Canad. Math. Bull. Vol. **52** (4), 2009 pp. 598–612

Numerical Semigroups That Are Not Intersections of *d*-Squashed Semigroups

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Abstract. We say that a numerical semigroup is *d-squashed* if it can be written in the form

$$S = \frac{1}{N}\langle a_1, \dots, a_d\rangle \cap \mathbb{Z}$$

for $N, a_1, \dots, a_d$ positive integers with $\gcd(a_1, \dots, a_d) = 1$. Rosales and Urbano have shown that a numerical semigroup is 2-squashed if and only if it is proportionally modular.

Recent works by Rosales *et al.* give a concrete example of a numerical semigroup that cannot be written as an intersection of 2-squashed semigroups. We will show the existence of infinitely many numerical semigroups that cannot be written as an intersection of 2-squashed semigroups. We also will prove the same result for 3-squashed semigroups. We conjecture that there are numerical semigroups that cannot be written as the intersection of *d*-squashed semigroups for any fixed *d*, and we prove some partial results towards this conjecture.

1 Introduction

Numerical semigroups (*i.e.,* subsemigroups of the positive integers whose complement in $\mathbb{Z}^+$ contains only finitely many elements) have been the object of intensive study in the last decade, not only in relation to semigroup rings which are Krull rings, but also in connection to solutions of diophantine equations.

In this field, an especially interesting question is whether it is possible to write numerical semigroups as an intersection of some kind of "elementary" building blocks. The best-known choices for building blocks are the irreducible numerical semigroups. Recall that a numerical semigroup is *irreducible* if it cannot be expressed as an intersection of two numerical semigroups properly containing it. Every numerical semigroup S admits a decomposition $S = S_1 \cap S_2 \cap \cdots \cap S_n$ with S_i irreducible for all i (see [4]). Examples of irreducible numerical semigroups are those generated by pairs of coprime natural numbers, but it is known (see [4, 5]) that there exist irreducible numerical semigroups whose minimal sets of generators have arbitrary cardinality.

We can then consider a generalization of these 2-generated numerical semigroups as basic building blocks, as follows (though these semigroups need not be irreducible).

Received by the editors August 29, 2006; revised March 15, 2009.

The first and third authors were partially supported by the DGI and European Regional Development Fund, jointly, through Project MTM2004-00149, by PAI III grant FQM-298 of the Junta de Andalucía. The third author was also partially supported by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya. The second author was partially supported by an Erasmus-Socrates grant of the European Community. The fourth author was partially supported by an NSERC Discovery Grant.

AMS subject classification: Primary: 20M14; secondary: 06F05, 46L80.

Keywords: numerical semigroup, squashed semigroup, proportionally modular semigroup.

Definition 1.1 We say that a numerical semigroup is *d-squashed* if it can be written in the form

$$S = \frac{1}{N}\langle a_1, \dots, a_d\rangle \cap \mathbb{Z}$$

for $N, a_1, \dots, a_d$ positive integers with $\gcd(a_1, \dots, a_d) = 1$.

In [8] it was shown that the class of 2-squashed semigroups is exactly the class of semigroups defined in [7], called *proportionally modular* semigroups. Proportionally modular semigroups arise as the solution to certain diophantine inequalities.

Further, 2-squashed semigroups are related to some questions about K-theory of C^*-algebras which we shall discuss below. For arbitrary $d \geq 2$, these semigroups are related to combinatorial questions of algebraic geometry.

From now on, we write $\mathbb{N}$ for $\{1, 2, \dots\}$, and $\mathbb{Z}^+$ for $\{0\} \cup \mathbb{N}$. In Section 2, we show the existence of infinitely many semigroups that cannot be written as the intersection of 2-squashed semigroups. At the end of this section, we outline the connection with K-theory of C^*-algebras, providing a negative answer to a question of Toms [10]. Finally, in Section 3, we consider the question of whether there exist semigroups that cannot be written as an intersection of d-squashed semigroups, for fixed $d > 2$. We prove some preliminary results for general d, and then show that there exist semigroups that cannot be written as an intersection of 3-squashed semigroups. We offer the following conjecture.

Conjecture 1.2 *For any $d \geq 2$, there exist numerical semigroups $S \subset \mathbb{Z}^+$ that cannot be written as an intersection of d-squashed semigroups.*

2 Existence of Examples for $d = 2$

We will use an argument of geometric inspiration to prove that there exist infinitely many numerical semigroups that cannot be written as the intersection of 2-squashed semigroups. The basic fact we need (which first occurred to us in a geometric context) can be obtained as a direct consequence of (arithmetic) results of Rosales *et al.* [7, 8]. Recall that a numerical semigroup S is called *proportionally modular* if $S = T \cap \mathbb{Z}^+$, where T is a submonoid of $\mathbb{R}^+$ generated by a closed interval [7, p. 285].

Proposition 2.1 *A numerical semigroup S is a 2-squashed semigroup if and only if there exist rational numbers $0 < \alpha < \beta < \infty$ such that $S = \bigcup_{i=0}^{\infty} [i\alpha, i\beta] \cap \mathbb{Z}$.*

Proof By [8, Theorem 5], a numerical semigroup S is a 2-squashed semigroup if and only if it is proportionally modular. Then the result holds by [7, Lemma 12, Theorem 13, Remark 2]. ∎

Remark 2.2 The underlying geometric idea (which we will exploit subsequently) is that the elements of a 2-squashed semigroup S can be seen as the y-coordinates of lattice points in a cone of $\mathbb{R}^2$.

