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The Treatment of Proportion in Elementary Geometry.

By Professor GIBSON.

I do not think any apology is needed for asking the Society to consider the treatment of Proportion in Elementary Geometry. Although the fifth book of Euclid's *Elements* appears in all editions of Euclid, I know of no school or college where it is read; I know of no examination for which it is prescribed, and I have never seen an examination paper which contained a question based upon it, except in regard to its definitions. Indeed I believe it is not unfair to say that even among teachers themselves a thorough knowledge of Euclid's fifth book is very rare.

In a country where respect for Euclidian methods borders on superstition, this is surely a striking state of matters, and there can, I think, be little doubt that most teachers are far from satisfied with the practice now usually adopted of getting over the difficulties of the Euclidian theory. The usual practice, so far as I can determine, is to go over the definitions of the fifth book, and to prove the first and the thirty-third propositions of the sixth book in Euclid's manner, but to adopt the arithmetic or algebraic proofs of the theorems of the fifth book, some attempt being made to connect Euclid's definition of proportion with that given in Algebra. The version of Book V. in the text-book of the A.I.G.T. does not seem to have fared better than Euclid's own; at any rate, I have not met any teacher who adopts its proofs of the theorems required in the application of proportion to geometry.

The difficulty of the situation is increased by the fact that Euclid's treatment of proportion is *in itself* admirable, and while he did not, I think, give the full development of the conception of a ratio, as distinguished from a proportion, of which his method is capable, all recent researches into the representation of continuous magnitude by number have only put in a stronger light the intrinsic excellence of his method. It is therefore from no disparagement of the scientific value of the Euclidian theory of proportion that I urge

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a different method of dealing with it; but it is quite impossible to overlook the fact that in elementary teaching Euclid's method has broken down. The current practice of using only his definitions seems to me a very unsatisfactory makeshift, especially because I believe that we have ready to our hand an equally rigorous method, that is free from the chief difficulties of Euclid's. If we had to teach pupils whose intellect was sufficiently matured to deal intelligently with abstract conceptions, then there would be no great need for abandoning the Euclidian theory; but we can never expect to have such pupils.

In teaching Euclid's definitions of ratio and proportion, the prime difficulty that I have found has been to connect in a satisfactory way Euclid's definitions with those the pupil has been accustomed to use in arithmetic. And just at this point, I think, the unsatisfactoriness of Euclid's method for elementary teaching is most clearly seen. I suppose we may take it as axiomatic that we should, as far as possible, appeal to the conceptions already present in the pupil's mind and, it may be, already partially reduced to the definiteness required for mathematical work, and we should meet new difficulties, not so much by discarding the old conceptions as by developing them in such a way that the new form shall be seen to be but a generalisation of the old form. All through his mathematical training the pupil is guided on these lines; at a very early stage the primary conception of number as an integer is extended so as to include the conception of number as a fraction, and the usual operations on integers are carried over with the same names to the fractional number, though at first sight the names themselves seem often very unsuitable. In algebra this process is carried much farther, but at every stage the propriety of the extension is shown by a proof of the identity of certain fundamental elements common to all the stages.

Now in arithmetic "the ratio of one whole number to another is measured by the fraction which the one is of the other; and the ratio of one quantity to another is the ratio of the two whole numbers that express these quantities in terms of the same unit" (*E.M.S. Proc.*, VI., p. 98). It is certainly not easy to see the connection between ratio defined in this way and ratio as defined in Euclid's fifth book. In the text of Euclid there may be said to be two definitions, of which the first is "ratio is the (or a) relation of two magnitudes of the same kind to one another in respect of quantuplicity." It may be remarked that this definition has a rather curious history. Barrow, a most learned and ingenious defender of the Euclidian theory, says, after devoting a lecture to the exposition and defence of the definition, that Euclid perhaps gave it only as "a prelude for method or ornament's sake to the more accurate definitions of *the same*, *a greater*, and *a less ratio* . . . that he might insinuate a certain general idea of ratio into the minds of the learners by this *metaphysical* definition ; I say, *metaphysical*, for it is not properly mathematical, since it has no dependence upon it, nor is, or I believe can be, deduced in the Mathematics." (*Geom. Lect. XVIII.*) Simson says that he fully believes the definition is not Euclid's, but is the addition of some unskilful editor. On the other hand, it is this definition which has bulked most largely in recent discussions. Thus a reviewer writes, "we are inclined to think no treatise on geometrical proportion complete which does not give a thorough discussion of the theory of proportion based on *quantuplicity*." (*Math. Gaz.*, *April* 1896, p. 23).

