}\big((\mathrm{Id} \times \varphi)(X_1)\big),$$

where the first equivalence is by Lemma 3.4, the second is due the fact that $U \cong U \times \{0\} \cong \widetilde{U}$ and $\mathcal{S}(U) \cong \mathcal{S}(U \times \{0\}) \cong \mathcal{S}(\widetilde{U})$, and the third follows from the fact that $\widetilde{\varphi}|_{X_1} = \varphi$. As always, $I^U_{\mathrm{Sch}}(X)$ is the ideal in $\mathcal{S}(U)$ of Schwartz functions identically vanishing on X. As $\widetilde{U}$ is closed in $U \times \mathbb{R}^{n_2}$ (as it is defined by polynomial equalities on $U \times \mathbb{R}^{n_2}$); then by Theorem 2.13 and Proposition 2.17, we get that

$$\mathcal{S}(\widetilde{U})/I^{\widetilde{U}}_{\mathrm{Sch}}\big((\mathrm{Id} \times \varphi)(X_1)\big) \cong \mathcal{S}(U \times \mathbb{R}^{n_2})/I^{U \times \mathbb{R}^{n_2}}_{\mathrm{Sch}}\big((\mathrm{Id} \times \varphi)(X_1)\big).$$

Applying Lemma 3.4 again for the open subset $U \times V \subset U \times \mathbb{R}^{n_2}$, we get that

$$\mathcal{S}(U \times \mathbb{R}^{n_2})/I^{U \times \mathbb{R}^{n_2}}_{\mathrm{Sch}}\big((\mathrm{Id} \times \varphi)(X_1)\big) \cong \mathcal{S}(U \times V)/I^{U \times V}_{\mathrm{Sch}}\big((\mathrm{Id} \times \varphi)(X_1)\big),$$

and thus we obtain

$$\mathcal{S}(X_1) \cong \mathcal{S}(U \times V)/I_{\mathrm{Sch}}^{U \times V}\big((\mathrm{Id} \times \varphi)(X_1)\big).$$

Repeating the above construction using the diagram

$$\begin{tikzcd} X_2 \arrow[r, hook] & V \arrow[r, bend left, "0\times\mathrm{Id}"] \arrow[r, bend right, "\phi\times\mathrm{Id}"'] & \mathbb{R}^{n_2} \times V \end{tikzcd}$$

yields

$$\mathcal{S}(X_2) \cong \mathcal{S}(U \times V)/I_{\mathrm{Sch}}^{U \times V}\big((\varphi^{-1} \times \mathrm{Id})(X_2)\big).$$

Clearly $(\mathrm{Id} \times \varphi)(X_1) = (\varphi^{-1} \times \mathrm{Id})(X_2)$, and so $\mathcal{S}(X_1) \cong \mathcal{S}(X_2)$. Note that the isomorphism constructed is in fact the pull back by φ from $\mathcal{S}(X_2)$ onto $\mathcal{S}(X_1)$. This proves (i).

The proof of (ii) is the same as the proof of (i), where one should consider tempered functions instead of Schwartz functions, and use Lemma 3.5 and Theorem 2.14 instead of Lemma 3.4 and Theorem 2.13. ∎

Definition 3.7 Let X be a real affine algebraic variety, and let $i: X \hookrightarrow \mathbb{R}^n$ be a closed embedding. A function $f: X \to \mathbb{R}$ is called *a Schwartz function on* X if $i_* f := f \circ i^{-1} \in \mathcal{S}(i(X))$. Denote the space of all Schwartz functions on X by $\mathcal{S}(X)$, and define a topology on $\mathcal{S}(X)$ by declaring a subset $U \subset \mathcal{S}(X)$ to be open if $i_*(U) \subset \mathcal{S}(i(X))$ is an open subset. By Lemma 3.6(i), $\mathcal{S}(X)$ is well defined (independent of the embedding chosen).

Remark 3.8 (i) If $X \cong \mathbb{R}^m$, then Definition 3.7 coincides with the classical one.
(ii) If X is smooth, then Definition 3.7 coincides with Definition 2.8.

Theorem 3.9 *Let M be an affine algebraic variety, and let $X \subset M$ be a closed subset. Then the restriction from M to X defines an isomorphism $\mathcal{S}(X) \cong \mathcal{S}(M)/I_{\mathrm{Sch}}^{M}(X)$ (with the quotient topology), where $I_{\mathrm{Sch}}^{M}(X)$ is the ideal in $\mathcal{S}(M)$ of functions identically vanishing on X.*

Proof Take some closed embedding $M \hookrightarrow \mathbb{R}^n$; then $X \hookrightarrow M \hookrightarrow \mathbb{R}^n$ are closed embeddings. Then

$$\mathcal{S}(M)/I_{\mathrm{Sch}}^{M}(X) = \big(\mathcal{S}(\mathbb{R}^n)/I_{\mathrm{Sch}}(M)\big)/I_{\mathrm{Sch}}^{M}(X) \cong \mathcal{S}(\mathbb{R}^n)/I_{\mathrm{Sch}}(X) = \mathcal{S}(X). \qquad \blacksquare$$

Remark 3.10 In particular, for any $\phi \in \mathcal{S}(M)$, one has that $\phi|_X \in \mathcal{S}(X)$, and this restriction map $\mathcal{S}(M) \to \mathcal{S}(X)$ is onto.

Definition 3.11 Let X be an affine algebraic variety. A function $f: X \to \mathbb{R}$ is called *a tempered function on* X, if there exists a closed embedding $i: X \hookrightarrow \mathbb{R}^n$ such that $i_* f := f \circ i^{-1}$ is a restriction of a tempered function on $\mathbb{R}^n$ to $i(X)$. By Lemma 3.6(ii) in that case, this property holds for any closed embedding. The set of all tempered functions forms a unitary algebra, which we denote by $\mathcal{T}(X)$.

Proposition 3.12 *Let X be an affine algebraic variety, $t \in \mathcal{T}(X)$ and $s \in \mathcal{S}(X)$. Then $t \cdot s \in \mathcal{S}(X)$.*

Proof Consider some closed embedding $i: X \hookrightarrow \mathbb{R}^n$ and identify $i(X)$ with X (by definitions of tempered and Schwartz functions the choice of the embedding does not matter). There exist $T \in \mathcal{T}(\mathbb{R}^n)$ and $S \in \mathcal{S}(\mathbb{R}^n)$ such that $t = T|_X$ and $s = S|_X$. By Proposition 2.10, $T \cdot S \in \mathcal{S}(\mathbb{R}^n)$, and so $(T \cdot S)|_X = t \cdot s \in \mathcal{S}(X)$. ■

Corollary 3.13 (Proposition 2.3) *Let X be an affine algebraic variety, and U an open subset of X. Then $(U, \mathcal{R}_X|_U)$ is an affine algebraic variety, and we can define $\mathcal{S}(U)$.*

There is a canonical way of defining a Euclidean topology on an algebraic variety; see [BCR, Remark 3.2.15(a)]. In what follows, when using the notion support, we always mean the support in this topology, rather than in Zariski topology.

Proposition 3.14 (tempered partition of unity) *Let X be an affine algebraic variety, and let $\{V_i\}_{i=1}^m$ be a finite open cover of X.*

(i) *There exist tempered functions $\{\beta_i\}_{i=1}^m$ on X, such that*

$$\operatorname{supp}(\beta_i) \subset V_i \quad \textit{and} \quad \sum_{i=1}^m \beta_i = 1.$$

(ii) *We can choose $\{\beta_i\}_{i=1}^m$ in such a way that for any $\varphi \in \mathcal{S}(X)$, $(\beta_i \varphi)|_{V_i} \in \mathcal{S}(V_i)$.*

Proof Consider some closed embedding $X \hookrightarrow \mathbb{R}^n$. For any $1 \le i \le m$, let $U_i \subset \mathbb{R}^n$ be some open subset such that $V_i = X \cap U_i$. Define $U_{m+1} := \mathbb{R}^n \smallsetminus X$ we get that $\{U_i\}_{i=1}^{m+1}$ is an open cover of $\mathbb{R}^n$ (by affine algebraic submanifolds). By Theorem 2.11, there exist $\{\alpha_i\}_{i=1}^{m+1}$, tempered functions on $\mathbb{R}^n$ such that $\operatorname{supp}(\alpha_i) \subset U_i$, $\sum_{i=1}^{m+1} \alpha_i = 1$, and $\{\alpha_i\}_{i=1}^{m+1}$ can be chosen in such a way that $(\alpha_i \psi)|_{U_i} \in \mathcal{S}(U_i)$ for any $\psi \in \mathcal{S}(\mathbb{R}^n)$. For $1 \le i \le m+1$ define $\beta_i := \alpha_i|_X$. Clearly, for $1 \le i \le m$, $\operatorname{supp}(\beta_i) \subset V_i$. Since $\alpha_{m+1}|_X = 0$, $\beta_{m+1}|_X = 0$, and so $\sum_{i=1}^m \beta_i = 1$. By definition, $\{\beta_i\}_{i=1}^m$ are tempered functions on X. This proves (i).

Now consider $\varphi \in \mathcal{S}(X)$. By definition, there exists $\widetilde{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ such that $\varphi = \widetilde{\varphi}|_X$ and for $1 \le i \le m$, $(\alpha_i \widetilde{\varphi})|_{U_i} \in \mathcal{S}(U_i)$. By Theorem 3.9, as V_i is closed in U_i, we get that $((\alpha_i \widetilde{\varphi})|_{U_i})|_{V_i} \in \mathcal{S}(V_i)$. But $((\alpha_i \widetilde{\varphi})|_{U_i})|_{V_i} = (\beta_i \varphi)|_{V_i}$, and so (ii) is proved. ■

Definition 3.15 Let $X \subset \mathbb{R}^n$ be an algebraic set and let $y \in X$ be some point. A function $f: X \to \mathbb{R}$ is *flat at y* if there exists $F \in C^\infty(\mathbb{R}^n)$ with $f = F|_X$, such that the Taylor series of F at y is identically zero. If f is flat at y for any $y \in Z$ (where $Z \subset X$ is some subset), we say that f is *flat at Z*.

3.0.1 An Important Remark

As the Taylor series is only dependent on the Euclidean local behaviour of functions, one can replace $\mathbb{R}^n$ above by any Euclidean open subset of $\mathbb{R}^n$ containing X. This is done in Section A.2, and is used when it is more convenient.

3.0.2 Warning

That f is flat at Z means that f is flat at y for any $y \in Z$. It does not mean, a-priori, that there exists $F \in C^\infty(\mathbb{R}^n)$ with $f = F|_X$, such that all the Taylor series of F, at any point $y \in Z$ are identically zero. Lemma 3.16 addresses this matter.

The proofs of Lemmas 3.16 and 3.17 are given in Appendix A, as they require some tools from subanalytic geometry.

Lemma 3.16 *Let X be a compact (in the Euclidean topology) algebraic set in $\mathbb{R}^n$, and let $Z \subset X$ be some (Zariski) closed subset. Define $U := X \setminus Z$ and*

$$W_Z := \{\phi: X \to \mathbb{R} | \exists \widetilde{\phi} \in C^\infty(\mathbb{R}^n) \text{ such that } \widetilde{\phi}|_X = \phi \text{ and } \phi \text{ is flat at } Z\},$$
$$(W_Z^{\mathbb{R}^n})^{\mathrm{comp}} := \{\phi \in C^\infty(\mathbb{R}^n) | \phi \text{ is compactly supported and is flat at } Z\}.$$

Then for any $f \in W_Z$, there exists $\widetilde{f} \in (W_Z^{\mathbb{R}^n})^{\mathrm{comp}}$ such that $\widetilde{f}|_X = f$.

Lemma 3.17 *Let $\varphi: X_1 \to X_2$ be a biregular isomorphism between two algebraic sets $X_1 \subset \mathbb{R}^{n_1}$ and $X_2 \subset \mathbb{R}^{n_2}$. If $f: X_2 \to \mathbb{R}$ is flat at some $p \in X_2$, then $\varphi^* f := f \circ \varphi$ is flat at $\varphi^{-1}(p)$.*

Definition 3.18 Let X be an affine algebraic variety, and let $f: X \to \mathbb{R}$ be some function. We say that f *is flat at* $p \in X$ if there exists a closed embedding $i: X \hookrightarrow \mathbb{R}^n$ such that $i_* f := f \circ i^{-1}: i(X) \to \mathbb{R}$ is flat at $i(p)$. By Lemma 3.17, in that case this property holds for any closed embedding.