For K a set of integers, we write $\min(K)$ and $\max(K)$ for the minimum and maximum values in K. We have the following theorem.

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Theorem 2.3 *Let K be a finite set of non-negative integers which is not the set of y-values of the lattice points inside any compact, convex body in $\mathbb{R}^2$, and suppose further that* $\max(K) < 2\min(K)$*. Then the semigroup generated by K is not 2-squashed.*

Proof Let S be the semigroup generated by K. Note that, since $\max K < 2\min K$, we have that $S \cap [\min K, \max K] = K$.

Suppose S is 2-squashed. By the previous remark, there is a cone C in $\mathbb{R}^2$ of the form $C = \{(x, y) \mid \alpha x \leq y \leq \beta x\}$ with $\alpha > 0$, such that the set of y-values of lattice points in C is S. Now consider the part of C whose y-values lie between $\min K$ and $\max K$. This is a compact, convex set in $\mathbb{R}^2$, and the y-values of lattice points in the set must be $S \cap [\min K, \max K] = K$, which is a contradiction. ∎

Thus, to find a non 2-squashed semigroup, it suffices to find a set of non-negative integers K with $\max K < 2\min K$ and with the property that K cannot be the set of lattice points in a compact, convex body in $\mathbb{R}^2$. The next lemma, which we shall need again when we consider the 3-squashed case, accomplishes this.

Given a finite set of integers K, let $K^{(1)}$ denote the maximal consecutive sequence of integers in K beginning with the minimal element of K, let $K^{(2)}$ be the next consecutive sequence of integers in K, etc. Let $r(K)$ denote the number of these subsets which appear (so $K^{(r(K))}$ is the last one). For example, if $K = \{0, 1, 4, 7, 48, 49\}$, $K^{(1)} = \{0, 1\}$, $K^{(2)} = \{4\}$, $K^{(3)} = \{7\}$, and $K^{(4)} = \{48, 49\}$, so $r(K) = 4$.

Lemma 2.4 *A finite set of integers K is not the set of y-values of the lattice points in a compact, convex body in $\mathbb{R}^2$ if it has the following properties:*

(i) $|K^{(1)}| \geq 2$.
(ii) $r(K) \geq 4$.
(iii) $|K^{(r(K))}| = 2$.
(iv) *The only elements of K larger than* $(\min(K) + \max(K))/2$ *are those in $K^{(r(K))}$.*
(v) $|K|/(\max(K) - \min(K)) < \frac{1}{8}$.

Further, a set K satisfying these properties cannot be the set of y_3-values of the lattice points in any compact convex body contained in a plane in $\mathbb{R}^3$.

It will be clear from the proof that these conditions are not by any means the only possible set of conditions for which one could prove such a lemma, but they are sufficient for our purposes.

To give a concrete example: the lemma asserts that the set $K = \{0, 1, 4, 7, 48, 49\}$ cannot be the set of y-values of lattice points in a compact convex body in $\mathbb{R}^2$. Furthermore, it cannot be the set of y_3-values of the lattice points in any compact convex body contained in a plane in $\mathbb{R}^3$.

Proof Suppose that R is a compact, convex body in $\mathbb{R}^2$. Let K be the set of y-values of the lattice points in R. Assume further that K satisfies conditions (i)–(v). We will now find a contradiction.

To begin with, note that K contains at least six integers, by conditions (i), (ii), and (iii), and that therefore by (v) $\max(K) - \min(K) > 48$.

Write b_0 for $\min(K)$ and b_3 for $\max(K)$, and let $P_0 = (a_0, b_0)$ be a lattice point in R guaranteed by the definition of K. Similarly, let $P_1 = (a_1, b_0 + 1)$, $P_2 = (a_2, b_3 - 1)$,

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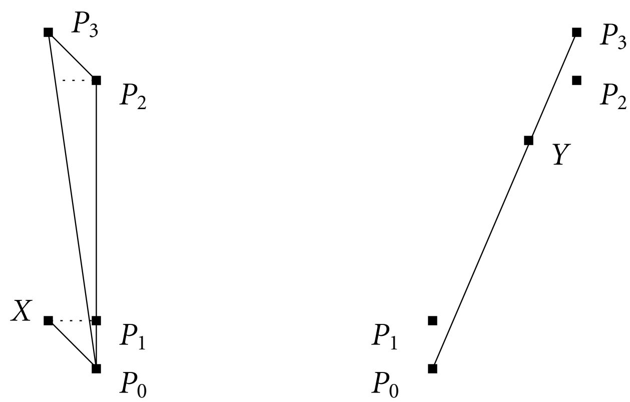

Figure 1: Examples of the non-parallel and parallel cases.

$P_3 = (a_3, b_3)$ be lattice points of R whose existence is guaranteed by the definition of K together with (i) and (iii). We now consider two cases, depending on whether or not P_0P_1 and P_2P_3 are parallel.

Suppose first that they are not parallel. Let Q be the convex hull of the P_i. Then Q is a lattice polygon, so the number of lattice points in Q is given by Pick's theorem:

$$|Q \cap \mathbb{Z}^2| = \text{area}(Q) + \frac{|\partial Q|}{2} + 1.$$

Here ∂Q denotes the lattice points on the boundary of Q. From Pick's theorem we can derive the inequality $|Q \cap \mathbb{Z}^2| > \text{area}(Q)$.

