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A COVERING PROPERTY OF FINITE GROUPS

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Finite groups G possessing a proper subgroup U such that for each element g of G there exists an automorphism of G mapping g into U are considered. The question of how the structure of U determines the structure of G is examined. For example, if G is soluble and U is nilpotent then G is nilpotent.

A well known exercise asks one to prove that for a finite group G and a proper subgroup U of G, G is not the set-theoretical union of the G-conjugates of U. Replacing the inner automorphisms by the group of all automorphisms of G one is led to consider groups satisfying the following condition:

$$(*) \qquad G = \bigcup_{\alpha \in \mathrm{Aut}(G)} U^{\alpha}$$

for a suitably chosen proper subgroup U of G. Call G a $*$-group if some U exists satisfying $(*)$. If we want to refer to the particular subgroup U we shall sometimes call the pair (G, U) a $*$-group if G and U satisfy $(*)$.

In §1 we shall give some examples and the idea of when induction can be applied. In §2 structure theorems for soluble $*$-groups are proved. For example, if U has a Sylow tower (is nilpotent) then G has a Sylow tower (is nilpotent). Another result yields supersolubility of G if

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$|U|$ is odd and all Sylow subgroups of U are cyclic. The last section is devoted to the question of whether solubility of U implies solubility of G. A reduction theorem is proved and some simple groups are discussed.

All groups in this paper are finite. All unexplained notation is standard (see, for example, [1] or [3]).

1. Introduction

DEFINITION. (a) Let $U \leq G$ be groups. The pair (G, U) is called a $*$-group if and only if $U \neq G$ and $G = \bigcup_{\alpha \in \mathrm{Aut}(G)} U^{\alpha}$.

(b) The group G is a $*$-group if there is $U \leq G$ such that (G, U) is a $*$-group.

EXAMPLES. (a) Let G be an elementary abelian p-group of order at least p^2. Then (G, U) is a $*$-group for every nontrivial subgroup U of G.

(b) The quaternion group of order eight is a $*$-group.

The induction for $*$-groups is described by:

LEMMA 1. *Let (G, U) be a $*$-group and C be a characteristic subgroup of G. Then*

(a) $(C, U \cap C)$ is a $*$-group unless $C \leq U$,

(b) $(G/C, UC/C)$ is a $*$-group unless $UC = G$.

Proof. This follows easily by considering restrictions of automorphisms of G on C or G/C.

DEFINITION. Let (G, U) be a $*$-group. Call (G, U) *reduced* if U does not contain a nontrivial characteristic subgroup of G.

We immediately have:

LEMMA 2. *Let (G, U) be a $*$-group. Let $D = \bigcap_{\alpha \in \mathrm{Aut}(G)} U^{\alpha}$. Then $(G/D, U/D)$ is a reduced $*$-group.*

We now give a construction principle for $*$-groups. We shall need the following property of relatively free groups in some variety.

LEMMA 3 ([4]). *Let G be relatively free in some variety of*

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groups. Then G *has a generating set such that every mapping of this set into* G *can be extended to an endomorphism of* G .

From this the following is immediate.

LEMMA 4. *Let* G *be a noncyclic finite* p*-group, relatively free in some variety. Then*

(a) $\mathrm{Aut}(G)$ *acts transitively on the bases of* G (a base of G is an ordered tuple of group elements whose images in $G/\Phi(G)$ form a basis of the vector space $G/\Phi(G)$),

(b) *every nontrivial characteristic subgroup of* G *is contained in* $\Phi(G)$.

COROLLARY. *Any noncyclic relatively free* p*-group* G *is a* $*$*-group.*

Proof. Let $x \in G \backslash \Phi(G)$ and define $U = \langle x, \Phi(G) \rangle$. The corollary follows immediately from Lemma 4.

REMARK. The examples just constructed are not reduced unless G is elementary abelian. However, some computations yield examples of non-abelian reduced $*$-groups which are p-groups. We only state the result.

THEOREM 1. *Let* p *be any odd prime. Let* G *be the relatively free group in the free generators* g_1, g_2, g_3 *in the variety of groups of exponent* p *and nilpotency class two. Then*

(a) $|G| = p^6$, (G, U) *is a reduced* $*$*-group where* $U = \langle g_1, [g_1, g_2] \rangle$,

(b) *if* (G, V) *is a* $*$*-group then there exists* $W \leq V$ *such that* (G, W) *is a* $*$*-group and* $W \cong U$.