But, passing over such contradictory estimates, it may be asked —Is the definition such as to make the general conception of ratio more definite and more suitable for mathematical purposes than it is without the definition? Or again, take the other definition, namely "magnitudes are said to have a ratio to one another when they can, being multiplied, exceed the one the other," which is usually and correctly interpreted to mean that magnitudes can have a ratio to each other only when they are of the same kind. Does the former definition give any clearer idea than the latter? Certainly not, until the word *quantuplicity* is defined, and even when this has been done, there seems a good deal to accomplish. Now what is the meaning of the Greek word πηλικότης translated *quantuplicity*? Barrow translates it by *quantity*, and objects to the translation *tantuplicity* ; he quotes with approval a Greek scholiast who says he thinks Euclid designedly put *according to quantity*, rather than *according to quality*, because all ratios are not capable of being expressed by number. Again, to elucidate the meaning of the word, the *Syllabus* of the *A.I.G.T.* states that "the quantuplicity of A with respect to B may be estimated by examining how the multiples of A are distributed among the multiples of B when both are arranged in ascending order of magnitude, and the series of multiples continued without limit." It would seem as if we were defining one term by another which is certainly not simpler, as it should be, according to all the rules of logic.

But translate the word as we may, what is to be understood by the definition? Take the simple case when A is five-thirds of B; what is the ratio of A to B according to the definition? One interpreter of this order of ideas says "that relation in virtue of which A is a fraction of B—a fraction being defined as the ratio of two numbers—is called the ratio of A to B when they are commensurable." (*Nixon*, *Euc. Rev.*, p. 223, 1st Ed.) This is surely an extraordinary statement; notice that the ratio of two numbers is not defined as a fraction, but *the fraction* is defined as the *ratio* of two numbers, so that we apparently are defining the fundamental conception of a ratio by a ratio itself. Suppose, however, that we say the relation that we are considering between A and B is the same as the relation between the numbers 5 and 3, we must determine which of the many relations between 5 and 3 is to be fixed upon as the relation to be denominated their ratio. It is assumed to be neither their sum, their difference, nor their product; the next simplest is the quotient of 5 by 3, and this relation is expressed by the fraction $\frac{5}{3}$. Does the definition then mean that when A is equal to five-thirds of B, the ratio of A to B is that relation between the numbers 5 and 3 which is expressed by taking the quotient of 5 by 3? This statement is, I should say, intolerably prolix, but unless this be the meaning of the definition, I am quite at a loss to say what it means. And I think it is just possible to render the Greek words, without putting any strain upon them, so to bring out this meaning more fully, thus "ratio is a sort of quotient-relation between two homogeneous magnitudes." If it were worth while to go into the matter, I think it might be shown that Eutocius, the commentator on Archimedes, explains the word πηλικότης to mean the same as our word "quotient"; the translation suggested also gives the natural signification to the particle ποιά.

In the case of commensurables, then, that is when $nA = mB$, m, n being integers, the ratio of A to B is the quotient relation between m and n; but when A and B are incommensurable, the above method of expounding the definition is at fault, since there are no

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integers such that $nA = mB$. What, then, is to be understood by *quantuplicity?* If we examine the infinite sequence of multiples of A and B, we may find for any multiple of A, say nA, two multiples of B, say mB and $(m+1)B$ between which the multiple nA lies. But the ratio of A to B is neither the quotient relation between m and n, nor that between $m+1$ and n, so that we do not know what the ratio is; there is, in fact, *no quotient relation* between A and B so long as we are restricted to integers. The word *quantuplicity* is not a whit simpler than *ratio*, and the statement quoted above from the *Syllabus* of the manner of estimating the quantuplicity of A with respect to B seems to me to assume that quantuplicity is a relation between numbers. Now so long as number means *rational number*, there is no relation between the incommensurable magnitudes A and B that can be said to be the same as the relation between two numbers. By a stretch of language the relation may be said to be greater than the relation between m and n, but less than that between $m+1$ and n; but even so, it does not tell us what the relation *is*.