Consider the area of the triangle $P_0P_1P_3$. Its vertical length is $b_3 - b_0$; its area will be half that length times the horizontal distance from P_1 to P_0P_3. Similarly, the area of $P_0P_2P_3$ is $\frac{1}{2}(b_3 - b_0)$ times the horizontal distance from P_2 to P_0P_3. Let $X = (a_0 + a_3 - a_2, b_0 + 1)$ be the lattice point such that P_0X is parallel to and has the same length as P_2P_3. See Figure 1. The horizontal distance from P_2 to P_0P_3 is, by symmetry, equal to that from X to P_0P_3. X and P_1 are lattice points on the same horizontal line and by our assumption that P_0P_1 and P_2P_3 are not parallel, X and P_1 are not the same point. Thus, if they lie on the same side of P_0P_3, one of them must be horizontal distance at least 1 away from it, while if they lie on opposite sides of P_0P_3, the sum of their horizontal distances is at least 1, and thus at least one of these horizontal distances is at least $\frac{1}{2}$. It follows that the area of Q is at least $\frac{1}{4}(b_3 - b_0)$, and thus that Q contains at least this many lattice points.

However, we know that $|K|$ is less than half that amount. Thus, there must be some $k \in K$ which is the y-value of at least three lattice points of Q. Thus it follows that the line $y = k$ intersects Q (and hence R) in a line of width at least 2. By convexity, it follows that the line $y = j$ intersects R in a line of width at least 1 whenever $(\min(K) + k)/2 \le j \le (\max(K) + k)/2$. Thus, there is an interval of half the length of K over which the width of R is at least 1. Therefore, K contains all the lattice points

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in this interval. But since the length of this interval is $(\max(K) - \min(K))/2$, even allowing for round-off error, condition (v) must be violated.

Now consider the case where P_0P_1 and P_2P_3 are parallel. After a horizontally shearing lattice homomorphism, we may assume that $a_1 = a_0$, and therefore that also $a_2 = a_3$. Without loss of generality, assume $a_3 > a_0$. The line $x = a_0$ intersects R in a segment of length at least 1, and the line $x = a_3$ also intersects R in a segment of length at least 1. By convexity, the same is true for any line $x = k$ with $a_0 \leq k \leq a_3$. It follows that there is at least one lattice point in R with each x-value between a_0 and a_3. Note that if we fix an integer a such that $a_0 \leq a \leq a_3$, then the set $\{y \mid (a, y) \in R\}$ is a consecutive set of lattice points in K. Since $r(K) \geq 4$, there must be at least four different integers in $[a_0, a_3]$, so $a_3 - a_0 \geq 3$.

Consider the line segment ℓ from (a_0, b_0) to (a_3, b_3). Let $m = (b_3 - b_0)/(a_3 - a_0)$ be the slope of ℓ. By convexity, all the points of this line are in R.

Let $c = a_0 + \frac{3}{4}(a_3 - a_0)$, and $d = b_0 + \frac{3}{4}(b_3 - b_0)$, so (c, d) lies on ℓ. Let c' be the integer closest to c. Define d' by requiring that (c', d') lie on ℓ.

Since $|c - c'| \leq \frac{1}{2} \leq \frac{1}{6}(a_3 - a_0)$, we know that $|d - d'| \leq \frac{1}{6}(b_3 - b_0)$. Thus

$$b_0 + \left(\frac{3}{4} - \frac{1}{6}\right)(b_3 - b_0) \leq d' \leq b_0 + \left(\frac{3}{4} + \frac{1}{6}\right)(b_3 - b_0).$$

Since $b_3 - b_0 > 48$, this implies that

$$\frac{b_0 + b_3}{2} + 4 < d' < b_3 - 4.$$

Since the vertical line defined by $x = c'$ intersects R in a segment of length at least 1, there is a lattice point in R whose y-value is within 1 of d'. This contradicts (iv), completing the proof of the main claim.

Now suppose that R is a compact, convex body contained in a plane in $\mathbb{R}^3$, and suppose again, for the sake of contradiction, that K consists of the y_3-values of the lattice points in R. We focus our attention on the plane in which R lies, and apply the first claim. The crucial point we need is that y_3 is a lattice coordinate on this plane (that is, y_3 restricts to a surjective map from the lattice points in this plane to $\mathbb{Z}$), which must be true given our assumptions since K contains two consecutive integers. ■

Corollary 2.5 *There are infinitely many subsemigroups of* $\mathbb{N}$ *that are not* 2*-squashed.*

Proof Let $K = \{0, 1, 4, 7, 48, 49\}$. As already remarked, K satisfies the hypotheses of Lemma 2.4, so it cannot be the set of y-values of lattice points in a compact, convex body in $\mathbb{R}^2$. We cannot directly use K in Theorem 2.3, since it does not satisfy the additional hypothesis that $\max K < 2 \min K$. However, any $K_n = K + n$ for $n \geq 50$, will satisfy the hypotheses of the theorem. Since the subgroups generated by K_n and $K_{n'}$ are different if n and n' are distinct positive numbers, we have generated infinitely many non 2-squashed semigroups. ■

We should remark that these examples are not the first known examples of semigroups that cannot be written as an intersection of 2-squashed semigroups. In [7, Example 28] it was shown, by using [7, Algorithm 24, Algorithm 27], that the numerical semigroup $S = \langle 4, 6, 7 \rangle$ cannot be written as intersection of 2-squashed semigroups.

2.1 Applications to K-Theory

Since the early seventies, K-theory has been successfully used as a tool for classifying C^*-algebras. During this time, interest has been focused not only on the search for invariants in order to classify certain classes of C^*-algebras (AF C^*-algebras and irrational rotation algebras, among others), but also on self-contained considerations about determining the range of these invariants. This interest is closely related to the search for pathological behaviors in the structure of these algebras and feeds into the parallel development of a rich theory of ordered groups.