Proposition 3.19 (Extension by zero) *Let X be an affine algebraic variety, and U an open subset of X. By Corollary* 3.13, *$\mathcal{S}(U)$ is defined. Then the extension by zero to X of a Schwartz function on U is a Schwartz function on X, which is flat at $X \setminus U$.*

Proof Since X is affine, we can choose some closed embedding $X \hookrightarrow \mathbb{R}^n$, and so we can think of X as an algebraic set. According to Proposition 2.2, there exists $F \in \mathbb{R}[x_1, \dots, x_n]$ such that X is the zero locus of F (denote $X = \mathrm{zeros}(F)$). $U \subset X$ is Zariski open in X, *i.e.*, $Z := X \setminus U$ is Zariski closed in X, thus Z is Zariski closed in $\mathbb{R}^n$. As before, there exists $G \in \mathbb{R}[x_1, \dots, x_n]$ such that $Z = \mathrm{zeros}(G)$. Define $V := \mathbb{R}^n \setminus Z$. Note that $U = X \setminus Z$ is a closed subset of V, as $X \setminus Z = X \cap V$. As V is open in $\mathbb{R}^n$, by Proposition 2.3, V is an affine variety. Consider some closed embedding $V \hookrightarrow \mathbb{R}^m$. By Theorem 3.9, $\mathcal{S}(U) \cong \mathcal{S}(V)|_U$. Let $h \in \mathcal{S}(U)$; then there exists $\bar{h} \in \mathcal{S}(V)$ such that $h = \bar{h}|_U$. As V is an open subset of the affine algebraic manifold $\mathbb{R}^n$, by Theorem 2.15, the extension of $\bar{h}$ by zero to $\mathbb{R}^n$ (denote it by $\widehat{h}$) is a Schwartz function on $\mathbb{R}^n$ that is flat on Z. Finally, defining $\widetilde{h} := \widehat{h}|_X$, we get that $\widetilde{h} \in \mathcal{S}(X)$ (by Definition 3.7), $\widetilde{h}|_U = h$ (by definition), and $\widetilde{h}$ is flat on $X \setminus U$ (as $\widehat{h}$ is an extension of $\widetilde{h}$ to $\mathbb{R}^n$ that is flat at $X \setminus U$). ■

Lemma 3.20 *Let X be a compact (in the Euclidean topology) algebraic set in $\mathbb{R}^n$, then*

$$\mathcal{S}(X) = \left\{ f: X \to \mathbb{R} : \exists \widetilde{f} \in C^\infty(\mathbb{R}^n) \text{ such that } \widetilde{f}|_X = f \right\}.$$

Proof The inclusion $\subset$ is trivial, as $\mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$. For the inclusion $\supset$, take some $g \in \{f: X \to \mathbb{R} : \exists \widetilde{f} \in C^\infty(\mathbb{R}^n) \text{ such that } \widetilde{f}|_X = f\}$. Let $\widetilde{g}$ be some $C^\infty(\mathbb{R}^n)$ function satisfying $\widetilde{g}|_X = g$. Let $\rho \in C^\infty(\mathbb{R}^n)$ be a compactly supported (in the Euclidean topology) function such that $\rho|_X = 1$ (it is standard to show such ρ exists by convolving the characteristic function of some bounded open subset containing X with some appropriate approximation of unity). Then $\rho \cdot \widetilde{g}$ is a smooth compactly supported function on $\mathbb{R}^n$, hence $\rho \cdot \widetilde{g} \in \mathcal{S}(\mathbb{R}^n)$. Moreover, $(\rho \cdot \widetilde{g})|_X = \widetilde{g}|_X = g$, and so $g \in \mathcal{S}(X)$. ∎

Lemma 3.21 *Let X be a compact (in the Euclidean topology) algebraic set in $\mathbb{R}^n$, and let $Z \subset X$ be a Zariski closed subset. Define $U := X \setminus Z$, $W_Z := \{\phi \in \mathcal{S}(X) | \phi \text{ is flat on } Z\}$ and $W_Z^{\mathbb{R}^n} := \{\phi \in \mathcal{S}(\mathbb{R}^n) | \phi \text{ is flat on } Z\}$. Then for any $f \in W_Z$, there exists $\widetilde{f} \in W_Z^{\mathbb{R}^n}$ such that $\widetilde{f}|_X = f$.*

Proof This is immediate from Lemmas 3.16 and 3.20. ∎

Proposition 3.22 *Let X be an affine algebraic variety, and let $Z \subset X$ be some closed subset. Define $U := X \setminus Z$ and $W_Z := \{\phi \in \mathcal{S}(X) | \phi \text{ is flat on } Z\}$. Then restriction from X to U of a function in W_Z is a Schwartz function on U,* i.e., $\mathrm{Res}^U_X(W_Z) \subset \mathcal{S}(U)$.

Proof We first prove the case where X is complete, and then deduce the non-complete case from the complete case.

Consider X as an algebraic subset in $\mathbb{R}^n$.

Case 1: X is complete. By Remark 2.6, X is compact in the Euclidean topology in $\mathbb{R}^n$. Define $U^{\mathbb{R}^n} := \mathbb{R}^n \setminus Z$ and $W_Z^{\mathbb{R}^n} := \{\phi \in \mathcal{S}(\mathbb{R}^n) | \phi \text{ is flat on } Z\}$. Z is closed in $\mathbb{R}^n$, and so $U^{\mathbb{R}^n}$ is open in $\mathbb{R}^n$. As $U = U^{\mathbb{R}^n} \cap X$, we get that U is closed in $U^{\mathbb{R}^n}$. We show that indeed $\mathrm{Res}^U_X(W_Z) \subset \mathcal{S}(U)$ by showing the existence of the following 3 maps:

$$\begin{tikzcd}
& W_Z^{\mathbb{R}^n} \arrow[dl, "\mathrm{Res}^X_{\mathbb{R}^n}"', "(1)"] \arrow[dr, "\mathrm{Res}^{U^{\mathbb{R}^n}}_{\mathbb{R}^n}", "(2)"'] & \\
W_Z \arrow[dr, dotted, "\mathrm{Res}^U_X"'] & & \mathcal{S}(U^{\mathbb{R}^n}) \arrow[dl, "\mathrm{Res}^U_{U^{\mathbb{R}^n}}", "(3)"'] \\
& \mathcal{S}(U) &
\end{tikzcd}$$

Clearly a restriction of a function in $W_Z^{\mathbb{R}^n}$ to X lies in W_Z, *i.e.*, $\mathrm{Res}^X_{\mathbb{R}^n}$ is well defined. By Lemma 3.21, $\mathrm{Res}^X_{\mathbb{R}^n}$ is onto. Let $g \in W_Z^{\mathbb{R}^n}$. Then, by Theorem 2.15, $g|_{U^{\mathbb{R}^n}} \in \mathcal{S}(U^{\mathbb{R}^n})$, *i.e.*, map (2) is well defined. Let $h \in \mathcal{S}(U^{\mathbb{R}^n})$. Then, by Theorem 3.9, $h|_U \in \mathcal{S}(U)$, *i.e.*, map (3) is well defined. Thus, Proposition 3.22 holds if X is complete.

Case 2: X is non-complete. Consider a one point compactification, *i.e.*, a pair $(\dot{X}, i)$ as in Proposition 2.7, and take some $f \in W_Z \subset \mathcal{S}(X)$. As $i: X \to i(X)$ is a biregular isomorphism, $i_* f := f \circ i^{-1} \in \mathcal{S}(i(X))$. As $i(X)$ is open in $\dot{X}$, by Proposition 3.19 there exists $\dot{f} \in \mathcal{S}(\dot{X})$ such that $i_* f = \dot{f}|_{i(X)}$ ($\dot{f}$ is the extension by zero to $\dot{X}$ of $i_* f$).

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Let $p := \dot{X} \setminus i(X)$ (by Proposition 2.7, p is a single point, hence it is closed in $\dot{X}$). We claim that $i(Z) \cup \{p\}$ is closed in $\dot{X}$. Indeed, $\dot{X} \setminus (i(Z) \cup \{p\}) = i(X) \setminus i(Z)$ is open in $i(X)$ (as Z is closed in X and i is a biregular isomorphism of X and $i(X)$), and $i(X)$ is open in $\dot{X}$. Now define $U' := \dot{X} \setminus (i(Z) \cup \{p\})$, is open in $\dot{X}$. By Case 1, $\mathrm{Res}_{\dot{X}}^{U'}(\dot{f}) \in \mathcal{S}(U')$. Observe that $i^{-1}|_{U'}$ is a biregular isomorphism of U' and U, and so $(i^{-1}|_{U'})_* \mathrm{Res}_{\dot{X}}^{U'}(\dot{f}) \in \mathcal{S}(U)$. But $(i^{-1}|_{U'})_* \mathrm{Res}_{\dot{X}}^{U'}(\dot{f}) = (i^{-1}|_{U'})_*((i_* f)|_{U'}) = f|_U$, thus $f|_U \in \mathcal{S}(U)$. ■

Theorem 3.23 *Let X be an affine algebraic variety, and let $Z \subset X$ be some closed subset. Define $U := X \setminus Z$ and $W_Z := \{\phi \in \mathcal{S}(X) | \phi$ is flat on $Z\}$. Then W_Z is a closed subspace of $\mathcal{S}(X)$, and extension by zero $\mathcal{S}(U) \to W_Z$ (denote $\mathrm{Ext}_U^X \colon \mathcal{S}(U) \to W_Z$) is an isomorphism of Fréchet spaces whose inverse is the restriction of functions (denoted by $\mathrm{Res}_X^U \colon W_Z \to \mathcal{S}(U)$).*

Proof As $W_Z = \bigcap_{z \in Z} \{\phi \in \mathcal{S}(X) | \phi \text{ is flat on } z\}$ is an intersection of closed sets, it is a closed subspace of $\mathcal{S}(X)$ and thus a Fréchet space (see Proposition 2.16).

By Proposition 3.19, the extension of a function in $\mathcal{S}(U)$ by zero to X is a function in $\mathcal{S}(X)$ that is flat at Z, *i.e.*, $\mathrm{Ext}_U^X(\mathcal{S}(U)) \subset W_Z$. Furthermore, we claim that Ext_U^X is continuous.

Indeed, take some closed embedding $X \hookrightarrow \mathbb{R}^n$ and consider X as an algebraic set. Define $W = \mathbb{R}^n \setminus Z$. Then W is an affine algebraic manifold containing U (which is an affine algebraic variety), and $U = W \cap X$, *i.e.*, U is closed in W. Now take some closed embedding $W \hookrightarrow \mathbb{R}^N$; then by Theorem 3.9 and Proposition 2.17, $\mathcal{S}(U) \cong \mathcal{S}(W)/I_{\mathrm{Sch}}^W(U)$. As W is open in $\mathbb{R}^n$, by Theorem 2.15, $\mathrm{Ext}_W^{\mathbb{R}^n}$ is a closed embedding $\mathcal{S}(W) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$ (and in particular it is a continuous map). Then we can write $I_{\mathrm{Sch}}^W(U) = I_{\mathrm{Sch}}^{\mathbb{R}^n}(X) \cap \mathcal{S}(W)$. In particular, $I_{\mathrm{Sch}}^W(U)$ is closed in $\mathcal{S}(\mathbb{R}^n)$. Finally, as the embedding $\mathcal{S}(W) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$ is continuous, the map

$$\mathcal{S}(U) \cong \mathcal{S}(W)/\big(I_{\mathrm{Sch}}^{\mathbb{R}^n}(X) \cap \mathcal{S}(W)\big) \longrightarrow \mathcal{S}(\mathbb{R}^n)/I_{\mathrm{Sch}}^{\mathbb{R}^n}(X) = \mathcal{S}(X)$$

is continuous as well, *i.e.*, Ext_U^X is continuous.

By Proposition 3.22 the restriction of a function in W_Z to U is a Schwartz function on U; *i.e.*, $\mathrm{Res}_X^U(W_Z) \subset \mathcal{S}(U)$.

