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MULTIPLIERS IN CONTINUOUS VECTOR-VALUED FUNCTION SPACES

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We identify the multiplier space of continuous vector-valued function spaces as the bounded continuous multiplier-valued space in the strong topology.

1. INTRODUCTION

In this note, we study the multipliers of vector-valued function spaces on a locally compact Hausdorff space S. We focus our attention on the space $C_0(S, X)$ of X-valued continuous functions vanishing at infinity on S, where X is a Banach space. For the case when $X = A$ a commutative Banach algebra with a bounded approximate identity, and $S = G$ is a locally compact abelian group, Lai [2] has characterised the multiplier space $\mathcal{M}(C_0(G, A))$ to be $C_{s,b}(G, \mathcal{M}(A))$, the set of bounded strongly continuous functions from G to the multipliers of A. Thus

$$\text{Hom}_{C_0(G,A)}(C_0(G, A), C_0(G, A)) \cong C_{s,b}(G, \mathcal{M}(A)). \tag{1}$$

Note that $C_0(G, A)$ is a Banach algebra under the usual pointwise product, in which case it may be identified as the ε-tensor product $C_0(G)\check{\otimes}_\varepsilon A$. In [2], Lai considered the case where $C_0(G, A)$ is an $L^1(G, A)$-module under convolution product. In this case, the multipliers are

$$\text{Hom}_{L^1(G,A)}(C_0(G, A), C_0(G, A)) \cong M(G, A) \tag{2}$$

where $M(G, A)$ is the space of bounded regular A-valued measures, provided that A has an identity of norm 1.

If X is an A-module Banach space, some further characterisations for multipliers of vector-valued Bochner integrable function spaces are given in Lai [3] (see also Lai and Chang [4]). There are two interesting open problems raised in Lai [3, p.62]. Under what conditions can one characterise the multipliers of the form

$$\text{Hom}_{C_0(G,A)}(C_0(G, X), C_0(G, X)) = ? \tag{P1}$$

$$\text{Hom}_{L^1(G,A)}(C_0(G, X), C_0(G, X)) = ? \tag{P2}$$

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These problems can be considered in more general situations. This note is focused on the first problem. The second problem will be studied in another article.

Throughout the paper we let A be a commutative Banach algebra, and let X and Y be (left) Banach A-modules (shortly, A-modules). We write

$$(3)\qquad \mathrm{Hom}_A(X, Y) = \{T \in \mathcal{L}(X, Y) \colon T(a \cdot x) = a \cdot Tx \quad \text{for any } a \in A,\ x \in X\}$$

where $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X to Y. If $X = Y = A$, then $\mathrm{Hom}_A(A, A) = \mathcal{M}(A)$ is the usual multiplier space of A. Let S be a locally compact Hausdorff space (not necessarily a topological group). Let $C_b(S, X)$ be the linear space of X-valued bounded continuous functions on S.

Then $C_b(S, X)$ the space of X-valued bounded continuous functions in S, and

$$(4)\qquad \|f\|_\infty = \sup_{s\in S} \|f(s)\|_X \quad \text{for} \quad f \in C_b(S, X)$$

defines a norm for $C_b(S, X)$ and $(C_b(S, X), \|\ \|_\infty)$ is a Banach space. We let by

$$C_0(S, X) = \{f \in C_b(S, X);\ \lim_{s\to\infty} f(s) = 0\}.$$

$C_0(S, X)$ is a closed subspace of $C_b(S, X)$. If X is an A-module, it is evident that $C_0(S, X)$ is a $C_0(S, A)$-module under pointwise product. For each closed subspace W of $\mathcal{L}(X, Y)$, we define

$$(5)\qquad C_{s,b}(S, W) = \{F \colon S \to W; F \text{ is strongly continuous and bounded}\}.$$

We now define the norm on $C_{s,b}(S, W)$ by

$$(6)\qquad \|F\|_\infty = \sup_{s\in S} \|F(s)\| = \sup_{\substack{s\in S\\ \|x\|\leqslant 1}} \|F(s)x\|_Y$$

where $\|\ \|_Y$ means the Banach norm of Y. Then $C_{s,b}(S, W)$ is a Banach space under the norm $\|\ \|_\infty$ defined in (6). If $X = \mathbb{C}$, the complex number field, we write $C_0(S, \mathbb{C}) = C_0(S)$.

The main purpose of this note is to prove that

$$(7)\qquad \mathrm{Hom}_{C_0(S,A)}(C_0(S, X), C_0(S, Y)) \cong C_{s,b}(S, \mathrm{Hom}_A(X, Y)).$$

2. Multipliers for continuous vector-valued functions

The symbols S, X, Y, A, $\mathcal{L}(X, Y)$, $C_0(S, X)$ and $C_{s,b}(S, W)$, with $W \subset \mathcal{L}(X, Y)$ a closed subspace, are explained in the previous section.

We begin with the following characterisation which will be useful later.