Among numerous questions related to the structural regularity of simple C^*-algebras is the following one: is it possible to construct a simple C^*-algebra A whose ordered K_0-group is not endowed with the unperforation property (a sort of torsion-freeness for ordered groups)?

This question was answered in the affirmative by J. Villadsen [11]. Subsequent refinements, due to Rørdam and Villadsen [3], Elliott and Villadsen [1], and Toms [10], allow one to restrict the K-theoretical scope of these examples by constructing a simple C^*-algebra A such that $(K_0(A), K_0(A)^+) \cong (\mathbb{Z}, S)$, where $S \subseteq \mathbb{Z}^+$ is a submonoid such that $\mathbb{Z}^+ \setminus S$ is a finite set.

In this context, the following question arises naturally: is it possible to find such an algebra for any such monoid S? While the first approaches were not promising, Toms, for any prime numbers $q_1, \dots, q_n$, integers $m_1, \dots, m_n \in \mathbb{N}$ with $\gcd(q_i, m_i) = 1$, and $N \in \mathbb{N}$ with $\gcd(q_i, N) = \gcd(m_i, N) = 1$, provided a simple C^*-algebra A with $(K_0(A), K_0(A)^+) \cong (\mathbb{Z}, S)$, where

$$S = \frac{1}{N}\Big(\bigcap_{i=1}^{n} \langle q_i, m_i \rangle\Big) \cap \mathbb{Z}.$$

The obvious question is

Toms: Is any submonoid $S \subset \mathbb{Z}^+$ with $\mathbb{Z}^+ \setminus S$ finite of this particular form?

Toms showed that the result holds for 2-generated numerical semigroups [9, 10]. However, it is clear that any semigroup of Toms' form can be written as an intersection of 2-squashed semigroups. Thus, by the results we have just discussed, $\langle 4, 6, 7 \rangle$ provides a concrete counterexample to Toms' question, while Corollary 2.5 guarantees that there exist infinitely many numerical semigroups which are counterexamples to Toms' question.

3 A Generalized Conjecture

In this section we consider the question of whether there exist numerical semigroups which cannot be written as an intersection of d-squashed semigroups for fixed $d > 2$. The first part of the argument applies for all d, but to finish the argument we must specialize to the case $d = 3$.

For now, we will be operating inside $\mathbb{Z}^d$. We will sometimes think of $\mathbb{Z}^d$ sitting as lattice points inside $\mathbb{R}^d$. Let $y_1, \dots, y_d$ be the coordinate functions on $\mathbb{Z}^d$ (or $\mathbb{R}^d$). For greater clarity, though, when we discuss $\mathbb{R}^2$, we will consider its coordinate functions to be x and y.

We begin by proving a d-dimensional analogue of Proposition 2.1.

Lemma 3.1 *If S is a d-squashed semigroup, then there is a d-dimensional simplicial cone C_S in the positive orthant of $\mathbb{R}^d$ such that $y_d(C_S \cap \mathbb{Z}^d) = S$.*

Proof Suppose $S = (\frac{1}{N}\langle a_1, \dots, a_d\rangle) \cap \mathbb{Z}^+$. It is easy to construct a suitable cone if $S = \mathbb{Z}^+$, so assume otherwise.

Let $L = \{(z_1, \dots, z_d) \in \mathbb{Z}^d \mid N \text{ divides } \sum a_i z_i\}$. Then L is a free abelian group. Since it is contained in $\mathbb{Z}^d$, its rank is at most d, but since it contains N times each of the standard basis vectors, its rank is at least d. So L is a d-dimensional sublattice of $\mathbb{Z}^d$.

Now consider the map $g\colon L \to \mathbb{Z}$ defined by $g(z_1, \dots, z_d) = \frac{1}{N}\sum a_i z_i$. This map is onto because the greatest common divisor of the a_i is 1. We can therefore define a new set of coordinates for L (and hence also for $\mathbb{R}^d$), denoted by x_i, such that $x_d = g$. (In other words, we choose a set of maps $x_i\colon L \to \mathbb{Z}$ such that $(x_1, \dots, x_d)$ is an isomorphism from L to $\mathbb{Z}^d$, and such that $x_d = g$.) We can also view $\{x_i\}$ and $\{y_i\}$ as two different systems of coordinates on $\mathbb{R}^d$.

Now consider C, the subset of $\mathbb{R}^d$ defined by the condition that $y_i \geq 0$ for all i. This is a full-dimensional simplicial cone. (With respect to the y_i coordinate system, it is exactly the positive orthant.) Therefore, it is defined by d inequalities, which we can express in our new coordinate system by linear functionals $f_i(x_1, \dots, x_d) \geq 0$. By definition, S consists of the values of x_d on lattice points in C. So we would like to take C (considered with respect to the x_i coordinate system) to be the cone satisfying the statement of the lemma. However, we may not be able to do that.