By definition, $\mathrm{Res}_X^U \circ \mathrm{Ext}_U^X \colon \mathcal{S}(U) \to \mathcal{S}(U)$ is the identity operator on $\mathcal{S}(U)$ and $\mathrm{Ext}_U^X \circ \mathrm{Res}_X^U \colon W_Z \to W_Z$ is the identity operator on W_Z. Thus, Ext_U^X is a continuous (linear) bijection. Then, by Banach open mapping Theorem 2.18, Ext_U^X is an isomorphism of Fréchet spaces. ■

Corollary 3.24 *Let X be an affine algebraic variety. A Schwartz function $f \in \mathcal{S}(X)$ is flat at $p \in X$ if and only if $f|_{X \setminus \{p\}} \in \mathcal{S}(X \setminus \{p\})$.*

Proof Apply Theorem 3.23 to $Z = \{p\}$. ■

Remark 3.25 By the same argument for an arbitrary function $f \in C^\infty(X)$ (*i.e.*, a function that is a restriction of a smooth function from an open neighborhood of some closed embedding of X) and any $p \in X$, the following conditions are equivalent:

(i) f is flat at p.
(ii) There exists a smooth compactly supported function ρ on some affine space in which X is closely embedded such that ρ is identically 1 on some open neighborhood of p and

$$(f \cdot \rho)|_{X \smallsetminus \{p\}} \in \mathcal{S}(X \smallsetminus \{p\}).$$

(iii) For any smooth compactly supported function ρ on any affine space in which X is closely embedded such that ρ is identically 1 on some open neighborhood of p, one has

$$(f \cdot \rho)|_{X \smallsetminus \{p\}} \in \mathcal{S}(X \smallsetminus \{p\}).$$

Theorem 3.23 also implies that Proposition 3.14(ii) holds for any partition of unity:

Corollary 3.26 *Let X be an affine algebraic variety, and let $V \subset X$ be some open subset of X. Then for any $\beta \in \mathcal{T}(X)$ such that* $\operatorname{supp}(\beta) \subset V$ *and for any $\varphi \in \mathcal{S}(X)$, one has $(\beta \cdot \varphi)|_V \in \mathcal{S}(V)$.*

Proof By Proposition 3.12, $\beta \cdot \varphi \in \mathcal{S}(X)$. Consider X as an algebraic subset of some $\mathbb{R}^n$. By definition, $\operatorname{supp}(\beta)$ is a closed subset of $\mathbb{R}^n$ in the Euclidean topology. There exists some Zariski open $\widetilde{V} \subset \mathbb{R}^n$ such that $V = \widetilde{V} \cap X$. As $\operatorname{supp}(\beta) \subset \widetilde{V}$, which is also an open subset of $\mathbb{R}^n$ in the Euclidean topology, the function $\beta \cdot \varphi$ is flat on $X \smallsetminus V$. Thus, by Theorem 3.23, $(\beta \cdot \varphi)|_V \in \mathcal{S}(V)$. ∎

Remark 3.27 Theorem 3.23 suggests the following point of view on Schwartz functions. Given an affine algebraic variety X, we take some affine compactification of it; *i.e.,* we consider X as an open subset of some complete affine variety Y (we used one point compactification, but this is not necessary). Then a Schwartz function on X is just a smooth function on Y (in the sense that it is the restriction to Y of a smooth function on the ambient space of Y), that is flat on $Y \smallsetminus X$. This point of view is convenient, as it involves only local properties; the condition of "rapidly decaying at infinity" is translated to the condition of flatness at "all points added in infinity" in the compactification process. This is also true in the Nash category (by Theorem 2.15 and, more generally, [AG, Theorem 5.4.1]), and the easiest example is the case where $X = \mathbb{R}$, where one can identify $\mathbb{R}$ with the unit circle without a point.

Definition 3.28 Let X be an affine algebraic variety. Define the *space of tempered distributions on* X as the space of continuous linear functionals on $\mathcal{S}(X)$. Denote this space by $\mathcal{S}^*(X)$.

For instance, in the case $X = \mathbb{R}^n$, any tempered function $t \in \mathcal{T}(\mathbb{R}^n)$ gives rise to a tempered distribution ξ_t, defined by $\xi_t(s) := \int s \cdot t dx$ for any $s \in \mathcal{S}(\mathbb{R}^n)$. Not all tempered distributions arise in such a manner, *e.g.,* Dirac's Delta.

Theorem 3.29 *Let X be an affine algebraic variety, and let $U \subset X$ be some Zariski open subset. Then* $\operatorname{Ext}_U^X\colon \mathcal{S}(U) \hookrightarrow \mathcal{S}(X)$ *is a closed embedding, and the restriction morphism $\mathcal{S}^*(X) \to \mathcal{S}^*(U)$ is onto.*

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Proof The first part of the theorem is just a restatement of Theorem 3.23 (substituting $Z = X \smallsetminus U$). The second part follows from the fact that $\mathcal{S}(X)$ is a Fréchet space (3.3) and from the Hahn–Banach Theorem (2.19). ∎

4 Sheaf and Cosheaf Properties

This section is devoted to proving that tempered functions and tempered distributions form sheaves (Propositions 4.3 and 4.4) and that Schwartz functions form a cosheaf (Proposition 4.5). The precise definition of a cosheaf is given right before Proposition 4.5.

Lemma 4.1 (Restrictions of tempered functions to closed and to open subsets) *Let X be an affine algebraic variety, and let $U \subset X$ be some open subset. Then*

$$\operatorname{Res}_X^{X\smallsetminus U}(\mathcal{T}(X)) = \mathcal{T}(X \smallsetminus U) \quad \textit{and} \quad \operatorname{Res}_X^U(\mathcal{T}(X)) \subset \mathcal{T}(U).$$

Proof Consider some closed embedding of X in some affine space; *i.e.,* consider X as an algebraic subset of $\mathbb{R}^n$. Then $\mathcal{T}(X) := \operatorname{Res}_{\mathbb{R}^n}^X(\mathcal{T}(\mathbb{R}^n))$. In these settings $X \smallsetminus U$ is also an algebraic subset of $\mathbb{R}^n$, and so

$$\mathcal{T}(X \smallsetminus U) := \operatorname{Res}_{\mathbb{R}^n}^{X\smallsetminus U}(\mathcal{T}(\mathbb{R}^n)) = \operatorname{Res}_X^{X\smallsetminus U}(\operatorname{Res}_{\mathbb{R}^n}^X(\mathcal{T}(\mathbb{R}^n))) = \operatorname{Res}_X^{X\smallsetminus U}(\mathcal{T}(X)).$$

This proves the first part of Lemma 4.1.

As $U \subset X$ is open, there exists an open subset $\widetilde{U} \subset \mathbb{R}^n$ such that $U = \widetilde{U} \cap X$, and U is closed in $\widetilde{U}$. In particular $\widetilde{U}$ is an affine algebraic submanifold of $\mathbb{R}^n$. Let $t \in \mathcal{T}(X)$. By definition there exists $T \in \mathcal{T}(\mathbb{R}^n)$ such that $T|_X = t$. By Proposition 2.12 $T|_{\widetilde{U}} \in \mathcal{T}(\widetilde{U})$. By Proposition 2.3 $\widetilde{U}$ is open affine, thus by the first part of the lemma, $t|_U = (T|_{\widetilde{U}})|_U \in \mathcal{T}(U)$. ∎

Corollary 4.2 *Let X be an affine algebraic variety, and let $U, V \subset X$ be two open subsets of X such that $U \subset V$. Then $\operatorname{Res}_V^U(\mathcal{T}(V)) \subset \mathcal{T}(U)$.*

Proof By Proposition 2.3, V is an affine algebraic variety, so we can apply Lemma 4.1 to the affine algebraic variety V and its open subset U. ∎

Proposition 4.3 *Let X be an affine algebraic variety. The assignment of the space of tempered functions to any open $U \subset X$, together with the restriction of functions, form a sheaf on X.*

Proof By Corollary 4.2, the above is a pre-sheaf. Clearly, the axiom of uniqueness holds.

Now let $t_i \in \mathcal{T}(U_i)$ be such that for any $i, j \in I$, $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$. Clearly, there exists a unique function $t: U \to \mathbb{R}$ such that for any $i \in I$, $t|_{U_i} = t_i$. In order to prove that the existence axiom holds, it is thus left to show that $t \in \mathcal{T}(U)$. As always, we can consider X as an algebraic subset of $\mathbb{R}^n$. By Proposition 2.4, we can assume $|I| < \infty$ by choosing some subcover and showing $t|_{U_i} = t_i$ only for indices i in this subcover (as the functions we begin with agree on the intersections, this will automatically hold for all the other indices we omitted). By standard induction on the number of indices

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(*i.e.*, the number of sets in the chosen finite subcover), it is enough to show that the following holds.

> Let $X \subset \mathbb{R}^n$ be an algebraic subset and let $U_1, U_2 \subset X$ be two open subsets. Assume that for any $i \in \{1,2\}$ we are given $t_i \in \mathcal{T}(U_i)$ such that $t_1|_{U_1 \cap U_2} = t_2|_{U_1 \cap U_2}$. Then there exists a function $t \in \mathcal{T}(U_1 \cup U_2)$ such that $t|_{U_1} = t_1, t|_{U_2} = t_2$.

Clearly, there exists a (unique) function $t: U_1 \cup U_2 \to \mathbb{R}$ such that $t|_{U_i} = t_i$. It is left to show that $t \in \mathcal{T}(U_1 \cup U_2)$. Indeed, there exist open sets $\widetilde{U}_i \subset \mathbb{R}^n$ such that $U_i = \widetilde{U}_i \cap X$, and U_i is closed in $\widetilde{U}_i$. Then, by Lemma 3.5, there exist $T_i \in \mathcal{T}(\widetilde{U}_i)$ such that $t_i = T_i|_{U_i}$. Define $U = U_1 \cup U_2$ and $\widetilde{U} = \widetilde{U}_1 \cup \widetilde{U}_2$. As $\widetilde{U}$ is an affine algebraic manifold and $\{\widetilde{U}_1, \widetilde{U}_2\}$ is an open cover of $\widetilde{U}$, by Theorem 2.11 there exist $\alpha_1, \alpha_2 \in \mathcal{T}(\widetilde{U})$ such that $\operatorname{supp}(\alpha_i) \subset \widetilde{U}_i$ and $\alpha_1 + \alpha_2 = 1$. Define $T'_i(x) := \operatorname{Ext}^{\widetilde{U}}_{\widetilde{U}_i}(T_i)$ for $i = 1, 2$. Define a new function on $\widetilde{U}$ by $T := \alpha_1 \cdot T'_1 + \alpha_2 \cdot T'_2$.

By Proposition 2.12, in order to show that $T \in \mathcal{T}(\widetilde{U})$, it is enough to show that $T|_{\widetilde{U}_i} \in \mathcal{T}(\widetilde{U}_i)$ for $i \in \{1,2\}$. Let us show this for $i = 1$ (symmetrical arguments work for $i = 2$): $T|_{\widetilde{U}_1} = \alpha_1|_{\widetilde{U}_1} \cdot T'_1|_{\widetilde{U}_1} + (\alpha_2 \cdot T'_2)|_{\widetilde{U}_1} = \alpha_1|_{\widetilde{U}_1} \cdot T_1 + (\alpha_2 \cdot T'_2)|_{\widetilde{U}_1}$. As the space of tempered functions is an algebra, it is enough to show each of these three functions belongs to $\mathcal{T}(\widetilde{U}_1)$. By Proposition 2.12, $\alpha_1|_{\widetilde{U}_1} \in \mathcal{T}(\widetilde{U}_1)$. By construction, $T_1 \in \mathcal{T}(\widetilde{U}_1)$. In order to show that $(\alpha_2 \cdot T'_2)|_{\widetilde{U}_1} \in \mathcal{T}(\widetilde{U}_1)$ we use Proposition 2.12 again. As $\{\widetilde{U}_1 \cap \widetilde{U}_2, \widetilde{U}_1 \smallsetminus (\operatorname{supp}(\alpha_2) \cap \widetilde{U}_1)\}$ is an open cover of $\widetilde{U}_1$, it is enough to show that

$$\big((\alpha_2 \cdot T'_2)|_{\widetilde{U}_1}\big)|_{\widetilde{U}_1 \cap \widetilde{U}_2} \in \mathcal{T}(\widetilde{U}_1 \cap \widetilde{U}_2),$$
$$\big((\alpha_2 \cdot T'_2)|_{\widetilde{U}_1}\big)|_{\widetilde{U}_1 \smallsetminus (\operatorname{supp}(\alpha_2) \cap \widetilde{U}_1)} \in \mathcal{T}(\widetilde{U}_1 \smallsetminus (\operatorname{supp}(\alpha_2) \cap \widetilde{U}_1)).$$

The later is obvious as $((\alpha_2 \cdot T'_2)|_{\widetilde{U}_1})|_{\widetilde{U}_1 \smallsetminus (\operatorname{supp}(\alpha_2) \cap \widetilde{U}_1)} = 0$, and the first also holds, since both $\alpha_2|_{\widetilde{U}_1 \cap \widetilde{U}_2} \in \mathcal{T}(\widetilde{U}_1 \cap \widetilde{U}_2)$ and $T_2|_{\widetilde{U}_1 \cap \widetilde{U}_2} \in \mathcal{T}(\widetilde{U}_1 \cap \widetilde{U}_2)$, as $\widetilde{U}_1 \cap \widetilde{U}_2$ is open in both $\widetilde{U}$ and in $\widetilde{U}_2$ (and again by Proposition 2.12). Finally, as $T \in \mathcal{T}(\widetilde{U})$ and $U_1 \cup U_2 = U \subset \widetilde{U}$ is a closed subset, by Lemma 4.1, $t = T|_{U_1 \cup U_2} \in \mathcal{T}(U_1 \cup U_2)$. ■

Proposition 4.4 *Let X be an affine algebraic variety. The assignment of the space of tempered distributions to any open $U \subset X$, together with restrictions of functionals from $\mathcal{S}^*(U)$ to $\mathcal{S}^*(V)$ for any other open $V \subset U$, form a flabby sheaf on X.*

Proof Any open $U \subset X$ is an affine algebraic variety (by Proposition 2.3), and any open $V \subset X$ contained in U is open in U. Thus, by Proposition 3.19, the above is a pre-sheaf.