Another result we shall only state deals with the nilpotency class of a $*$-group. We have:

THEOREM 2. *Let* (G, U) *be a* $*$*-group,* $|U| = p^k$ *, where* p *is a prime. Then* G *is nilpotent of class at most* k . *Moreover if the class of* G *equals* k *then every characteristic subgroup of* G *is a member of the descending central series of* G .

2. Soluble *-groups

In this section we deal with the influence of the structure of U to the structure of a soluble *-group (G, U). We shall apply the following deep result of Shult.

THEOREM 3 ([5]). *Let X be any p-soluble group, p odd, and suppose that* $\operatorname{Aut}(X)$ *acts transitively on the set of subgroups of order p of X. Then the Sylow p-subgroups of X are abelian.*

COROLLARY. *Let (G, U) be a *-group and U be a cyclic p-group, where p is an odd prime. Then G is homocyclic, that is, G is isomorphic with a direct sum of groups isomorphic with U.*

Proof. By Shult's result G is abelian. The conclusion now follows easily.

We now state and prove our first main result.

THEOREM 4. *Let (G, U) be a *-group with G soluble. If U is p-closed then G is p-closed.*

Proof. Let G be a counterexample of least possible order. Then by Lemma 1 *(b)* either $G = UO_p(G)$ or $\left(G/O_p(G), UO_p(G)/O_p(G)\right)$ is a *-group. In the first case $G/O_p(G) \cong U/\left(U \cap O_p(G)\right)$ is p-closed, so G is p-closed. In the second case $G/O_p(G)$ is p-closed by minimality of G, unless $O_p(G) = 1$. So in our counterexample $O_p(G) = 1$.

Let C be a minimal characteristic subgroup of G, so C is an elementary abelian q-group for some prime $q \neq p$. Again, by Lemma 1 *(b)* either $G = UC$ or $(G/C, UC/C)$ is a *-group. So in both cases G/C is p-closed. Let $P/C := \Omega_1\left(Z\left(O_p(G/C)\right)\right)$. Hence P is characteristic in G. As $O_p(P) \leq O_p(G) = 1$, P is not p-closed. So $P \nleq U$ as U is p-closed. Hence, by Lemma 1 *(a)*, $(P, U \cap P)$ is a *-group. If $P \neq G$, P is p-closed by minimality. So $P = G$. Hence U is p-closed by assumption and q-closed, so U is abelian. Let H be a complement of C in G, so $G = CH$ semidirect.

Now for any $c \in C$ there exists $1 \neq h \in H$ such that $[c, h] = 1$. Indeed, let $c \in C$. By assumption there exists $\alpha \in \operatorname{Aut}(G)$ such that

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$c^{\alpha} \in U$. Let $y \in U$ be an element of order p . As U is abelian, $[c^{\alpha}, y] = 1$ so $[c, y^{\alpha^{-1}}] = 1$. Let $y^{\alpha^{-1}} = y_1 y_2$ where $y_1 \in C$, $y_2 \in H$. Then $1 = [c, y_1 y_2] = [c, y_2][c, y_1]^{y_2} = [c, y_2]$ as C is abelian. But $h := y_2 \neq 1$ as y has order p and C is a q-group.

As the orders of C and H are coprime, C is a completely reducible H-module. Let $C = \oplus C_i$ be a decomposition of C into a direct sum of irreducible H-modules C_i . Let $1 \neq c_i \in C_i$ and $c := c_1 + \ldots$. Then, by our previous remark, there exists $1 \neq h \in H$ with $[c, h] = 1$. Let $H_0 := \langle h \rangle \trianglelefteq H$ as H is abelian. So $1 \neq c \in C_C(H_0)$. But the C_i are H-invariant and so $1 \neq c_i \in C_{C_i}(H_0)$. But $C_{C_i}(H_0)$ is an H-invariant subspace of C_i . By irreducibility H_0 centralises each C_i , so $[H_0, C] = 1$. This contradicts the faithful action of H .

COROLLARY. *Let (G, U) be a *-group, G being soluble.*

(a) If U has a Sylow tower then G has a Sylow tower.