The problem we face is that C might not lie in the positive orthant (again, with respect to the x_i coordinate system). Consider the $x_d = 1$ slice of C,

$$D = \{w \in C \mid x_d(w) = 1\}.$$

A priori, if we take an affine slice of a cone, we can get an unbounded set. However, if that happens in our case, then all the slices $x_d = k$ for $k \in \mathbb{N}$ would be unbounded, and, in particular, would contain lattice points. Therefore the x_d-values of the lattice points of C would be all of $\mathbb{Z}^+$, so $S = \mathbb{Z}^+$, contrary to our initial assumption. Thus, D is bounded. We can therefore choose $v = (v_1, \dots, v_{d-1}, 0) \in \mathbb{Z}^d$ such that $v + D$ lies in the positive orthant. Let C' be the cone over $v + D$. We claim that C' satisfies the statement of the lemma, with respect to the coordinate system given by the x_i. The x_d-values of lattice points in C' agree with those of C (and hence with S), and C' lies in the positive orthant, so C' satisfies our conditions, and we are done. ■

We have now shown that any d-squashed semigroup can be obtained by taking some simplicial cone C in the positive orthant of $\mathbb{R}^d$, intersecting it with $\mathbb{Z}^d$, and then taking its projection onto one coordinate axis.

If S is a d-squashed semigroup not containing some $n \in \mathbb{N}$, then we know that the intersection of C with the hyperplane $y_d = n$ must not contain any lattice points. If $d = 2$ (contrary to our running assumption in this section), this would tell us a lot about C, because in this case the "hyperplane" is a line, and so the intersection of C with this line is an interval of $\mathbb{R}$ which contains no lattice points. Such intervals are relatively easy to understand.

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However, since we are assuming $d > 2$, we will have to work harder. Let us step back, and consider what we might hope to be able to prove, given some convex body inside $\mathbb{R}^{d-1}$ which contains no lattice points.

Our first hope might be that such a set would have to be bounded, as is true of one-dimensional convex bodies containing no lattice points. But that is not true, even in two dimensions. It is easy to see that a two-dimensional convex set containing no lattice points can be arbitrarily large in one direction, provided it is reasonably skinny. In fact, it is easy to convince oneself that any two-dimensional convex set which contains no lattice points must have *some* direction in which it is narrow. (See Lemma 3.5 below for a precise statement.) It turns out that a similar statement holds in arbitrary dimension.

Khintchine's Flatness Theorem ([2, 2.2]) *If P is a convex body in $\mathbb{R}^{d-1}$ which contains no lattice points, then there is some surjective linear map $\phi\colon \mathbb{Z}^{d-1} \to \mathbb{Z}$ such that, extending ϕ to $\mathbb{R}^{d-1}$, the difference between the maximum and minimum values of ϕ on P is bounded by some function $q(d)$ that depends only on d and not on P.*

Applying this theorem, we determine that the intersection of C with the hyperplane $y_d = n$ is skinny in some direction, namely, the direction inside $\mathbb{R}^{d-1}$ perpendicular to the hyperplanes on which ϕ is constant. Now consider what happens when we intersect C with a hyperplane parallel to $y_d = n$, say, $y_d = n'$. Since C is a cone, its thickness in the ϕ direction is a linear function of n'. If we like, we can direct our attention to the part of C lying between $y_d = 0$ and $y_d = n/(8q(d))$, and here the thickness of C will be less than $\frac{1}{8}$. From this, we will conclude that (under one additional technical assumption) there is a $d-1$-simplex R contained in $\mathbb{R}^d$ such that $y_d(C \cap \mathbb{Z}^d) \cap [0, n/8q(d)] = y_d(R \cap \mathbb{Z}^d) \cap [0, n/8q(d)]$. In less technical language, we will show that, at least in the interval $[0, n/8q(d)]$, S has to look like the projection onto a coordinate axis of the lattice points in a $d-1$-dimensional simplex. In other words, we can replace the d-dimensional cone from Lemma 3.1 by a $d-1$-dimensional simplex (provided we restrict our attention to an interval within S). This might not seem like a big improvement, but in fact, it will allow us to prove the theorem we want, at least in the case $d = 3$.

Before we do that, though, we state and prove a somewhat technical lemma which we shall need shortly. In untechnical language, this lemma says that if we have a convex shape P in $\mathbb{R}^2$ which is skinny in the x-direction, and such that when we project P onto the y axis, obtaining an interval I, most of the lattice points in I have some lattice point in P which projects onto them, then we can choose those lattice points in P so that they all lie on a line.

Lemma 3.2 *Let P be a convex compact set in $\mathbb{R}^2$. Let I be the set of y-values of points in P. Let K be the set of y-values of lattice points in P. Suppose the following conditions hold.*

(i) *For any fixed $m \in I$, the length of the line segment $\{(x, m) \in P\}$ is less than $\frac{1}{8}$.*
(ii) $|K|/(|\mathbb{Z} \cap I|) \geq \frac{3}{4}$.

Then K is a set of consecutive integers corresponding to lattice points on a line in P.

A diagram showing a possible P appears in Figure 2.

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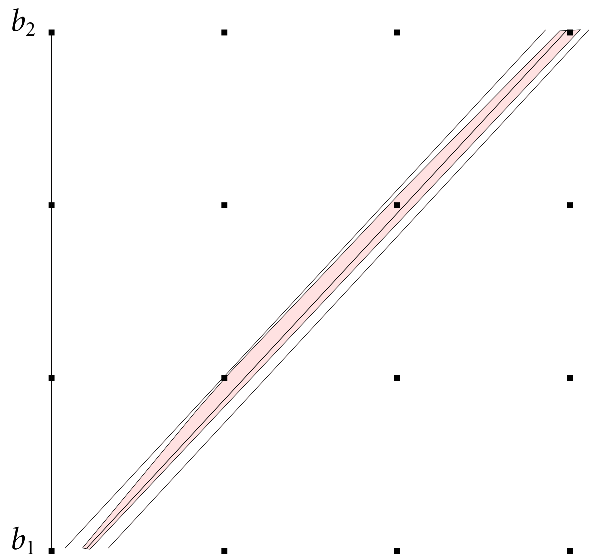

Figure 2: A convex body P satisfying (i) and (ii).