Let $\{U_i\}_{i \in I}$ be some open cover of U. By Proposition 2.4 there exists a finite open cover $\{U_i\}_{i=1}^k$. Note that U is an affine algebraic variety (by Proposition 2.3). Then, by Proposition 3.14, for any $1 \le i \le k$, there exists $\beta_i \in \mathcal{T}(U)$ such that $\operatorname{supp}(\beta_i) \subset U_i$, $\sum_{i=1}^k \beta_i = 1$ and for any $s \in \mathcal{S}(U)$, $(\beta_i \cdot s)|_{U_i} \in \mathcal{S}(U_i)$.

Now let $\xi, \zeta \in \mathcal{S}^*(U)$ be such that for any $i \in I$, $\xi|_{\mathcal{S}(U_i)} = \zeta|_{\mathcal{S}(U_i)}$. In particular, as U_i is open in U (and as $\{1, 2, \dots, k\} \subset I$), for any $1 \le i \le k$, $\xi|_{\mathcal{S}(U_i)} = \zeta|_{\mathcal{S}(U_i)}$. Let $s \in \mathcal{S}(U)$. Note that $s = \sum_{i=1}^k (\beta_i \cdot s)$, so we can calculate

$$\xi(s) - \zeta(s) = \xi\Big(\sum_{i=1}^k (\beta_i \cdot s)\Big) - \zeta\Big(\sum_{i=1}^k (\beta_i \cdot s)\Big) = \sum_{i=1}^k \big(\xi(\beta_i \cdot s) - \zeta(\beta_i \cdot s)\big) = 0,$$

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where the second equality follows from linearity of ξ and ζ, and the third equality follows from the facts that $(\beta_i \cdot s)|_{U_i} \in \mathcal{S}(U_i)$ and $\xi|_{\mathcal{S}(U_i)} = \zeta|_{\mathcal{S}(U_i)}$. Thus, we have shown that the axiom of uniqueness holds.

Now let $\xi_i \in \mathcal{S}^*(U_i)$ be such that for any $i, j \in I$, $\xi_i|_{\mathcal{S}(U_i \cap U_j)} = \xi_j|_{\mathcal{S}(U_i \cap U_j)}$. In particular, as $U_i \cap U_j$ is open in U (and as $\{1, 2, \ldots, k\} \subset I$), for any $1 \le i \le k$, $\xi_i|_{\mathcal{S}(U_i \cap U_j)} = \xi_j|_{\mathcal{S}(U_i \cap U_j)}$. We define a functional $\xi \in \mathcal{S}^*(U)$ by the following formula for any $s \in \mathcal{S}(U)$:

$$\xi(s) = \xi\Big(\sum_{i=1}^{k}(\beta_i \cdot s)\Big) := \sum_{i=1}^{k} \xi_i(\beta_i \cdot s).$$

We claim that for any $\alpha \in I$, one has $\xi|_{\mathcal{S}(U_\alpha)} = \xi_\alpha$. Indeed, $\{U_\alpha \cap U_i\}_{i=1}^k$ is an open cover of the affine algebraic variety U_α. Note that $\{\beta_i|_{U_\alpha}\}_{i=1}^k$ is "a partition of unity" of U_α as defined in Proposition 3.14, *i.e.,* $\beta_i|_{U_\alpha} \in \mathcal{T}(U_\alpha)$ (this follows from Proposition 4.3), $\sum_{i=1}^k \beta_i|_{U_\alpha} = 1$, $\operatorname{supp}(\beta_i|_{U_\alpha}) \subset U_\alpha \cap U_i$ and (by Corollary 3.26) for any $s \in \mathcal{S}(U_\alpha)$, one has $(\beta_i|_{U_\alpha} \cdot s)|_{U_\alpha \cap U_i} \in \mathcal{S}(U_\alpha \cap U_i)$. Also note that for any $1 \le i \le k$, one has $\xi_\alpha|_{\mathcal{S}(U_\alpha \cap U_i)} = \xi_i|_{\mathcal{S}(U_\alpha \cap U_i)}$. Finally, we are ready to calculate (for any $s \in \mathcal{S}(U_\alpha)$, where we also think of s as a function in $\mathcal{S}(U)$, by the usual extension by zero):

$$\xi_\alpha(s) = \xi_\alpha\Big(\sum_{i=1}^{k}\beta_i \cdot s\Big) = \sum_{i=1}^{k}\xi_\alpha(\beta_i \cdot s) = \sum_{i=1}^{k}\xi_i(\beta_i \cdot s) = \xi(s);$$

i.e., the axiom of existence holds. ∎

We recall the definition of a cosheaf on a topological space. For simplicity we assume our cosheaves take values in the category of real vector spaces, but this can be replaced by any other Abelian category with arbitrary coproducts.

A *pre-cosheaf* F on a topological space X is a covariant functor from $\operatorname{Top}(X)$ to the category of real vector spaces, where $\operatorname{Top}(X)$ is the category whose objects are the open sets of X, and whose morphisms are the inclusion maps. A *cosheaf* on a topological space X is a pre-cosheaf such that for any open $U \subset X$ and any open cover $\{U_i\}_{i\in I}$ of U, the following sequence is exact:

$$\bigoplus_{(i,j)\in I^2} F(U_i \cap U_j) \xrightarrow{\operatorname{Ext}_1} \bigoplus_{i\in I} F(U_i) \xrightarrow{\operatorname{Ext}_2} F(U) \longrightarrow 0,$$

where the k-th coordinate of $\operatorname{Ext}_1(\bigoplus_{(i,j)\in I^2} \xi_{i,j})$ is $\sum_{i\in I} \operatorname{Ext}_{U_k\cap U_i}^{U_k}(\xi_{k,i} - \xi_{i,k})$, and $\operatorname{Ext}_2(\bigoplus_{i\in I}\xi_i) := \sum_{i\in I}\operatorname{Ext}_{U_i}^{U}(\xi_i)$. When exactness is proved in Proposition 4.5, all calculations will be quickly reduced to finite subcovers. A cosheaf is *flabby* if for any two open subsets $U, V \subset X$ such that $V \subset U$, the morphism $\operatorname{Ext}_V^U \colon F(V) \to F(U)$ is injective.

Proposition 4.5 *Let X be an affine algebraic variety. The assignment of the space of Schwartz functions to any open $U \subset X$, together with the extension by zero, Ext_U^V, from U to any other open $V \supset U$, form a flabby cosheaf on X.*

Proof Any open $V \subset X$ is an affine algebraic variety (by Proposition 2.3), and any open $U \subset X$ contained in V is open in V. Thus, by Theorem 3.29, the above is a pre-cosheaf. It is left to show exactness.

Let $\{U_i\}_{i\in I}$ be some open cover of U, and let $s \in \mathcal{S}(U)$. By Proposition 2.4, there exists a finite subcover $\{U_i\}_{i=1}^k$. By Proposition 2.3 and Corollary 3.13, we can apply Proposition 3.14 on U, and so there exist $\beta_1, \dots \beta_k \in \mathcal{T}(U)$ such that $\sum_{i=1}^k \beta_i = 1$ and for any $1 \le i \le k$, $\operatorname{supp}(\beta_i) \subset U_i$, and $(\beta_i \cdot s)|_{U_i} \in \mathcal{S}(U_i)$. Then we can write: $s = \sum_{i=1}^k \beta_i \cdot s = \sum_{i=1}^k \operatorname{Ext}_{U_i}^U((\beta_i \cdot s)|_{U_i})$, and so Ext_2 is onto. It is left to show that $\ker(\operatorname{Ext}_2) = \operatorname{Im}(\operatorname{Ext}_1)$.

Assume we are given a finite subset $J \subset I$, and for any $i \in J$, $s_i \in \mathcal{S}(U_i)$ such that $\sum_{i\in J} \operatorname{Ext}_{U_i}^U(s_i) = 0$. It is sufficient to prove that for any $i > j \in J$ (for some linear order on J), there exists $s_{i,j} \in \mathcal{S}(U_i \cap U_j)$ such that for any $i \in J$,

$$s_i = \sum_{i>j\in J} \operatorname{Ext}_{U_i\cap U_j}^{U_i}(s_{i,j}) - \sum_{i<j\in J} \operatorname{Ext}_{U_i\cap U_j}^{U_i}(s_{j,i}).$$

We prove this claim by induction on $|J|$. For $|J| = 2$ one has $s_1|_{U_1\cap U_2} = -s_2|_{U_1\cap U_2}$, so defining $s_{2,1} = s_2|_{U_1\cap U_2}$, the claim holds. The only non trivial fact to verify is that $s_2|_{U_1\cap U_2} \in \mathcal{S}(U_1 \cap U_2)$; indeed, by Proposition 3.19, $-\operatorname{Ext}_{U_1}^{U_1\cup U_2}(s_1)$ is a Schwartz function on $U_1 \cup U_2$ that is flat on $U_1 \cup U_2 \smallsetminus U_1$, and $\operatorname{Ext}_{U_2}^{U_1\cup U_2}(s_2)$ is a Schwartz function on $U_1 \cup U_2$ that is flat on $U_1 \cup U_2 \smallsetminus U_2$. But as $\operatorname{Ext}_{U_2}^{U_1\cup U_2}(s_2) = -\operatorname{Ext}_{U_1}^{U_1\cup U_2}(s_1)$, we have that $\operatorname{Ext}_{U_2}^{U_1\cup U_2}(s_2)$ is flat on $(U_1 \cup U_2) \smallsetminus (U_1 \cap U_2)$. Then by Theorem 3.23,

$$s_2|_{U_1\cap U_2} = (\operatorname{Ext}_{U_2}^{U_1\cup U_2}(s_2))|_{U_1\cap U_2}$$

is a Schwartz function on $U_1 \cap U_2$.

Now assume the claim holds for any J of cardinality up to k, and let

$$J = \{1, 2, \dots, k, k+1\}.$$

Without loss of generality, we can assume $U = \bigcup_{i=1}^{k+1} U_i$, and so for any $1 \le i \le k+1$, we have $s_i \in \mathcal{S}(U_i)$ such that $\sum_{i=1}^{k+1} \operatorname{Ext}_{U_i}^U(s_i) = 0$. Define $\widetilde{U} := \bigcup_{i=1}^k U_i$. Note that $s_{k+1}|_{U_{k+1}\smallsetminus(U_{k+1}\cap\widetilde{U})} = 0$. As $\{U_i\}_{i=1}^k$ is an open cover of the affine $\widetilde{U}$, by Proposition 3.14, there exist $\{\beta_i\}_{i=1}^k \subset \mathcal{T}(\widetilde{U})$ such that for any $1 \le i \le k$, $\operatorname{supp}(\beta_i) \subset U_i$ and $\sum_{i=1}^k \beta_i = 1$.

Let $x \in U_{k+1} \smallsetminus (U_{k+1} \cap \widetilde{U})$. By Proposition 3.19, for any $1 \le i \le k$, $\operatorname{Ext}_{U_i}^U(s_i)$ is flat at x. Then $\operatorname{Ext}_{U_{k+1}}^U(s_{k+1}) = -\sum_{i=1}^k \operatorname{Ext}_{U_i}^U(s_i)$ is also flat at x. Applying Theorem 3.23 (note that $U \smallsetminus (U_{k+1} \smallsetminus (\widetilde{U} \cap U_{k+1})) = \widetilde{U}$), we get that $(\operatorname{Ext}_{U_{k+1}}^U(s_{k+1}))|_{\widetilde{U}} \in \mathcal{S}(\widetilde{U})$. Now by Corollary 3.26, we have

$$\left(\beta_i|_{U_i} \cdot \operatorname{Ext}_{U_i\cap U_{k+1}}^{U_i}(s_{k+1}|_{U_i\cap U_{k+1}})\right) = \beta_i|_{U_i} \cdot (\operatorname{Ext}_{U_{k+1}}^{\widetilde{U}}(s_{k+1}))|_{U_i} \in \mathcal{S}(U_i).$$

Define for any $1 \le i \le k$,

$$\gamma_i := s_i + \left(\beta_i|_{U_i} \cdot \operatorname{Ext}_{U_i\cap U_{k+1}}^{U_i}(s_{k+1}|_{U_i\cap U_{k+1}})\right).$$

Note that $\gamma_i \in \mathcal{S}(U_i)$ and that $\sum_{i=1}^k \operatorname{Ext}_{U_i}^{\widetilde{U}}(\gamma_i) = 0$. Thus, by induction hypothesis, for any $1 \le j < i \le k$, there exist $s_{i,j} \in \mathcal{S}(U_i \cap U_j)$ such that for any $1 \le i \le k$,

$$\gamma_i = \sum_{i>j\ge 1} \operatorname{Ext}_{U_i\cap U_j}^{U_i}(s_{i,j}) - \sum_{i<j\le k} \operatorname{Ext}_{U_i\cap U_j}^{U_i}(s_{j,i}).$$