It seems to me, then, that it is impossible to get out of the definition any precise meaning for the case of incommensurable magnitudes. Besides, I quite agree with Barrow that the definition is *metaphysical*; it is only by a considerable strain on the meaning of the word *quantuplicity* that we can get a precise meaning even for the simple case of commensurable magnitudes. And when all is said and done, the definition is not of the slightest use; nothing in the subsequent development of Euclid's theory depends on it, nor does it appear in the definition of proportion. I think it is a complete mistake, when treating ratio in Euclid's manner, to define it in any other way than as "a relation between like magnitudes." What precisely that relation is can only appear after the definition of equality of ratios; when the test of equality has been given, there remains the task of assigning the laws of operation to which it is subject. When that task is completed, it is seen that a ratio possesses in every respect the properties of number, meaning by number, rational and irrational number. No doubt Euclid does not carry its developments to its farthest limits; but it is to be observed that Euclid does not speak of ratios as we do. He does not say "two ratios are equal," they are only "the same," and he considers it necessary *to prove* that ratios obey the general test of equality.

This course was necessary to him from the abstract way in which he defines a ratio, and it seems to me to support Barrow's contention that he did not consider a ratio to be a quantity at all. It belonged to the category of "relations" and not to that of "quantities"; in the language of Barrow, "ratio is not a genus or kind of quantity, nor anything subject to quantity, or anywhere properly attributed to quantity directly by itself, but agrees no otherwise with it, than by a *catachresis* or *metonymy*." (*Lect.* XX., p. 368.) To the obvious objection that the mode of expression, "one ratio is equal to, greater than or less than another ratio" suggests that ratio is a quantity. Barrow replies, the expression really means that when the ratios are reduced to having a common consequent, the *antecedent* of the one is equal to, greater than or less than the *antecedent* of the other. (p. 377.)

It is quite unnecessary to follow Barrow's defence of Euclid any further; but I have discussed Euclid's definition of ratio at considerable length in order to show that when we approach the consideration of ratio from his standpoint, we have to put out of sight altogether the arithmetical conception of a ratio. Whether the quantuplicity definition be Euclid's or not, its absence from Book V. would not render the slightest change necessary in any part of the Book.

But, further, there is a vagueness that is most undesirable in elementary teaching in speaking of a ratio as "a relation." There may perhaps be nothing wrong in saying that the ratio of two magnitudes is a certain relation between them, but all through mathematics it is the *quantitative value* of the ratio that is really meant by the word. Thus we define the sine of an angle as a ratio, and we say that sin 30° is $\frac{1}{2}$. In pure geometry this quantitative idea may not be so prominent, but even there it seems to me to be the radical idea; when we want to make the first proposition of Book VI. quite definite, we are accustomed to say, "if the base be doubled, the triangle is doubled; if the base be halved, the triangle is halved" and so on; it is the quantitative aspect of the relation that is of importance. Hence I contend that for elementary teaching it is essential that the numerical aspect of ratio should be insisted upon, both because that is the important element in mathematics and because it is directly in line with the ordinary use of the word in arithmetic and in common life.

Let us now consider the method usually adopted for proving the theorems in proportion required in the applications of proportion to geometry. Only the definitions of Book V. are supposed to be learned by the pupil; the proofs of the theorems are established algebraically, it being first shown that Euclid's test of proportionality leads to the algebraical test. The reason alleged for this procedure is that Euclid's test is applicable to all magnitudes, whether commensurable or incommensurable, while the algebraical is, in the usual phraseology, applicable "strictly speaking" to commensurable magnitudes alone. Now the only intelligible meaning I can give to this language is, that the ratio of two commensurable magnitudes is a number, or may be expressed by a number, while the ratio of two incommensurable magnitudes is not a number, and can not be expressed by a number. Inexpressibility as a number is, as I understand, the reason for adopting a definition of ratio in terms of quantuplicity.

It is to be borne in mind that Euclid's definition of proportion or of equality of ratios *is in his order of ideas not a theorem capable of proof;* it is a complete misapprehension of Euclid's position to say, as is sometimes done, that his definition of proportion is a theorem. Euclid does not define a ratio as a number, and his definition does not confer properties on it. His definition of equality of ratios is the first step in the process of endowing the abstract relation with definite properties. We are therefore not entitled to assume that a ratio is a quantity homogeneous with number and possessing the same laws of combination as numbers; this is the final and not the first stage in this theory. Seeing that he does not define a ratio as a quantity subject to the laws of algebraic operation and homogeneous with number, it is quite illegitimate to reason about a ratio as if it were an algebraic quantity until the proof has been given that this thing called a ratio, which by hypothesis is not a number, is actually subject to the operative laws of number. Euclid himself is perfectly consistent; he does not even assume that two ratios which are each equal to a third are equal to each other.