Proof We may assume that $|K| > 1$, since otherwise the statement is trivially true.

First, we show that P is contained inside a strip defined by two parallel lines: $H = \{(x, y) \mid ty + c \leq x \leq ty + d\}$, with $d - c = \frac{1}{4}$. Choose a point $(a_1, b_1) \in P$ with b_1 minimal. Similarly, choose a point $(a_2, b_2) \in P$ with b_2 maximal. Let $t = (a_2 - a_1)/(b_2 - b_1)$. Now observe that every point on the line segment between (a_1, b_1) and (a_2, b_2) lies in P, by convexity. By condition (i), every point of P is at a horizontal distance of no more than $\frac{1}{8}$ from this line. Thus every point of P lies between this line translated to the right by $\frac{1}{8}$ and this line translated to the left by $\frac{1}{8}$. (These lines are also shown in Figure 2.)

Now let P' be the parallelogram defined by these two translated lines, and also the lines $y = b_1$ and $y = b_2$. Then P' contains P. Let K' be the set of y-values of lattice points in P'. We will now proceed to prove that the conclusion of the lemma holds for the pair (P', K') instead of (P, K). Since $K \subset K'$, observe that condition (ii) holds for K'.

Define $f\colon K' \to \mathbb{Z}$ by saying that $f(z)$ is the x-coordinate of the lattice point in P' whose y-coordinate is z. (Since P' is thin in the x-direction, $f(z)$ is uniquely determined.)

Let $X = \{z \mid \{z, z+1\} \subset K'\}$. We will show that X is non-empty, in other words, that there are some two consecutive integers in K'. Suppose otherwise. Then $I \cap \mathbb{Z}$ consists of, say, p lattice points. If K' contained no two consecutive lattice points, K' would contain at most $\lceil (p+1)/2 \rceil$ lattice points, which is at most $(p+2)/2$. Thus, the ratio of the number of points in K' to the number of points in $I \cap \mathbb{Z}$ is at most $(p+2)/2p$. For $p > 4$, this ratio is less than $\frac{3}{4}$, violating our assumption. But it is also easy to see that one of our assumptions must be violated if $p \leq 4$.

Having established that X is non-empty, we claim that $f(z+1) - f(z)$ will be constant for $z \in X$.

If $z \in X$, then the x-coordinate of a point of P' with y-coordinate $z+1$ must be between $f(z) + t - \frac{1}{4}$ and $f(z) + t + \frac{1}{4}$. Since these differ by less than 1, there is a unique integer in $[t - \frac{1}{4}, t + \frac{1}{4}]$, which must be $f(z+1) - f(z)$ for all $z \in X$. Write w for this integer.

Let L_i be the line $\{(wy + i, y) \mid y \in \mathbb{R}\}$. Let Z_i be the set of y-values of points in $L_i \cap H$, and let Y_i be the set of y-values of lattice points in $L_i \cap P'$. Observe that by our construction of w, if z and $z+1$ are both in K', then z and $z+1$ both lie in the same set Y_i.

Now, we claim that there is actually only a single non-empty Y_i. We will prove this by showing that if more than one were non-empty, condition (ii) would have to be violated.

If $t = w$, then the lines L_i are parallel to the boundaries of H. Since the width of H is $\frac{1}{4}$, only one line L_i lies inside H. So suppose $t \neq w$.

The length of Z_i does not depend on i. This is clear geometrically, but we will give an argument which also determines the length. Fix i, and let z be the smallest value in Z_i. Then $(wz + i, z)$ is the lowest point of $L_i \cap H$. See Figure 3. The highest point in $L_i \cap H$ will be the point $(wz' + i, z')$, where $|(wz' + i) - (wz + i) - t(z' - z)| = \frac{1}{4}$, because the (horizontal) width of H is $\frac{1}{4}$. Solving this for $z' - z$, which is the length of Z_i, we find $z' - z = 1/(4|w - t|)$. Clearly, this amount does not depend on i. We denote it ℓ. Since Y_i is contained in Z_i, the number of lattice points in Y_i is at most $\ell + 1$.

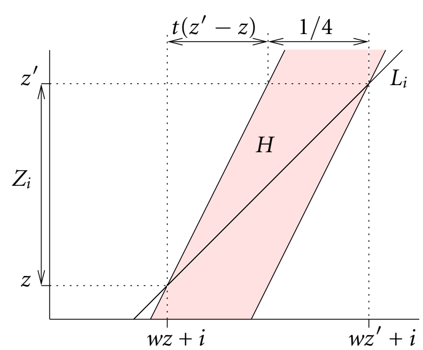

Figure 3

It is not immediately obvious from the definition that different Z_i do not overlap. In fact, even more is true: they are far apart from each other. Let the lowest point of $L_i \cap H$ be $(wz + i, z)$. Let z'' be the smallest value in Z_{i-1}. See Figure 4. By a similar argument to the above, $(w - t)(z'' - z) = 1$, so $|z'' - z| = 4\ell$. Thus, the separation between Z_i and Z_{i-1} is 3ℓ.