For any $1 \le i \le k$, define $s_{k+1,i} := \beta_i|_{U_{k+1}\cap U_i} \cdot s_{k+1}|_{U_{k+1}\cap U_i}$. Then

$$\gamma_i = s_i + \operatorname{Ext}_{U_{k+1}\cap U_i}^{U_i}(s_{k+1,i}),$$

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where $U_{k+1} \cap U_i$ is open in U_i. As both γ_i and s_i lie in $\mathcal{S}(U_i)$, so does

$$\operatorname{Ext}_{U_{k+1}\cap U_i}^{U_i}(s_{k+1,i}) = \gamma_i - s_i.$$

We claim that $s_{k+1,i} \in \mathcal{S}(U_{k+1} \cap U_i)$. Denoting $f := \beta_i|_{U_i} \cdot \operatorname{Ext}_{U_i \cap U_{k+1}}^{U_i}(s_{k+1}|_{U_i \cap U_{k+1}})$, we saw above that $f \in \mathcal{S}(U_i)$. Thus, as $s_{k+1,i} = f|_{U_i \cap U_{k+1}}$, by Theorem 3.23 we need to show that f is flat at

$$U_i \smallsetminus (U_{k+1} \cap U_i) = (U_{k+1} \cup U_i) \smallsetminus U_{k+1}.$$

Define $g := \operatorname{Ext}_{U_{k+1}}^{U_{k+1}\cup U_i}(s_{k+1})$. Then, by Theorem 3.23, $g \in \mathcal{S}(U_i \cup U_{k+1})$, and g is flat at $(U_i \cup U_{k+1}) \smallsetminus U_{k+1}$. In particular, $\widetilde{g} := g|_{U_i}$ is flat at $(U_i \cup U_{k+1}) \smallsetminus U_{k+1}$. Let $x \in (U_i \cup U_{k+1}) \smallsetminus U_{k+1}$, and let $\rho \in \mathcal{S}(U_i)$ be "a bump function around x", *i.e.,* a restriction to U_i of a smooth compactly supported function on some affine space in which U_i is closely embedded such that $\rho = 1$ on some Euclidean open neighborhood of x. Then, by Corollary 3.24, $(\rho \cdot \widetilde{g})|_{U_i \smallsetminus \{x\}} \in \mathcal{S}(U_i \smallsetminus \{x\})$. By Proposition 4.3, $\beta_i|_{U_i \smallsetminus \{x\}} \in \mathcal{T}(U_i \smallsetminus \{x\})$. Thus, by Proposition 3.12, $(\beta_i|_{U_i} \cdot \rho \cdot \widetilde{g})|_{U_i \smallsetminus \{x\}} \in \mathcal{S}(U_i \smallsetminus \{x\})$. On the one hand, by Theorem 3.23,

$$\operatorname{Ext}_{U_i \smallsetminus \{x\}}^{U_i}\big((\beta_i|_{U_i} \cdot \rho \cdot \widetilde{g})|_{U_i \smallsetminus \{x\}}\big) \in \mathcal{S}(U_i).$$

On the other hand, $(\beta_i|_{U_i} \cdot \rho \cdot \widetilde{g})|_{U_i}$ is a continuous function on U_i that equals $\operatorname{Ext}_{U_i \smallsetminus \{x\}}^{U_i}((\beta_i|_{U_i} \cdot \rho \cdot \widetilde{g})|_{U_i \smallsetminus \{x\}})$ on $U_i \smallsetminus \{x\}$. We deduce that

$$\operatorname{Ext}_{U_i \smallsetminus \{x\}}^{U_i}\big((\beta_i|_{U_i} \cdot \rho \cdot \widetilde{g})|_{U_i \smallsetminus \{x\}}\big) = (\beta_i|_{U_i} \cdot \rho \cdot \widetilde{g})|_{U_i},$$

and so $(\beta_i|_{U_i} \cdot \rho \cdot \widetilde{g})|_{U_i}$ is flat at x. Finally, as flatness is a Euclidean local property, and as ρ equals 1 on some Euclidean neighborhood of x, it follows that $\beta_i|_{U_i} \cdot \widetilde{g} = f$ is flat at x.

Then it is easily seen that for any $1 \le i \le k$, we have

$$s_i = \sum_{i > j \in \{1,2,\dots,k+1\}} \operatorname{Ext}_{U_i \cap U_j}^{U_i}(s_{i,j}) - \sum_{i < j \in \{1,2,\dots,k+1\}} \operatorname{Ext}_{U_i \cap U_j}^{U_i}(s_{j,i}).$$

It is left to check that $s_{k+1} = \sum_{i=1}^{k} \operatorname{Ext}_{U_{k+1}\cap U_i}^{U_{k+1}}(s_{k+1,i})$. Indeed,

$$\begin{aligned}\sum_{i=1}^{k} \operatorname{Ext}_{U_{k+1}\cap U_i}^{U_{k+1}}(s_{k+1,i}) &= \sum_{i=1}^{k} \operatorname{Ext}_{U_{k+1}\cap U_i}^{U_{k+1}}(\beta_i|_{U_{k+1}\cap U_i} \cdot s_{k+1}|_{U_{k+1}\cap U_i}) \\ &= s_{k+1} \cdot \sum_{i=1}^{k} \operatorname{Ext}_{U_{k+1}\cap U_i}^{U_{k+1}}(\beta_i|_{U_{k+1}\cap U_i}) = s_{k+1}. \qquad \blacksquare\end{aligned}$$

5 The General Case

For any (not necessarily affine) algebraic variety X, denote the space of all real valued functions on X by $\operatorname{Func}(X, \mathbb{R})$.

Lemma 5.1 *Let X be an algebraic variety, and let $X = \bigcup_{i=1}^{k} X_i = \bigcup_{i=k+1}^{l} X_i$ be two open affine covers. There are natural maps $\phi_1 \colon \bigoplus_{i=1}^{k} \operatorname{Func}(X_i, \mathbb{R}) \to \operatorname{Func}(X, \mathbb{R})$ and*

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$\phi_2: \bigoplus_{i=k+1}^{l} \mathrm{Func}(X_i, \mathbb{R}) \to \mathrm{Func}(X, \mathbb{R})$. *Then*

$$\phi_1\Big(\bigoplus_{i=1}^{k} \mathcal{S}(X_i)\Big) \cong \bigoplus_{i=1}^{k} \mathcal{S}(X_i)/\operatorname{Ker}(\phi_1|_{\oplus_{i=1}^{k}\mathcal{S}(X_i)})$$

has a natural structure of a Fréchet space, and there is an isomorphism of Fréchet spaces $\phi_1(\bigoplus_{i=1}^{k} \mathcal{S}(X_i)) \cong \phi_2(\bigoplus_{i=k+1}^{l} \mathcal{S}(X_i))$.

Proof It follows from Proposition 2.17 that

$$\phi_1\Big(\bigoplus_{i=1}^{k} \mathcal{S}(X_i)\Big) \cong \Big(\bigoplus_{i=1}^{k} \mathcal{S}(X_i)\Big)/\operatorname{Ker}(\phi_1|_{\oplus_{i=1}^{k}\mathcal{S}(X_i)})$$

is indeed a Fréchet space. A direct sum of Fréchet spaces is clearly a Fréchet space, and the kernel of $\phi_1|_{\oplus_{i=1}^{k}\mathcal{S}(X_i)}$ is a closed subspace, as $\bigoplus_{i=1}^{k} s_i \in \operatorname{Ker}(\phi_1|_{\oplus_{i=1}^{k}\mathcal{S}(X_i)})$ if and only if for any $x \in X$, $\sum_{i\in J_x} s_i(x) = 0$, where $J_x := \{1 \le i \le k | x \in X_i\}$; *i.e.*, the kernel is given by infinitely many "closed conditions".

Note that $X = \bigcup_{i=1}^{k} \bigcup_{j=k+1}^{l} X_i \cap X_j$ is an open cover of X by affine algebraic varieties. There is a natural map $\phi_3: \bigoplus_{i=1}^{k} \bigoplus_{j=k+1}^{l} \mathrm{Func}(X_i \cap X_j, \mathbb{R}) \to \mathrm{Func}(X, \mathbb{R})$. It is therefore enough to prove that $\phi_1(\bigoplus_{i=1}^{k} \mathcal{S}(X_i)) \cong \phi_3(\bigoplus_{i=1}^{k} \bigoplus_{j=k+1}^{l} \mathcal{S}(X_i \cap X_j))$. Note that $\{X_i \cap X_j\}_{j=k+1}^{l}$ is an open cover of X_i, and let ϕ^i denote the natural map

$$\phi^i: \bigoplus_{j=k+1}^{l} \mathrm{Func}(X_i \cap X_j, \mathbb{R}) \longrightarrow \mathrm{Func}(X_i, \mathbb{R}).$$

As $\phi_3 = \phi_1 \circ \bigoplus_{i=1}^{k} \phi^i$, it is enough to prove that

$$\bigoplus_{j=k+1}^{l} \mathcal{S}(X_i \cap X_j)/\operatorname{Ker}(\phi^i|_{\oplus_{j=k+1}^{l}\mathcal{S}(X_i\cap X_j)}) \cong \mathcal{S}(X_i).$$

We have an equality of sets by Proposition 4.5 and its proof. By Theorem 3.23, the extension $\mathrm{Ext}_{X_i\cap X_j}^{X_i}(\mathcal{S}(X_i\cap X_j)) \subset \mathcal{S}(X_i)$ is a closed embedding $\mathcal{S}(X_i\cap X_j) \hookrightarrow \mathcal{S}(X_i)$, and in particular it is continuous. Thus we have, by Theorem 2.18, an isomorphism of Fréchet spaces. ∎

Definition 5.2 Let X be an algebraic variety, let $X = \bigcup_{i=1}^{k} X_i$ be some open affine cover and consider the natural map $\phi: \bigoplus_{i=1}^{k} \mathrm{Func}(X_i, \mathbb{R}) \to \mathrm{Func}(X, \mathbb{R})$. Define *the space of Schwartz functions on* X by $\mathcal{S}(X) := (\bigoplus_{i=1}^{k} \mathcal{S}(X_i))/\operatorname{Ker}(\phi|_{\oplus_{i=1}^{k}\mathcal{S}(X_i)})$, with the natural quotient topology. By Lemma 5.1, this definition is independent of the cover chosen, and $\mathcal{S}(X)$ is a Fréchet space.

Theorem 5.3 *Let X be an algebraic variety, and let $Z \subset X$ be some Zariski closed subset. Then the restriction from X to Z defines an isomorphism $\mathcal{S}(Z) \cong \mathcal{S}(X)/I_{\mathrm{Sch}}^{X}(Z)$ (with the quotient topology), where $I_{\mathrm{Sch}}^{X}(Z)$ is the ideal in $\mathcal{S}(X)$ of functions identically vanishing on Z.*

Proof This easily follows from the fact that if $X = \bigcup_{i=1}^{n} X_i$ is an affine open cover then $Z = \bigcup_{i=1}^{n} Z \cap X_i$ is an affine open cover, and from Theorem 3.9. ∎

Lemma 5.4 *Let X be an algebraic variety, and let $t: X \to \mathbb{R}$ be some function. Then the following conditions are equivalent:*

(i) *There exists an open affine cover $X = \bigcup_{i=1}^{k} X_i$ such that $t|_{X_i} \in \mathcal{T}(X_i)$ for any $1 \le i \le k$.*

(ii) *For any open affine cover $X = \bigcup_{i=1}^{k} X_i$ and any $1 \le i \le k$, $t|_{X_i} \in \mathcal{T}(X_i)$.*

Proof Clearly, (ii) implies (i). For the other side, assume there exist two open affine covers $X = \bigcup_{i=1}^{k} X_i = \bigcup_{j=k+1}^{l} X_j$ such that for any $k+1 \le j \le l$, $t|_{X_j} \in \mathcal{T}(X_j)$. Fix some $1 \le i \le k$. Note that $\{X_i \cap X_j\}_{j=k+1}^{l}$ is an open cover of X_i. By Proposition 4.3, as $t|_{X_j} \in \mathcal{T}(X_j)$ for any $k+1 \le j \le l$, we have $t|_{X_j \cap X_i} \in \mathcal{T}(X_j \cap X_i)$. Applying Proposition 4.3 once again, we get that $t|_{X_i} \in \mathcal{T}(X_i)$. ■

Definition 5.5 Let X be an algebraic variety. A real valued function $t: X \to \mathbb{R}$ is called *a tempered function on* X if it satisfies the equivalent conditions of Lemma 5.4. Denote the space of all tempered functions on X by $\mathcal{T}(X)$.