Proposition 1. (The pseudoscalar case) *Let Y be any Banach space. Then*

$$\mathrm{Hom}_{C_0(S)}(C_0(S), C_0(S, Y)) \cong C_b(S, Y). \tag{8}$$

Proof: Let $T \in \mathrm{Hom}_{C_0(S)}(C_0(S), C_0(S, Y))$ and $s \in S$. If $f, g \in C_0(S)$ with $f(s) \neq 0$ and $g(s) \neq 0$ then there is a neighbourhood $\mathcal{N}(s)$ of s in S such that

$$f(t) \neq 0 \text{ and } g(t) \neq 0 \text{ for any } t \in \mathcal{N}(s).$$

Thus as T is a multiplier, we have

$$T(g \cdot f)(t) = g(t)(Tf)(t) = f(t)(Tg)(t) = T(f \cdot g)(t)$$

and then
$$\frac{(Tg)(t)}{g(t)} = \frac{(Tf)(t)}{f(t)} \quad \text{for any} \quad t \in \mathcal{N}(s).$$

Now, for each $s \in S$ there exists $f \in C_0(S)$ with $f(s) \neq 0$. Define

$$F(s) = \frac{(Tf)(s)}{f(s)}. \tag{9}$$

By the previous argument, F is well-defined and clearly is an element of $C_b(S, Y)$.

Clearly if $f(s) \neq 0$ then $(Tf)(s) = F(s)f(s)$. The equality also holds when $f(s) = 0$. To see this, choose $g \in C_0(S)$ such that $g(s) \neq 0$. Then

$$g(s)(Tf)(s) = T(gf)(s) = f(s)(Tg)(s) = 0$$

and so $Tf(s) = 0$.

Now since $\|(Tf)(s)\|_Y = \|F(s)f(s)\|_Y \leqslant \|F\|_\infty \|f\|_\infty$, $\|T\| \leqslant \|F\|_\infty$. On the other hand, by Uryson's Lemma, we can choose an $f \in C_0(S)$ such that $\|f\|_\infty = |f(s)|$. So $\|F(s)\|_Y = (\|(Tf)(s)\|_Y)/(|f(s)|) \leqslant (\|T\| \, \|f\|_\infty)/(\|f\|_\infty) = \|T\|$ for all $s \in S$. Hence $\|F\|_\infty \leqslant \|T\|$, and consequently $\|F\|_\infty = \|T\|$. This shows that $\mathrm{Hom}_{C_0(S)}(C_0(S), C_0(S, Y))$ is isometrically embedded in $C_b(S, Y)$.

Conversely, for any $F \in C_b(S, Y)$, we define

$$T_F \colon C_0(S) \to C_0(S, Y)$$

by

$$T_F(f) = F \cdot f \quad \text{for any} \quad f \in C_0(S).$$

Then one can easily show that T_F is a multiplier from $C_0(S)$, to $C_0(S, Y)$, and that $\|F\|_\infty = \|T_F\|$. Hence the proof is completed. ∎

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If X is an A-module and $Ax = 0$ implies $x = 0$, we say that X is an order-free A-module. It is easy to check that if X and Y are A-modules, with Y order-free, then

$$T(ah) = aTh \quad \text{for} \quad a \in A,\, h \in C_0(S, X)$$

and

$$T(fh) = fTh \quad \text{for} \quad f \in C_0(S),\, h \in C_0(S, X),$$

whenever $T \in \mathrm{Hom}_{C_0(S,A)}(C_0(S, X), C_0(S, Y))$. Note that $C_0(S)\check{\otimes}_\varepsilon X = C_0(S, X)$ and $C_0(S)\check{\otimes}_\varepsilon A = C_0(S, A)$, where $\check{\otimes}_\varepsilon$ means the completion of ε-norm (the least reasonable cross norm) tensor product.

Now we can establish the main theorem as follows.

Theorem 2. *Let X and Y be A-modules, with Y order free. Then*

(7) $$\mathrm{Hom}_{C_0(S,A)}(C_0(S, X), C_0(S, Y)) \cong C_{s,b}(S, \mathrm{Hom}_A(X, Y)).$$

The correspondence between the multiplier T and the function F is given by the following relation:

(10) $$(Tg)(s) = F(s) \cdot g(s) \quad \text{for } s \in S \text{ and any } g \in C_0(S, X).$$

Proof: Let $T \in \mathrm{Hom}_{C_0(S,A)}(C_0(S, X), C_0(S, Y))$. We define a map $\Phi_Y : X \to \mathrm{Hom}_{C_0(S)}(C_0(S), C_0(S, Y))$ by

(11) $$\Phi_T(x)(f) = T(f \otimes x) \quad \text{for any } x \in X \text{ and } f \in C_0(S).$$