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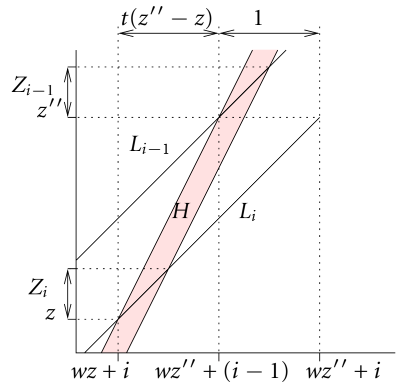

Figure 4

Roughly speaking, therefore, $\frac{1}{4}$ of the y-axis lies in some Z_i. Therefore, we would expect, on average, that the union of the Y_i would contain about $\frac{1}{4}$ of the lattice points in I. By (ii), though, the Y_i must contain at least $\frac{3}{4}$ of them. We now make this argument precise and show that it leads to the expected contradiction unless only one Y_i is non-empty.

Since there is some i with z and $z+1$ in Y_i, $\ell > 1$. The number of lattice points between Y_i and Y_{i-1} is at least $3\ell - 1$. Suppose that $p > 1$ is the number of Y_i that are non-empty. Then the number of lattice points in total in some Y_i is at most $p(\ell+1)$. The number of lattice points in I, but not in any Y_i, is at least $(p-1)3\ell$. Taking the ratio, we get

$$\frac{p(\ell+1)}{3(p-1)\ell} \leq \frac{p}{p-1}\frac{\ell+1}{3\ell}.$$

The first fraction on the right-hand side is a decreasing function of p, and so is maximized at $p = 2$ (since we assume $p > 1$). The second fraction is a decreasing function of ℓ, and we know that ℓ is more than 1. Thus, the ratio of the number of points from I that appear in some Y_i to those that do not appear is at most $4/3 < 2$. Thus the fraction of the lattice points in I that appear in some Y_i is at most $2/3 < 3/4$, violating (ii).

Thus there is only one Y_i which is non-empty, and it follows that K' is a consecutive sequence of integers, the y-values of a set of lattice points in P' that lie all on the line L_i.

Now $P \subset P'$, and $K \subset K'$. By the convexity of P, the intersection of P with the lattice points of P' must be a consecutive sequence of lattice points along the line, which implies the statement of the lemma. ■

Now we prove the lemma which we promised earlier.

Lemma 3.3 *Let S be a d-squashed semigroup and $n \notin S$. There is a value $\beta(d) < 1$ (depending only on d, not on n or S) such that if I is an interval contained in $[0, \beta(d)n]$,*

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and $|S \cap I|/|\mathbb{Z}^+ \cap I| > \frac{3}{4}$, then there is a $d-1$-dimensional simplex R in $\mathbb{R}^d$, such that, if we write V for the set of y_d-coordinates of lattice points in R, $S \cap I = V \cap I$.

Proof Let C be the cone guaranteed by Lemma 3.1. Let T be the intersection of C with the hypersurface $y_d = n$. Then T does not contain any lattice points, by assumption.

Appealing to Khintchine's Flatness Theorem, we obtain a surjective linear functional ϕ defined on the copy of $\mathbb{Z}^{d-1}$ lying inside $y_d = n$, such that the difference between the maximum and minimum values of ϕ on T is bounded by $q(d)$. Since ϕ is a surjective map from $\mathbb{Z}^d$ to $\mathbb{Z}$, we may reparameterize if necessary and assume that $\phi = y_1$; in other words, the direction in which T is narrow is the first coordinate direction.

Set $\beta(d) = 1/(8q(d))$. We will show that this choice of $\beta(d)$ satisfies the conditions of the theorem.

Let P be the part of the projection of C onto the (y_1, y_d)-plane having y_d-coordinate in I. Observe that, by our application of Khintchine's Flatness Theorem, P has thickness less than $\frac{1}{8}$ in the y_1 direction, so P satisfies condition (i) of Lemma 3.2.

Let K be the set of y_d-values of lattice points in P. Note that $K \supset S \cap I$, so K satisfies condition (ii) of Lemma 3.2.

We can therefore apply Lemma 3.2 to deduce that K consists of a string of consecutive numbers coming from lattice points all lying on a line in P.

Thus, the lattice points of $C \cap I$ all lie on some hyperplane in $\mathbb{R}^d$. The intersection of this hyperplane with C is either a simplex or a simplicial cone (with cone point not necessarily at the origin). In the former case, let R be this simplex. In the latter case, let R be the subset of this simplicial cone with $y_d \leq \beta(d)n$. Now R is a $d-1$-dimensional simplex inside $\mathbb{R}^d$. If we write V for the set of y_d-coordinates of lattice points in R, then $V \cap I = S \cap I$, as desired. ■

For the final part of the argument, we specialize to the case $d = 3$. Here, we have already concluded that if S is 3-squashed and $n \notin S$ (and S satisfies a certain density condition), then there is an interval $[0, \beta(d)n]$ within which S looks like the set of y_3-values of the lattice points in some two-dimensional simplex in $\mathbb{R}^3$. Now the crucial point is that this is a very restrictive condition, and in particular, we can apply Lemma 2.4 to prove our theorem.

Theorem 3.4 *There exist numerical semigroups that cannot be written as an intersection of 3-squashed semigroups.*

Proof Determine $\beta(3)$ as in Lemma 3.3. Let $b = \lceil 1/\beta(3) \rceil$ (where $\lceil x \rceil$ denotes the least integer $z \geq x$).