Now every method that I have seen of passing from Euclid's test of proportionality to the algebraic test assumes that a ratio is a quantity of the ordinary algebraic type. Thus take § 409 of Todhunter's *Algebra*, which is in substance identical with the corresponding exposition of Barrow, though Barrow states and Todhunter does not state the really unsatisfactory point of the demonstration. Todhunter begins thus:—"Let a, b, c, d be four magnitudes which are proportional according to Euclid's definition: then shall $\frac{a}{b}=\frac{c}{d}$. For if $\frac{a}{b}$ be not equal to $\frac{c}{d}$, one must be greater than the other, and it will be possible to find some fraction which lies between them." Now what is to be understood by the symbols $\frac{a}{b}$, $\frac{c}{d}$? Are these fractions? If so, they are not ratios —unless we assume either that a ratio is a fraction or else that the proof has been previously given that Euclid's ratio may be treated as a fraction. Again a, b, c, d are expressly called *magnitudes*, and yet without the slightest explanation one magnitude is divided by another. Even if a, b are taken to represent not the magnitudes but their measures, they will be, when the magnitudes are incommensurable, irrational numbers, and the whole basis of the Euclidian theory is that only rational numbers are to be employed. Look at the demonstration any way we please, its validity depends on the assumption that a ratio is a quantity of the same nature as a number and subject to the same laws of operation.

But if we go on to § 410 we find that the equation $\frac{a}{b}=\frac{c}{d}$ is "strictly speaking" not an equation at all, for it is said "the algebraical definition is, strictly speaking, confined to commensurable quantities." I will return, in a moment, to the conception latent in this sentence.

It seems to me, then, that the usual method of passing from Euclid's definition to the algebraical, and of then establishing the theorems of Book V. algebraically is thoroughly unsound, and the labour involved in trying to make a pupil understand Euclid's definitions is worse than wasted. But I am prepared to go even farther than this. Suppose that the theorems of Books V. and VI. have been acquired from the text of Euclid and that the pupil goes on to the study of trigonometry. There the trigonometrical functions are defined as ratios, and in every text-book with which I am acquainted, except De Morgan's, Euclid's ratio is treated exactly like an algebraic quantity. Now Euclid himself never reached the position that a ratio is in every case a number, and it was quite outside the range of Greek mathematics to investigate the laws of operation of mathematical symbols. At any rate, after all has been done that Euclid does in Books V. and VI., the problem still remains of co-ordinating ratios with numbers. Thus, consider what is implied in the statement "$\sin 45^\circ = \frac{1}{\sqrt{2}}$." By definition sin 45° is the ratio of the side of a square to its diagonal, and the fourth proposition of Book VI. is appealed to in order to show that the ratio is independent of the size of the square. But on what grounds is it stated that when the side is represented by 1 the diagonal may be represented by the symbol $\sqrt{2}$ and that the ratio is equal to $\frac{1}{\sqrt{2}}$. Euclid did not require to use such a symbol, but every application we make of geometry brings us face to face with the irrational number, and I contend that none of the standard text-books (for De Morgan's is now beyond the reach of most teachers, let alone their pupils) gives any reasonable exposition of the connection between Euclid's ratio and the irrational number. When the whole theory is based upon the supposed impossibility of representing continuous magnitude by number, it is surely necessary to say what is meant when continuous magnitudes are treated as done in trigonometry by the help of number. Some writers go an extreme length in denouncing the treatment of ratio from the numerical standpoint. Thus Nixon (*Euc. Rev.*, 1st Ed., p. 264) makes the statement:—"It is sometimes said that to *compound* ratios is the same as to *multiply* them. This, as a general statement, is quite wrong. The term "multiply" is an arithmetic term, and though applicable to the ratios of commensurable quantities, has no meaning in relation to the ratios of incommensurables." Yet the same writer in his excellent treatise on trigonometry, defines the trigonometrical functions as ratios and "multiplies" them, divides them and treats them in all respects as numbers. What is a pupil to think when he compares the statement in Book VI. with the treatment of ratios in the trigonometry?

The whole difficulty seems to me to lie in the conception of number; in Euclid's theory and in the usual expositions of it, number means rational number, and so long as the conception of number is thus restricted there is really no choice. It surely needs no argument to prove that a symbol which by hypothesis is not a number, has no place in a number-equation until the laws according to which it may be combined with numbers have been investigated. If, as seems to be the case, De Morgan and his followers consider that numerical value can only be expressed by rational numbers, and that arithmetic and algebra are only concerned with such numbers, then Todhunter's proof referred to above really proves nothing at all.