Proposition 5.6 *Let X be an algebraic variety, $t \in \mathcal{T}(X)$ and $s \in \mathcal{S}(X)$. Then $t \cdot s \in \mathcal{S}(X)$.*

Proof Let $X = \bigcup_{i=1}^{n} X_i$ be some open affine cover such that $s = \sum_{i=1}^{n} \mathrm{Ext}_{X_i}^{X}(s_i)$ for some $s_i \in \mathcal{S}(X_i)$. Then $t|_{X_i} \in \mathcal{T}(X_i)$, and by Proposition 3.12 $t|_{X_i} \cdot s_i \in \mathcal{S}(X_i)$. Thus, $t \cdot s = \sum_{i=1}^{n} \mathrm{Ext}_{X_i}^{X}(s_i \cdot t|_{X_i}) \in \mathcal{S}(X)$. ■

Definition 5.7 Let X be an algebraic variety. A function $f: X \to \mathbb{R}$ is called *flat at* $x \in X$ if there exists an affine open neighborhood $x \in X_i \subset X$ such that $f|_{X_i}$ is flat at x. It is called flat at $Z \subset X$ if it is flat at every $x \in Z$.

Remark 5.8 Equivalently, a function $f: X \to \mathbb{R}$ is called flat at $x \in X$ if for any affine open neighborhood $x \in X_i \subset X$ one has that $f|_{X_i}$ is flat at x. This easily follows by intersecting any two affine open neighborhoods.

Proposition 5.9 (Extension by zero for non affine varieties) *Let X be an algebraic variety, and let U be an open subset of X. Then the extension by zero to X of a Schwartz function on U is a Schwartz function on X that is flat at $X \smallsetminus U$.*

Proof Consider some affine open cover $X = \bigcup_{i=1}^{k} X_i$. Then $U = \bigcup_{i=1}^{k} (U \cap X_i)$ is an affine open cover of U. Take some $s \in \mathcal{S}(U)$. By definition $s = \sum_{i=1}^{k} \mathrm{Ext}_{U \cap X_i}^{U}(s_i)$, for some $s_i \in \mathcal{S}(U \cap X_i)$. As $U \cap X_i$ is open in X_i, by Proposition 3.19, $\mathrm{Ext}_{U \cap X_i}^{X_i}(s_i)$ is a Schwartz function on X_i, which is flat at $X_i \smallsetminus (U \cap X_i)$. Then

$$\begin{aligned}\mathrm{Ext}_{U}^{X}(s) = \mathrm{Ext}_{U}^{X}\Big(\sum_{i=1}^{k} \mathrm{Ext}_{U \cap X_i}^{U}(s_i)\Big) &= \sum_{i=1}^{k} \mathrm{Ext}_{U}^{X}\big(\mathrm{Ext}_{U \cap X_i}^{U}(s_i)\big) \\ &= \sum_{i=1}^{k} \mathrm{Ext}_{X_i}^{X}\big(\mathrm{Ext}_{U \cap X_i}^{X_i}(s_i)\big)\end{aligned}$$

is by definition a Schwartz function on X, and clearly it is flat on $X \smallsetminus U$. ■

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Lemma 5.10 (Restrictions of tempered functions to closed and to open subsets for non-affine varieties) *Let X be an algebraic variety, and let $U \subset X$ be some open subset. Then* $\mathrm{Res}_X^U(\mathcal{T}(X)) \subset \mathcal{T}(U)$ *and* $\mathrm{Res}_X^{X \smallsetminus U}(\mathcal{T}(X)) \subset \mathcal{T}(X \smallsetminus U)$.

Proof Consider some affine open cover $X = \bigcup_{i=1}^k X_i$. Then $U = \bigcup_{i=1}^k (U \cap X_i)$ is an affine open cover of U. Let $t \in \mathcal{T}(X)$, then by definition for any $1 \le i \le k$, $t|_{X_i} \in \mathcal{T}(X_i)$. By Proposition 4.3, $(t|_{X_i})|_{U \cap X_i} \in \mathcal{T}(U \cap X_i)$. Clearly, $(t|_{X_i})|_{U \cap X_i} = (t|_U)|_{U \cap X_i}$, thus $t|_U \in \mathcal{T}(U)$, *i.e.,* $\mathrm{Res}_X^U(\mathcal{T}(X)) \subset \mathcal{T}(U)$. Observe that $X \smallsetminus U = \bigcup_{i=1}^k ((X \smallsetminus U) \cap X_i)$ is an affine open cover of $X \smallsetminus U$. By Lemma 4.1, $(t|_{X_i})|_{(X \smallsetminus U) \cap X_i} \in \mathcal{T}((X \smallsetminus U) \cap X_i)$. Clearly $(t|_{X_i})|_{(X \smallsetminus U) \cap X_i} = (t|_{X \smallsetminus U})|_{(X \smallsetminus U) \cap X_i}$, and so $\mathrm{Res}_X^{X \smallsetminus U}(\mathcal{T}(X)) \subset \mathcal{T}(X \smallsetminus U)$. ∎

Corollary 5.11 *Let X be an algebraic variety, and let $U, V \subset X$ be two open subsets of X such that $U \subset V$. Then* $\mathrm{Res}_V^U(\mathcal{T}(V)) \subset \mathcal{T}(U)$.

Proof V is an algebraic variety, so we can apply Lemma 5.10 to the algebraic variety V and its open subset U. ∎

Proposition 5.12 *Let X be an algebraic variety. The assignment of the space of tempered functions to any open $U \subset X$ together with the restriction of functions, form a sheaf on X.*

Proof By Corollary 5.11, the above is a pre-sheaf. Clearly, the axiom of uniqueness holds.

Now let $t_i \in \mathcal{T}(U_i)$ be such that for any $i, j \in I$, $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$. Clearly there exists a (unique) function $t\colon U \to \mathbb{R}$ such that for any $i \in I$, $t|_{U_i} = t_i$. In order to prove that the existence axiom holds, it is thus left to show that $t \in \mathcal{T}(U)$. By Proposition 2.4 we can assume $|I| < \infty$ by choosing some subcover and showing $t|_{U_i} = t_i$ only for indices i in this subcover (as the functions we begin with agree on the intersections, this will automatically hold for all the other indices we omitted). By standard induction on the number of indices (*i.e.,* the number of sets in the chosen finite subcover), it is enough to show that the following holds:

Let X be an algebraic variety and let $U_1, U_2 \subset X$ be two open subsets. Assume that for any $i \in \{1, 2\}$ we are given $t_i \in \mathcal{T}(U_i)$ such that $t_1|_{U_1 \cap U_2} = t_2|_{U_1 \cap U_2}$. Then there exists a function $t \in \mathcal{T}(U_1 \cup U_2)$ such that $t|_{U_1} = t_1, t|_{U_2} = t_2$.

Clearly, there exists a (unique) function $t\colon U_1 \cup U_2 \to \mathbb{R}$ such that $t|_{U_i} = t_i$ for $1 \le i \le 2$. It is left to show that $t \in \mathcal{T}(U_1 \cup U_2)$. Consider some affine open cover $X = \bigcup_{j=1}^k X_j$. Then $U_i = \bigcup_{j=1}^k (U_i \cap X_j)$ is an affine open cover of U_i, and $U_1 \cup U_2 = \bigcup_{j=1}^k ((U_1 \cup U_2) \cap X_j)$ is an affine open cover of $U_1 \cup U_2$. As $t_i \in \mathcal{T}(U_i)$, one has $t_i|_{U_i \cap X_j} \in \mathcal{T}(U_i \cap X_j)$. As $(U_1 \cup U_2) \cap X_j$ is affine, and $\bigcup_{i=1}^2 U_i \cap X_j$ is an affine open cover of it, and as $t_1|_{U_1 \cap U_2} = t_2|_{U_1 \cap U_2}$; then, by Proposition 4.3, $t|_{(U_1 \cup U_2) \cap X_j} \in \mathcal{T}((U_1 \cup U_2) \cap X_j)$, *i.e.,* $t \in \mathcal{T}(U_1 \cup U_2)$. ∎

Further work In order to prove that the rest of the properties that were proved in the affine case also hold in the general case, the next natural step should be proving that a tempered partition of unity also holds in the non-affine case (*i.e.,* to prove a non-affine version of Proposition 3.14). Moreover, it seems that such a proposition would pave the way to proving all other properties. Our attempts at proving this in the algebraic context were not fruitful; however, we suggest the following idea: as the theory of Schwartz spaces etc. is now fully established for arbitrary Nash manifolds and for affine real algebraic varieties (and partially established for non-affine real algebraic varieties), one can try to construct this theory for some bigger category, such that the Nash category and the algebraic category form subcategories of this category. The first guess, of taking the semi-algebraic category to be the nominated one, will not work, as this category is in a sense "too flexible". As isomorphisms are not necessarily even smooth, in this category smooth functions are not pulled back by isomorphisms to smooth functions, and so there is no hope to define Schwartz spaces. We are currently working on generalizing the above theory to a category whose affine (local) models are closed semi-algebraic sets, but not all semi-algebraic morphisms are allowed. However, this category still generalizes both the Nash category and the algebraic category. If indeed such a theory can be established for this category, our hope (and guess) is that then proving partition of unity for non-affine varieties will be easier (as a consequence of the presence of "more" morphisms and "more" open subsets). Then the results for the non-affine algebraic case would just follow as a special case. A different direction of research we are pursuing is constructing a theory of Schwartz functions on arbitrary smooth manifolds, or more precisely defining the most general category whose local models are open subsets of $\mathbb{R}^n$, and on which Schwartz spaces can be defined consistently.

A The Whitney Extension Problem (Proofs of Two Key Lemmas)

The goal of this appendix is to prove Lemmas 3.16 and 3.17. All necessary preliminary results are also given. In this appendix we always consider the Euclidean topology on $\mathbb{R}^n$, unless otherwise stated.

A.1 Semi-analytic and Subanalytic Sets

One can define semi-analytic sets in $\mathbb{R}^n$ by (locally) using analytic functions; namely, $A \subset \mathbb{R}^n$ is semi-analytic if and only if for every point $p \in A$ there exist an open neighbourhood $p \in U \subset \mathbb{R}^n$ and finitely many analytic functions $f_{i,j}, g_{i,k}: U \to \mathbb{R}$, such that

$$A \cap U = \bigcup_{i=1}^{r} \left\{ x \in \mathbb{R}^n | \forall 1 \leq j \leq s_i, 1 \leq k \leq t_i : f_{i,j}(x) > 0, g_{i,k}(x) = 0 \right\}.$$

The images of semi-analytic sets under analytic maps (which we did not define) and even under standard projections, are not necessarily semi-analytic (see [L, Example at the end of section III]). This motivates the following definition (following [BM1, Definition 3.1]): $A \subset \mathbb{R}^n$ is *subanalytic* if and only if for every point $p \in \mathbb{R}^n$ there exists an open neighbourhood $p \in U \subset \mathbb{R}^n$ such that $A \cap U$ is a projection of a relatively compact (*i.e.,* bounded) semi-analytic set (and see equivalent definitions in [S, pp. 40 and 95]). A map $\nu: A \to B$ (where $A \subset \mathbb{R}^n, B \subset \mathbb{R}^m$ are subanalytic) is called *subanalytic* if its graph is a subanalytic set in $\mathbb{R}^{n+m}$.

Definition A.1 Let $X \subset \mathbb{R}^n$ be a subanalytic set, let $y \in X$ be some point, and let $m \in \mathbb{N}$. A function $f: X \to \mathbb{R}$ is *m-flat at y* if there exists $F \in C^m(\mathbb{R}^n)$ with $f = F|_X$, such that the Taylor polynomial of order m of F at y is identically zero. If f is m-flat at y for any $m \in \mathbb{N}$, we say that f is *flat at y* (*e.g.,* a function $f: \mathbb{R}^n \to \mathbb{R}$ is flat at some $y \in \mathbb{R}^n$ if the Taylor series of f at y is identically zero). If f is (m-)flat at y for any $y \in Z$ (where $Z \subset X$ is some subset), we say that f is *(m-)flat at Z*.

Defining m-flat functions (and all other notions of flatness) on (closed) subanalytic sets can be also done by 2 other approaches. The first is by defining the "Zariski m-paratangent bundle" and m-flat functions as restrictions of C^m functions on the ambient space that satisfy some natural condition expressed in terms of the Zariski m-paratangent bundle. The second is an intrinsic approach, using Glaeser extensions of real valued functions (see [BMP2, F]). [BMP2, Theorems 1.7, 1.8, and 1.9] imply the equivalence of all three approaches.

A.1.1 An Important Remark

As the Taylor polynomial is only dependent on the Euclidean local behaviour of functions, one can substitute $\mathbb{R}^n$ above by any Euclidean open subset of $\mathbb{R}^n$ containing X. This is done in A.2 and is used when it is more convenient.