Using Lemma 2.4, construct a set of integers K contained in an interval $[0, j]$ with j an integer, such that K cannot be the set of y_3-values of the lattice points in a compact convex set in a plane in $\mathbb{R}^3$. Let $J = K \cup \{j+1, j+2, \dots, 4j+3\}$. Clearly, since K is not the set of y_3-values of lattice points in a compact convex set in a plane in $\mathbb{R}^3$, neither is J, and the density of J in $\{0, 1, \dots, 4j+3\}$ is more than $\frac{3}{4}$.

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Let $n = b(b+1)(4j+5)$. Let $I = [b(4j+5)+1, (b+1)(4j+5)-1] \cap \mathbb{Z}^+$. Let $P = b(4j+5)+1+J$, which is contained in I.

Now let S be the subsemigroup of $\mathbb{Z}^+$ generated by P, $n-(I \setminus P)$, and $\{q \mid q > n\}$. Clearly, S is a numerical semigroup.

The sum of any b elements of P will be less than n, while the sum of any $b+1$ elements of P will be greater than n. The sum of an element of P and an element of $n-(I \setminus P)$ cannot be n, while if we add further elements of P or further elements of $n-(I \setminus P)$, the result will be larger than n. Thus S does not contain n.

Now if S can be written as an intersection of 3-squashed semigroups, it must be that there is some 3-squashed semigroup Q containing S and not containing n. If $Q \cap I$ strictly contains $S \cap I$, there is some $x \in Q \cap I$ which is not in P. But by our construction of S, $n-x$ is then in S, and hence in Q. So Q contains x and $n-x$, and therefore also n, which would be a contradiction. So $Q \cap I = S \cap I = P$. But by Lemma 3.3, P cannot be the intersection with I of a 3-squashed semigroup which does not contain n, so we have a contradiction. ■

In order to apply our theorem to obtain concrete examples of semigroups, we need to know $q(3)$. Because we found it difficult to locate a statement of the value of $q(3)$ in the literature, we include the following lemma:

Lemma 3.5 *The minimum possible value for $q(3)$ in Khintchine's Flatness Theorem is $q(3) = 2$.*

Proof First, we prove that one can take $q(3) = 2$ in the statement of the Flatness Theorem; in other words, we show that if a convex set C in $\mathbb{R}^2$ contains no lattice points, then there is some lattice coordinate with respect to which the width of C is at most 2.

We may as well assume that C is closed and bounded. Let I be the projection of C onto the x-axis. If C is narrow in the x-direction, then we are done, so we may assume that the length of I is at least 2.

Now consider the function on the interval I defined by setting $f(c)$ to be the length of the intersection of C with the line $x = c$. Since, by assumption, C contains no lattice points, we know that the value of c at any integer point in I is less than 1.

For any $a < b$ in I, consider the trapezoid T_{ab} whose left and right sides are the intersections of $x = a$ and $x = b$, respectively, with C. The trapezoid T_{ab} is contained in C. For $0 \le t \le 1$, the length of the intersection of T_{ab} with the line $x = ta+(1-t)b$ is $tf(a)+(1-t)f(b)$. Thus, $f(ta+(1-t)b) \ge tf(a)+(1-t)f(b)$. In other words, the function f is concave.

Now choose $d \in I$. Let d' be the largest integer less than d. Let $e = 2d'-d$, $f = 2d'-d+2$. Since the length of I is at least 2, at least one of e, f must lie in I. Without loss of generality, suppose that $e \in I$. Then

$$\frac{1}{2}(f(d)+f(e)) \le f\Big(\frac{1}{2}(d+e)\Big) = f(d') \le 1.$$

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Since $f(e) \geq 0$, we know that $f(d) \leq 2$. Since d was arbitrary, it follows that the width of C in the y-direction is at most 2.

Now we show that we cannot take $q(3)$ to be less than 2. Consider the triangle whose vertices are $A = (\epsilon, \epsilon)$, $B = (\epsilon, 2 - 2\epsilon)$, and $C = (2 - 2\epsilon, \epsilon)$. For ϕ a lattice coordinate other than the x-coordinate, consider the length of AB. Since $\phi((0, 0))$, $\phi((0, 1))$, and $\phi((0, 2))$ are distinct integers, the ϕ-length of AB tends to at least 2 as ϵ goes to zero. On the other hand, clearly the length of AC tends to 2 for the x-coordinate. Thus as ϵ goes to zero, the width of the triangle ABC with respect to any coordinate tends to at least 2, so we cannot take $q(3) < 2$ in Khintchine's Flatness Theorem. ■

We now proceed to give an example of the kind of semigroup whose existence is guaranteed by the theorem. Since $q(3) = 2$, $b = 16$. Let K be the set given after the statement of Lemma 2.4: $K = \{0, 1, 4, 7, 48, 49\}$. So

$$J = \{0, 1, 4, 7, 48, 49, 50, \dots, 199\}.$$

Now $j = 49$, and $n = 54672$. Write J^c for $\{0, \dots, 199\} \setminus J$. Let S be generated by $3217 + J$, $51455 - J^c$, and $\{54673, 54674, \dots\}$. Then $54672 \notin S$, and S cannot be written as an intersection of 3-squashed semigroups.

Acknowledgments Part of this work was done during a visit of the second author to the Departamento de Matemáticas de la Universidad de Cádiz (Spain), through a grant of the Erasmus-Socrates Program of the European Community. His contribution is contained in his Master's Thesis, done under the supervision of E. Pardo in Fall 2004. The second author thanks the host center for its warm hospitality. The first and third authors want to thank P. Garcia-Sánchez for turning our attention to the papers [7, 8]. The fourth author thanks Andrew Toms for discussing some of the matters touched on in this paper with him. We would like to thank the referees for their comments, which improved the paper.

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