(b) If U is nilpotent then G is nilpotent.

Proof. *(a)* follows from Theorem 4 and Lemma 1 by an easy induction argument.

(b) A group is nilpotent if and only if it is p-closed for all primes p , so *(b)* is immediate from Theorem 4.

Our next main theorem deals with the case when U satisfies the following conditions:

(Z) $|U|$ is odd and all Sylow subgroups of U are cyclic.

For the structure of groups satisfying (Z) see [3]. We shall need the following properties of (Z)-groups.

PROPOSITION 1. *U is metacyclic, $U = U'\langle t \rangle$ for some $t \in U$.*

PROPOSITION 2. *U is supersoluble, in particular U has a Sylow*

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tower and U *is* p*-closed where* p *is the greatest prime divisor of* $|U|$.

THEOREM 5. *Let* (G, U) *be a* $*$*-group,* U *satisfying* (Z). *Then*

(a) G *is supersoluble,*

(b) G *is metabelian.*

Proof. *(a)* Let G be a counterexample of least order. As the order of G is odd, G is soluble. So, by Theorem 4, G is p-closed for some prime p dividing the order of G. As the class of supersoluble groups is a saturated formation we have $\Phi(G) = 1$ by minimality of G, Lemma 1 *(b)* and a standard property of Frattini subgroups. So, as $\Phi(O_p(G)) \leq \Phi(G)$, $O_p(G)$ is elementary abelian.

We claim $O_q(G) = 1$ for all primes $q \neq p$. Indeed, assume that $O_r(G) \neq 1$ for some prime r. Then by Lemma 1 *(b)* either $(G/O_r(G), UO_r(G)/O_r(G))$ is a $*$-group or $G = UO_r(G)$. In the first case $G/O_r(G)$ is supersoluble by minimality, in the second case $G/O_r(G) \cong U/(U \cap O_r(G))$ is supersoluble by Proposition 2. But if $O_q(G) \neq 1$ for some prime $q \neq p$, G could be embedded into $G/O_p(G) \times G/O_q(G)$ which is supersoluble by our remarks above. So G would be supersoluble; a contradiction.

Now by property (*) and the fact that all subgroups of order r of U are conjugate we see that $\mathrm{Aut}(G)$ acts transitively on the subgroups of order r of G (r being any prime). So G is a $T(r)$-group in the sense of [2]. By [2], all nonnormal Sylow subgroups of G are cyclic. By the above only the Sylow p-subgroup of G is normal and so all Sylow subgroups of $G/O_p(G)$ are cyclic, so $G/O_p(G)$ is a (Z)-group.

Obviously $G \neq O_p(G)$ and so $Z/O_p(G) := (G/O_p(G))' < G/O_p(G)$. Moreover, by Proposition 1, there exists $t \in G$ with $G = Z\langle t \rangle$. As Z is characteristic in G and (G, U) is a $*$-group, we may assume that $t \in U$.

Z normalises every one dimensional subspace of $O_p(G)$. Indeed, by

Lemma 1 *(a)*, Z is supersoluble. As $1 \neq O_p(G) \leq Z$, there exists $1 \neq x \in O_p(G)$ with $\langle x \rangle \trianglelefteq Z$. Let $1 \neq y \in O_p(G)$. As $\mathrm{Aut}(G)$ acts transitively on the subgroups of order p of G, there exists $\alpha \in \mathrm{Aut}(G)$ with $\langle y \rangle = \langle x \rangle^\alpha$. As the restriction of α on Z yields an automorphism of Z, we get $\langle y \rangle \trianglelefteq Z$.

$\langle t \rangle$ normalises every one dimensional subspace of $O_p(G)$. Indeed, as $O_p(G)$ is elementary abelian, we get $|O_p(U)| = p$. Now $t \in U$ and so t normalises $O_p(U)$. Let $1 \neq y \in O_p(G)$. By property $T(p)$ we have $\alpha \in \mathrm{Aut}(G)$ with $\langle y \rangle = O_p(U)^\alpha$. So $\langle y \rangle^t = \left(O_p(U)^\alpha\right)^t = \left(O_p(U)^{t^{\alpha^{-1}}}\right)^\alpha$. As $G = Z\langle t \rangle$ we have $t^{\alpha^{-1}} = zt^n$ for some $z \in Z$ and some integer n. So $\langle y \rangle^t = \left(O_p(U)^{zt^n}\right)^\alpha = O_p(U)^\alpha = \langle y \rangle$ as Z normalises $O_p(U) \leq O_p(G)$ by the above. The conclusion follows.