Evidently (11) defines an element in $C_0(S, Y)$. As x is fixed, the operator $\Phi_T(x)$ defines a bounded linear operator from $C_0(S)$ into $C_0(S, Y)$, and is evidently a multiplier. By Proposition 1, there exists an element, say F_x, in $C_b(S, Y)$ such that

$$\Phi_T(x)(f) = F_x \cdot f.$$

Define $F(s)(x) = F_x(s)$ for $s \in S$. Then $F(s)$ is a linear operator from X into Y. Moreover, for $a \in A$ and $f \in C_0(S)$, we have

$$\begin{aligned} F(s)(ax) \cdot f(x) &= F_{ax}(s) f(s) \\ &= T(f \otimes ax)(s) \\ &= aT(f \otimes x)(s) \\ &= aF_x(s) f(s) \\ &= aF(s)(x) f(s) \quad \text{for any } s \in S \end{aligned}$$

or

$$F(s)(ax) = aF(s)(x).$$

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This show that $F(s) \in \mathrm{Hom}_A(X, Y)$. Consequently, $F \in C_{s,b}(S, \mathrm{Hom}_A(X, Y))$. Moreover

$$\begin{aligned}\|F(s)\| &= \sup_{\|x\|\leqslant 1} \|F(s)(x)\|_Y \\ &\leqslant \sup_{\|x\|\leqslant 1} \|F_x\|_\infty = \sup_{\substack{\|x\|\leqslant 1 \\ \|f\|_\infty \leqslant 1}} \|F_x \cdot f\|_\infty \\ &= \sup_{\|f\otimes x\|\leqslant 1} \|T(f\otimes x)\|_\infty \\ &= \|T\| \qquad \text{for all } s \in S.\end{aligned}$$

So
$$\|F\|_\infty \leqslant \|T\|.$$
But
$$\begin{aligned}\|T(f\otimes x)\|_\infty &= \|F_x \cdot f\|_\infty \leqslant \|F_x\|_\infty \|f\|_\infty \\ &\leqslant \|F\|_\infty \|x\| \|f\|_\infty \\ &= \|F\|_\infty \|f\otimes x\|_\infty\end{aligned}$$

for all $f \otimes x \in C_0(S)\check{\otimes}_\varepsilon X$. Hence

$$\|T\| \leqslant \|F\|_\infty.$$

Conversely, let $F \in C_{s,b}(S, \mathrm{Hom}_A(X, Y))$ and $f \in C_0(S)$. Then $F \cdot f$ is a continuous function on S vanishing at infinity, that is $F \cdot f \in C_0(S, \mathrm{Hom}_A(X, Y))$. For any $x \in X$ and $f \in C_0(S)$, F determines a bounded linear operator T from $C_0(S, X)$ to $C_0(S, Y)$ given by

$$T(f\otimes x)(s) = (F(s)x)f(s).$$

Clearly $\|T\| = \|F\|_\infty$.

Evidently, T is a multiplier, that is,

$$T((h\otimes a)\cdot(f\otimes x)) = (h\otimes a)T(f\otimes x)$$

for $h \otimes a \in C_0(S)\check{\otimes}_\varepsilon A$ and $f \otimes x \in C_0(S)\check{\otimes}_\varepsilon X$. In fact, since X and Y are A-modules,

$$\begin{aligned}T((h\otimes a)\cdot(f\otimes x)) &= T(hf\otimes ax) \\ &= F(\cdot)(ax)(hf)(\cdot) \\ &= ah(\cdot)F(\cdot)(x)f(\cdot) \\ &= (h\otimes a)T(f\otimes x).\end{aligned}$$

The isometry between F and T is routine. Therefore the proof is complete. ∎

Some special cases can be obtained from Theorem 2. If $X = Y = A$ has a bounded approximate identity, then Theorem 2 reduces to the result of Lai [**2**, Theorem 2].

Corollary 3. *Let A be a Banach algebra with a bounded approximate identity. Then*

$$\mathcal{M}(C_0(S, A)) \cong C_{s,b}(S, \mathcal{M}(A)). \tag{1}$$

If A has an identity, then $\mathrm{Hom}_A(Y, Y) = Y$, and hence we obtain the answer of (P1) (Lai [**3**, p.62]). That is

Corollary 4. *Let A be a Banach algebra with identity of norm 1. Then*

$$\mathrm{Hom}_{C_0(S,A)}(C_0(S, Y), C_0(S, Y)) \cong C_b(S, Y). \tag{12}$$

Note that the strong topology on Y agrees with the norm topology.

It is remarkable that if $S = \mathbb{R}^n$, then one can investigate the multipliers between spaces of continuously differentiable, vector-valued functions; that is, between the spaces

$$C_0^{(m)}(\mathbb{R}^n, X) \quad \text{and} \quad C_0^{(m)}(\mathbb{R}^n, Y).$$

To do this one needs to clarify the relation between $C_0^{(m)}(\mathbb{R}^n)\check{\otimes}_\varepsilon X$ and $C_0^{(m)}(\mathbb{R}^n, X)$. If S is a particular subset of the complex number space $\mathbb{C}$, one can also consider the multipliers for analytic functions on the unit disc with values in X. These characterisations are not so clear (see [**1**]).

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