A.1.2 Equivalence of Definitions 3.15 and A.1

Any algebraic set is also subanalytic, and so it seems we have two different definitions of flat functions at a point. Clearly if a function is flat at some point according to Definition 3.15 it is also flat according to Definition A.1. The other way is not trivial. By Definition A.1, f is flat at y means that for any $m \in \mathbb{N}$ there exists $F^m \in C^m(\mathbb{R}^n)$ such that $F^m|_X = f$ and the Taylor polynomial of order m of F^m at y is identically zero. This does not mean, a-priori, that there exists $F \in C^\infty(\mathbb{R}^n)$ such that $F|_X = f$ and the Taylor series of F at y is identically zero, *i.e.,* that f is flat at y according to Definition 3.15. According to [M], as X is algebraic it is formally semicoherent relative to the singleton $\{y\}$ (see [BM2, Definition 1.2] and discussion immediately after), which is equivalent, according to [BM2, Theorem 1.13], to the fact that f does extend to such an F. Later we will formulate it as $C^\infty(X;\{y\}) = C^{(\infty)}(X;\{y\})$. Thus, the two definitions of flat functions at a point in algebraic sets are equivalent.

A.2 Restrictions from Open Neighborhoods and Composite Functions

In this subsection we mainly follow the notation of [BMP1]. Let $X \subset \mathbb{R}^n$ be some subanalytic set, let $Z \subset X$ be a closed subset (in the induced topology from $\mathbb{R}^n$), and $k \in \mathbb{N}$. As said, flatness at a point is clearly a local property. Hence, it is natural to present the following spaces of functions:

$$C^k(X;Z) := \{f: X \to \mathbb{R} \mid \exists U \subset \mathbb{R}^n \text{ an open neighborhood of } X \text{ and } F \in C^k(U)$$
$$\text{such that } F|_X = f \text{ and } F \text{ is } k\text{-flat at } Z\}.$$

Similarly, for $k = \infty$, we have

$$C^\infty(X;Z) := \{f: X \to \mathbb{R} \mid \exists U \subset \mathbb{R}^n \text{ an open neighborhood of } X \text{ and } F \in C^\infty(U)$$

such that $F|_X = f$ and F is flat at $Z\}$.

Denote $\bigcap_{k\in\mathbb{N}} C^k(X;Z)$ by $C^{(\infty)}(X;Z)$. Clearly, $C^\infty(X;Z) \subset C^{(\infty)}(X;Z)$. In general $C^\infty(X;Z) \neq C^{(\infty)}(X;Z)$, remarkably, even when $Z = \varnothing$ (see, [P, Example 2]).

For any $k \in \mathbb{N} \cup \{\infty\}$, denote $C^k(X) := C^k(X;\varnothing)$; this coincides with the usual definition if X is smooth.

Let Ω be some open subset of $\mathbb{R}^m$. A continuous map $\varphi\colon \Omega \to \mathbb{R}^n$ is called semi-proper if for each compact subset K of $\mathbb{R}^n$ there exists a compact subset L of Ω such that $\varphi(L) = K \cap \varphi(\Omega)$. Let $\varphi\colon \Omega \to \mathbb{R}^n$ be a semi-proper real analytic map. In that case, $X := \varphi(\Omega)$ is a closed subanalytic subset of $\mathbb{R}^n$. For any $k \in \mathbb{N}$ define

$$\big(\varphi^* C^k(X)\big)^\wedge :=$$
$$\big\{ f \in C^k(\Omega) \bigm| \forall a \in X, \exists g \in C^k(X), f - \varphi^*(g) := f - g \circ \varphi \text{ is } k\text{-flat at } \varphi^{-1}(a) \big\},$$

and for $k = \infty$ define:

$$\big(\varphi^* C^\infty(X)\big)^\wedge :=$$
$$\big\{ f \in C^\infty(\Omega) \bigm| \forall a \in X, \exists g \in C^\infty(X), f - \varphi^*(g) := f - g \circ \varphi \text{ is flat at } \varphi^{-1}(a) \big\}.$$

Finally, for any closed subanalytic subset $Z \subset X$ and any $k \in \mathbb{N} \cup \{\infty\}$, define

$$\big(\varphi^* C^k(X;Z)\big)^\wedge := \big(\varphi^* C^k(X)\big)^\wedge \cap C^k\big(\Omega; \varphi^{-1}(Z)\big).$$

We use some properties of these spaces of functions, proved mainly in [BMP1]. Of special importance is the following theorem.

Theorem A.2 (Uniformization theorem [BM1, Theorem 5.1]) *Let M be a real analytic manifold, and let $X \subset M$ be a closed analytic subset. Then there is a real analytic manifold N and a proper real analytic mapping $\widetilde{\varphi}\colon N \to M$ such that $\widetilde{\varphi}(N) = X$.*

The following Lemma A.3 is stated in [BMP1]. For the reader's convenience we give here a detailed proof. We thank Prof. Pierre D. Milman for providing guidelines for this proof.

Lemma A.3 $\bigcap_{k\in\mathbb{N}} (\varphi^* C^k(X;Z))^\wedge = (\varphi^* C^\infty(X;Z))^\wedge$.

Proof The inclusion $\supset$ is clear. For the inclusion $\subset$, notice that

$$\bigcap_{k\in\mathbb{N}} C^k(\Omega; \varphi^{-1}(Z)) = C^\infty(\Omega; \varphi^{-1}(Z)),$$

as Ω is open (and in particular smooth). Thus, it is left to show that

$$\bigcap_{k\in\mathbb{N}} (\varphi^* C^k(X))^\wedge \subset (\varphi^* C^\infty(X))^\wedge.$$

Fix some $f \in \bigcap_{k\in\mathbb{N}} (\varphi^* C^k(X))^\wedge$, $x \in X$, and $\omega \in \varphi^{-1}(x)$. Define the set of k-jets at x whose pullbacks' k-jets at ω equal the k-jet of f, *i.e.,*

$$A^k := \{J \text{ is a } k\text{-jet at } x \mid \varphi^* J\text{'s } k\text{-jet at } \omega \text{ equals the } k\text{-jet of } f \text{ at } \omega\}.$$

By abuse of notation we considered jets as functions, *i.e.,* we chose a representative. This is independent of the choice made. Then the condition above can be reformulated as $\varphi^* J - f$ is k-flat at ω.

Then $A^k \neq \varnothing$, by the definition of f (*e.g.,* the k-jet of g corresponding to x in the definition of $(\varphi^* C^k(X))^\wedge$ is contained in A^k). For any $k \geq l \in \mathbb{N} \cup \{0\}$ define the projection of A^k to A^l by $A_l^k := pr_l(A^k) \subset A^l$ (*i.e.,* A_l^k is the the set of all l-jets that can be extended to k-jets in a way that is compatible with f). Also, define $A_l := \bigcap_{k \geq l} A_l^k \subset A^l$ (*i.e.,* A_l is the the set of all l-jets that can be extended to k-jets in a way that is compatible with f for any $k \geq l$). Assume the following hold for any $l \in \mathbb{N}$:

(a) $A_l \neq \varnothing$.

(b) $pr_l \colon A_{l+1} \to A_l$ is onto, *i.e.,* any l-jet in A_l can be extended to an $(l+1)$-jet in A_{l+1}.

In that case using the principle of dependent choices (see Section A.2.1) there exists a series of jets $\{J_l\}_{l \in \mathbb{N}}$ such that $J_l \in A_l$ and for any $l \in \mathbb{N}$, $pr_l(J_{l+1}) = J_l$. This series can be thought of as a formal power series on $\mathbb{R}^n$ (which we denote by G), where J_l is the truncated power series up to order l. By Borel's Theroem (see Section A.4) there exists a function $\widetilde{G} \in C^\infty(\mathbb{R}^n)$ such that $T_x^\infty \widetilde{G} = G$, where $T_x^\infty \widetilde{G}$ means the Taylor series of $\widetilde{G}$ at x. By construction we observe that $T_\omega^\infty(\varphi^* \widetilde{G}) = T_\omega^\infty f$, and so conclude that $f \in (\varphi^* C^\infty(X))^\wedge$ (as $\widetilde{G}|_X \in C^\infty(X)$ satisfies the desired condition).

Thus, it is left to prove that the assumptions (a) and (b) hold.

In order to see that $A_l \neq \varnothing$, observe that for any $k \geq l$, A_l^k is a finite dimensional affine space (it is finite dimensional as it lies in the finite dimensional space A^l). Moreover, these affine spaces form a decreasing sequence $A^l = A_l^l \supset A_l^{l+1} \supset A_l^{l+2} \supset \cdots$. As $A^k \neq \varnothing$ for any $k \geq l$, also $A_l^k \neq \varnothing$. The dimensions of these affine spaces form a decreasing sequence of non-negative integers, and so stabilizes. Thus, we conclude that $\bigcap_{k \geq l} A_l^k (= A_l) = A_l^{k_l} \neq \varnothing$ for some $k_l \in \mathbb{N}$, and so assumption (a) holds.

In order to see that $\mathrm{pr}_l \colon A_{l+1} \to A_l$ is onto, we take some $J \in A_l$, and define for any $k \geq l+1$: $B_{l+1}^k := \{H \in A_{l+1}^k \mid \mathrm{pr}_l(H) = J\}$. By the definition of A_l there exists a k-jet $J_k \in A^k$ such that $\mathrm{pr}_l(J_k) = J$, and so $\mathrm{pr}_{l+1}(J_k) \in A_{l+1}^k$ satisfies $\mathrm{pr}_l(\mathrm{pr}_{l+1}(J_k)) = J$, and so $B_{l+1}^k \neq \varnothing$. As $B_{l+1}^{l+1} \supset B_{l+1}^{l+2} \supset B_{l+1}^{l+3} \supset \cdots$ is a decreasing sequence of finite dimensional affine spaces it stabilizes, and so $\bigcap_{k \geq l+1} B_{l+1}^k \neq \varnothing$. Choose some $\mathcal{J} \in \bigcap_{k \geq l+1} B_{l+1}^k$. By definition, $\mathcal{J} \in A_{l+1}$, and by construction, $pr_l(\mathcal{J}) = J$. Thus, assumption (b) holds. ∎

A.2.1 The Principle of Dependent Choices (DC)

We assume a weak version of the axiom of choice, the principle of dependent choices: if E is a binary relation on a nonempty set A and for any $a \in A$ there exists an element $b \in A$ such that bEa, then there is a sequence $a_0, a_1, \ldots, a_n, \ldots$ in A such that $a_{n+1} E a_n$ for any $n \in \mathbb{N}$. The interested reader is referred to [J, p. 50].

Theorem A.4 (E. Borel [T, Theorem 38.1]) *Let Φ be an arbitrary formal power series in n indeterminates, with complex coefficients. Then there exists a function in $C^\infty(\mathbb{R}^n)$ whose Taylor expansion at the origin is identical to Φ.*

A.3 Proof of Lemma 3.16

Recall we want to prove the following. Let X be a compact (in the Euclidean topology) algebraic set in $\mathbb{R}^n$, and let $Z \subset X$ be some (Zariski) closed subset. Define

$$W_Z := \{\phi: X \to \mathbb{R} | \exists \widetilde{\phi} \in C^\infty(\mathbb{R}^n) \text{ such that } \widetilde{\phi}|_X = \phi \text{ and } \phi \text{ is flat at } Z\},$$
$$(W_Z^{\mathbb{R}^n})^{\text{comp}} := \{\phi \in C^\infty(\mathbb{R}^n) | \phi \text{ is compactly supported and is flat at } Z\},$$

and $U := X \smallsetminus Z$. Then for any $f \in W_Z$, there exists $\widetilde{f} \in (W_Z^{\mathbb{R}^n})^{\text{comp}}$ such that $\widetilde{f}|_X = f$.

We start the proof with a preliminary lemma.