The last two remarks show that $G = Z\langle t \rangle$ normalises every cyclic subgroup of $O_p(G)$, so G is supersoluble contradicting the choice of G. So *(a)* is proved.

(b) By *(a)* we get that G' is nilpotent. By Theorem 3 all Sylow subgroups of G are abelian, so G' is abelian.

3. Nonsoluble $*$-groups

This chapter is concerned with the question whether for a $*$-group (G, U) solubility of U implies solubility of G. We firstly prove a reduction theorem.

THEOREM 6. *Let (G, U) be a $*$-group, let U be soluble and G be not soluble. Then there exists a $*$-group (H, V) where H is simple and V is soluble.*

Proof. Let (G, U) be as in the assumption of the theorem where G has least possible order. We show that G is simple.

G is characteristically simple. Otherwise, let C be any nontrivial characteristic subgroup of G. Then by Lemma 1, $C \leq U$ or $(C, C \cap U)$

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is a *-group. In the second case C is soluble by minimality. So C is soluble in all cases. Analogously G/C is soluble, so G would be soluble.

Let $G = S \times \ldots \times S$ where S is nonabelian simple and let π_i be the canonical projection onto the ith coordinate. By assumption, for any $x \in S$ there exists $\alpha \in \mathrm{Aut}(G)$ such that $(x, \ldots, x)^\alpha \in U$. Now, by the well known structure of the automorphism group of characteristically simple groups we have $(x, \ldots, x)^\alpha = \left(x^{\alpha_1}, \ldots, x^{\alpha_k}\right)$ for suitable $\alpha_i \in \mathrm{Aut}(S)$. This implies that either $\left(S, \pi_i(U)\right)$ is a *-group for some index i or $\pi_i(U) = S$ for all i. In the first case we are done; the second case contradicts the solubility of U.

Theorem 6 suggests the investigation of *-groups (G, U) where G is simple. Obviously, G and U have the same exponent. By this remark the simple groups $\mathrm{PSL}(2, q)$, $\mathrm{Sz}(q)$ and the Ree groups are ruled out. Also the Mathieu groups are not *-groups. For example if $G = M_{12}$ then U must be M_{11}. However, by inspection of the centralizers of the elements of order three, one can show that $\left(M_{12}, M_{11}\right)$ is not a *-group. Also the alternating groups are not *-groups. Here we shall only prove that the alternating group of degree $n \geq 5$ does not contain a soluble subgroup U having the same exponent. Let $n = 2m$ be even. Then, by Bertrand's postulate, there are primes p, q with $m \leq p < q \leq 2m$. Let H be a $\{p, q\}$-Hall subgroup of U. So $|H| = pq$. By Sylow's Theorem H is cyclic. But the minimal degree of a permutation group containing an element of order pq is $p + q$ which is strictly greater than n, a contradiction. The case for n odd is similar.

So we are led to state the following:

CONJECTURE 1. Let (G, U) be a *-group. If U is soluble does it follow that G is soluble?

The conjecture above would be solved if we could establish

CONJECTURE 2. A nonabelian simple group G does not posses a soluble subgroup U with $\exp(U) = \exp(G)$.

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References

[1] Daniel Gorenstein, *Finite groups* (Harper and Row, New York, Evanston, and London, 1968).

[2] Wolfgang Hauptmann, "Gruppen mit einer Automorphismengruppe die transitiv auf den Untergruppen von Primzahlordnung operiert", *Mitt. Math. Sem. Giessen* 101 (1973).

[3] B. Huppert, *Endliche Gruppen I* (Die Grundlehren der mathematischen Wissenschaften, 134. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[4] Hanna Neumann, *Varieties of groups* (Ergebnisse der Mathematik und ihrer Grenzgebiete, 37. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[5] Ernest E. Shult, "On finite automorphic algebras", *Illinois J. Math.* 13 (1969), 625-653.

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