If we are ever to have a theory of proportion that proceeds on other lines than those of Euclid's theory, it is essential to abandon the restriction in regard to number. At bottom, the passage from the rational to the irrational number is identical with that from the ratio of commensurable to the ratio of incommensurable magnitudes; but the difficulties in the one case seem to me much less than in the other. It is a curious study to compare the treatment of irrational numbers in our text-books with that of ratio in geometry; the difficulties are in great part the same, but there is no attempt to deal with irrational numbers in the thorough way Euclid treats ratio. So far as I can make out, the irrational number is really nothing more than a mere symbol; one constantly finds such language as this, "that $\surd 2$ does not exist"; there is a perpetual confusion between $\surd 2$ and rational approximations to it, as if we could approximate to a thing which does not exist!

In De Morgan's "Elements of Trigonometry" this attitude is seen very clearly. On p. 2 it is stated that when the sides of a right-angled triangle contain a, b, c of any linear unit the equation $aa+bb=cc$ does not exist arithmetically when any of the sides is incommensurable with the linear unit. He goes on to explain that the true interpretation of the equation is the proportion

$$aa+bb:1 \quad ::cc:1$$

where $a:1$, $b:1$, $c:1$ are symbols denoting the ratios of the sides to the linear unit, and this interpretation is based on the fully developed Euclidian theory of proportion. There is, no doubt, much that is excellent in De Morgan's exposition, but it seems to me that De Morgan bases the exposition on the conception that numerical value can only be expressed by rational number. What his exposition of Euclid's theory really proves is, that the symbols for ratios are subject to exactly the same laws of operation as the symbols of rational numbers, and perform the same function in distinguishing all magnitudes that rational numbers do in distinguishing commensurable magnitudes. The equation

$$aa + bb = cc$$

exists just as much when a is the symbol for a ratio as when it is the symbol for a number; algebra is in this point of view a calculus of ratios. But there is really no need for this distinction between ratio and number, and as a matter of fact all through analysis the word number is used as equivalent to ratio. Once the fraction is admitted as a number, there is no reason for denying the name to the ratio of incommensurable magnitudes. The notion that numerical value can only be expressed by rational numbers seems to me quite untenable, and the use of such phrases as "arithmetical number" or "existing arithmetically" when the pupil has advanced so far in his studies as to apply algebra to trigonometry, fosters a totally false conception of the essential nature of a number-system. What is he to think when he comes to analytical geometry or the calculus if his number-system is restricted to rational numbers? The equation of a curve is not really an equation; how can a continuous function even be defined? We never meet in our text-books the statement that an independent variable is really a ratio and not a number at all; indeed, it would seem as if an equation to a curve or a differential equation were a mere misnomer. From the strict Euclidian point of view I cannot see that a ratio fails to satisfy any of the laws of algebra, and that it may not take its place, when once it has been fully developed, in any algebraic equation whatever; the whole of analysis would then be a calculus of ratios, most rigorously established, and "ratio" would be the equivalent of "real number" as that word is used in analysis.

I come back then to the original contention that the completely developed ratio of Euclid answers in every respect to the conception of number. For elementary teaching, however, I maintain that it is best to begin where Euclid ends, and before going into any complete theory of proportion the nature of the irrational number should be explained. In this way we follow the natural order; if magnitudes were commensurable there would, I suppose, be no question about treating ratio as a number, and seeing that in all parts of mathematics, not in geometry merely, the irrational number forces itself on our consideration, I think it is much better

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to extend the notion of number and adapt ratio to that, than to set up a new theory of ratio which will end, if it is to be of any use outside pure geometry, in ranking ratio as a number.

Addendum.—I have omitted the rest of the paper as read before the Society; in the omitted portion an attempt was made to present the conception of the irrational number in a form suitable for school use, but as the essentials of the presentation are to be found in easily accessible text-books, it has been thought unncessary to retain the discussion. I think the Society, as a body, should make an effort to place the teaching of proportion on a more logical basis; it is not creditable that the present state of matters should continue. No doubt, the difficulties are considerable; but it would at least be better frankly to acknowledge these, and, if it be found impossible either to develop Euclid's conception to its natural completion or to provide a substitute in the form of a proper treatment of the irrational number, it would be less hurtful to confine all proofs to cases of commensurable magnitudes and to abandon the utterly illogical method so much in vogue of passing from the Euclidian to the algebraic definition of proportion. In any case, I think that in approaching the theory of proportion in geometry, attention should first be confined to commensurable magnitudes, and that the full theory should be taken up after the pupil has gained some familiarity with the processes involved.