Lemma A.5 *There exists a pair (Ω, φ) such that $\Omega \subset \mathbb{R}^m$ is an open subset of $\mathbb{R}^m$ (in the Euclidean topology) and $\varphi: \Omega \to \mathbb{R}^n$ is a semiproper real analytic function such that $\varphi(\Omega) = X$. Moreover, for any $f \in C^\infty(X; \varnothing)$, $\varphi^* f := f \circ \varphi \in C^\infty(\Omega)$.*

Proof By Theorem A.2, there is a real analytic manifold N and a proper real analytic mapping $\widetilde{\varphi}: N \to \mathbb{R}^n$ such that $\widetilde{\varphi}(N) = X$. As N is a real analytic manifold, it has an open cover $\{N_i\}_{i \in I}$ such that for any $i \in I$, N_i is analytically diffeomorphic to an open subset of $\mathbb{R}^{d_i}$; *i.e.,* there exist analytical diffeomorphisms $\nu_i: N_i \to \mathbb{R}^{d_i}$ such that $\nu_i(N_i)$ is open in $\mathbb{R}^{d_i}$. As $\widetilde{\varphi}$ is proper and $X \subset \mathbb{R}^n$ is compact, $N = \widetilde{\varphi}^{-1}(X)$ is compact, and thus there exists a finite subcover $\{N_i\}_{i=1}^k$. We let $m := \max_{i=1}^k \{d_i\} + 1$. For any $1 \le i \le k$, define $\Omega_i = \nu_i(N_i) \times (i - \frac{1}{4}, i + \frac{1}{4})^{m-d_i} \subset \mathbb{R}^m$, and define $\psi_i: \Omega_i \to \Omega_i$ by $\psi_i(n_i, \alpha_1, \dots, \alpha_{m-d_i}) := (n_i, i, \dots, i)$, where $n_i \in \nu_i(N_i)$ and $\alpha_j \in (i - \frac{1}{4}, i + \frac{1}{4})$. Note that ψ_i is semiproper real analytic. Define $\Omega := \bigcup_{i=1}^k \Omega_i$. As $\Omega_1, \dots, \Omega_k$ are disjoint sets in $\mathbb{R}^m$, we can naturally define a semiproper function $\psi: \Omega \to \Omega$ by $\psi|_{\Omega_i} := \psi_i$. Clearly, $\Omega \subset \mathbb{R}^m$ is open. Now define a function $\nu^{-1}: \psi(\Omega) \to N$ by $\nu^{-1}(n_i, i, \dots, i) = \nu_i^{-1}(n_i)$, where $n_i \in \nu_i(N_i)$. Note that ν^{-1} is a proper map. Finally, defining $\varphi := \widetilde{\varphi} \circ \nu^{-1} \circ \psi$, we get that φ is a semiproper real analytic function satisfying $\varphi(\Omega) = X$. The "Moreover" part of Lemma A.5 is obvious. ∎

Proof of Lemma 3.16 Fix $f \in W_Z$. In particular, $f \in C^\infty(X; \varnothing)$. Then, by Lemma A.5, $\varphi^* f := f \circ \varphi \in C^\infty(\Omega)$. Denote $\widetilde{f} := \varphi^* f$. By definition, $\widetilde{f} \in (\varphi^*(C^\infty(X)))^\wedge$.

We now prove that $\widetilde{f} \in C^\infty(\Omega; \varphi^{-1}(Z))$. For any $z \in Z$ and any $k \in \mathbb{N}$, there exists $\bar{f}_z^k \in C^k(\mathbb{R}^n)$ such that $f = \bar{f}_z^k|_X$ and $\bar{f}_z^k$ is k-flat at z. Note that for any $z \in Z$ and any $k \in \mathbb{N}$, we have $\widetilde{f}(= f \circ \varphi) = \bar{f}_z^k \circ \varphi$, and in particular for any $z' \in Z$ and any $k' \in \mathbb{N}$ we have $\bar{f}_{z'}^{k'} \circ \varphi = \bar{f}_z^k \circ \varphi$. First, we prove that for any $k \in \mathbb{N}$, $\widetilde{f} \in C^k(\Omega; \varphi^{-1}(Z))$. Fix some $z \in Z$ and some $\widetilde{z} \in \varphi^{-1}(z)$. As we can write $\widetilde{f} = \bar{f}_z^k \circ \varphi$, the fact that $\widetilde{f}$ is k-flat at $\widetilde{z}$ follows immediately from Lemma A.6.

We have thus shown that for any $k \in \mathbb{N}$, $\widetilde{f} \in C^k(\Omega; \varphi^{-1}(Z))$. As we also had $\widetilde{f} \in (\varphi^*(C^k(X)))^\wedge$, we get that $\widetilde{f} \in (\varphi^* C^k(X; Z))^\wedge := (\varphi^*(C^k(X)))^\wedge \cap C^k(\Omega; \varphi^{-1}(Z))$. As this holds for any $k \in \mathbb{N}$ we get that $\widetilde{f} \in \bigcap_{k \in \mathbb{N}} (\varphi^* C^k(X; Z))^\wedge$. By Lemma A.3, $\bigcap_{k \in \mathbb{N}} (\varphi^* C^k(X; Z))^\wedge = (\varphi^* C^\infty(X; Z))^\wedge$, and by [BMP1, Theorem 1.3],

$$\left(\varphi^* C^\infty(X; Z)\right)^\wedge = \varphi^* C^{(\infty)}(X; Z),$$

so $\widetilde{f} \in \varphi^* C^{(\infty)}(X; Z)$.

As $\widetilde{f} \in \varphi^* C^{(\infty)}(X;Z)$, there exists $h \in C^{(\infty)}(X;Z)$ such that $\widetilde{f} = \varphi^* h = h \circ \varphi$. Since $\widetilde{f} = f \circ \varphi$ and φ is onto X, it follows that $h = f$, *i.e.*, $f \in C^{(\infty)}(X;Z)$. According to [M], as X is algebraic, it is formally semicoherent rel. Z (see [BM2, Definition 1.2] and discussion immediately after), which is equivalent, according to [BM2, Theorem 1.13], to the fact that $C^{\infty}(X;Z) = C^{(\infty)}(X;Z)$. We conclude that $f \in C^{\infty}(X;Z)$.

Recall that we started with some $f \in W_Z$, we showed that $f \in C^{\infty}(X;Z)$, and our goal is to show that f is the restriction of some function in $(W_Z^{\mathbb{R}^n})^{\mathrm{comp}}$. As $f \in C^{\infty}(X;Z)$ there exists an open $V \subset \mathbb{R}^n$ such that $X \subset V$, and $F\colon V \to \mathbb{R}$ such that $F \in C^{\infty}(V)$, F is flat at Z and $F|_X = f$. Without loss of generality, as X is compact in the Euclidean topology on $\mathbb{R}^n$, we can assume the set V is bounded in the Euclidean norm on $\mathbb{R}^n$.

Take some open $V' \subsetneq V$ containing X. Let $\rho \in C^{\infty}(\mathbb{R}^n)$ be a function supported in V' such that $\rho|_{V''} = 1$, where $V'' \subsetneq V'$ is some open subset containing X (it is standard to show such ρ exists by convolving the characteristic function of some open subset containing V'' and strictly contained in V', with some appropriate approximation of unity). Now define $\widetilde{F}\colon \mathbb{R}^n \to \mathbb{R}$ by $\widetilde{F}|_V := \rho \cdot F$ and $\widetilde{F}|_{\mathbb{R}^n \smallsetminus V} := 0$. Clearly $\widetilde{F}|_X = F|_X = f$, $\widetilde{F} \in C^{\infty}(\mathbb{R}^n)$, $\widetilde{F}$ is flat at Z (as $\widetilde{F}|_{V''} = F|_{V''}$), and $\widetilde{F}$ is compactly supported (in the Euclidean topology) in $\mathbb{R}^n$; *i.e.*, $\widetilde{F} \in (W_Z^{\mathbb{R}^n})^{\mathrm{comp}}$ and $\widetilde{F}|_X = f$. ■

A.4 Multivariate Faá Di Bruno Formula

The famous chain rule for deriving real valued functions from the real line states that $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$. This can be generalized to higher derivatives and higher dimensions, *i.e.*, partial derivatives of arbitrary order of composite multivariate functions. We are interested only in the following result.

Lemma A.6 (*cf.* [CS, Theorem 2.1]) *Let $x_0 \in \mathbb{R}^d$, $V \subset \mathbb{R}^d$ be some open neighborhood of x_0 and $g\colon V \to \mathbb{R}^m$, $g \in C^{\infty}(V, \mathbb{R}^m)$. Let $U \subset \mathbb{R}^n$ be some open neighborhood of $g(x_0)$ and $f\colon U \to \mathbb{R}$, $f \in C^{\infty}(U)$. Assume f is flat at $g(x_0)$,* i.e., *its Taylor series at $g(x_0)$ is identically zero. Then $f \circ g\colon g^{-1}(U) \to \mathbb{R}$ is flat at x_0.*

Proof of Lemma 3.17 By definition, for any $x \in X_1$, we have

$$\varphi(x) = \Big(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_{n_2}(x)}{g_{n_2}(x)} \Big),$$

where $f_1, \dots, f_{n_2}, g_1, \dots, g_{n_2} \in \mathbb{R}[X_1]$ and $g_i^{-1}(0) \cap X_1 = \varnothing$ for any $1 \le i \le n_2$. By abuse of notation we choose some representatives in $\mathbb{R}[x_1, \dots, x_{n_1}]$ and consider $f_1, \dots, f_{n_2}, g_1, \dots, g_{n_2}$ as functions in $\mathbb{R}[x_1, \dots, x_{n_1}]$. Define

$$U := \Big\{ x \in \mathbb{R}^{n_1} \mid \prod_{i=1}^{n_2} g_i(x) \neq 0 \Big\}.$$

Then U is open in $\mathbb{R}^{n_1}$, X_1 is a closed subset of U, and φ can be naturally extended to a regular map $\widetilde{\varphi}\colon U \to \mathbb{R}^{n_2}$ (by the same formula as φ). Note that U is an affine algebraic manifold.

Let $f\colon X_2 \to \mathbb{R}$ be some function that is flat at some $p \in X_2$. In particular, $f \in C^{\infty}(X_2; \{p\})$; *i.e.*, there exist an open subset $V \supset X_2$ and a function $F \in C^{\infty}(V)$ such that F is flat at p and $F|_{X_2} = f$. As $\widetilde{\varphi}: U \to \mathbb{R}^{n_2}$ is continuous, then $U' := \widetilde{\varphi}^{-1}(V)$ is an open subset of U, and so an open subset of $\mathbb{R}^{n_1}$. Clearly U' contains X_1. Denote by G the pullback of F to U' via $\widetilde{\varphi}$; *i.e.*, $G: U' \to \mathbb{R}$ is defined by $G := F \circ (\widetilde{\varphi}|_{U'})$. By Lemma A.6, as F is flat at p, G is flat at $\widetilde{\varphi}^{-1}(p)$, and in particular, $f \circ \varphi = G|_{X_1}$ is flat at $\varphi^{-1}(p)$. ∎

B Noetherianity of the Zariski Topology

First, as by definition any algebraic variety has a finite cover by affine varieties, it is enough to prove Proposition 2.4 for affine varieties. Second, it is enough to prove Proposition 2.4 for algebraic subsets of $\mathbb{R}^n$. Let $X \subset \mathbb{R}^n$ be an algebraic set, $U \subset X$ a Zariski open subset, and $\{U_\alpha\}_{\alpha \in I}$ an open cover of U. We prove Proposition 2.4 in 3 steps.

Proof

Step 1 Assume $X = U = \mathbb{R}^n$. By Proposition 2.2 for any $\alpha \in I$ there exists $f_\alpha \in \mathbb{R}[x_1, \dots, x_n]$ such that $U_\alpha = \{x \in \mathbb{R}^n | f_\alpha(x) \neq 0\}$. As $\mathbb{R}^n \smallsetminus \bigcup_{\alpha \in I} U_\alpha = \varnothing$, the zero locus of $\langle f_\alpha \rangle_{\alpha \in I}$ (the set of all points $x \in \mathbb{R}^n$ satisfying $f(x) = 0$ for any f in the ideal generated by all of the polynomials $\{f_\alpha\}_{\alpha \in I}$) is empty. By Hilbert's Basis Theorem, $\mathbb{R}[x_1, \dots, x_n]$ is Noetherian, and so there exist $g_1, \dots, g_m \in \mathbb{R}[x_1, \dots, x_n]$ such that $\langle f_\alpha \rangle_{\alpha \in I} = \langle g_1, \dots, g_m \rangle$. As the zero locus of this ideal is empty, $g := \sum_{l=1}^m g_l^2$ satisfies $g^{-1}(0) = \varnothing$. As $g \in \langle f_\alpha \rangle_{\alpha \in I}$ there exist $a_1, \dots, a_k \in \mathbb{R}[x_1, \dots, x_n]$ and $\alpha_1, \dots, \alpha_k \in I$ such that $g = \sum_{i=1}^k a_i \cdot f_{\alpha_i}$. This implies that $f_{\alpha_1}, \dots, f_{\alpha_k}$ have no common zeroes, and so $\bigcup_{i=1}^k U_{\alpha_i} = \mathbb{R}^n$.

Step 2 Assume $X = U$. There exist $\{V_\alpha\}_{\alpha \in I}$ open in $\mathbb{R}^n$ such that $U_\alpha = V_\alpha \cap X$. Then $\{V_\alpha\}_{\alpha \in I} \cup (\mathbb{R}^n \smallsetminus X)$ is an open cover of $\mathbb{R}^n$. By Step (1) it has a finite subcover, and so intersecting this subcover with X we get a finite subcover of U.

Step 3 The general case. By Proposition 2.3, U is itself an affine algebraic variety. Moreover, $\{U_\alpha \cap U\}_{\alpha \in I}$ is an open cover of U. By considering some closed embedding of U in some $\mathbb{R}^m$, we are reduced to Step (2). ∎

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