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FREE ORTHOLATTICES

GÜNTER BRUNS
IN MEMORIAM: ERNST BRUNS

Introduction. It has been known for some time but does not seem to be anywhere in the literature that the variety of all ortholattices is generated by its finite members (see (4.2) of this paper). This is well known to imply that the word problem for free ortholattices is solvable. On the other hand, it is also known that the solution obtained this way is of no practical use. The main purpose of this paper is to present a workable solution.

Our solution consists in reducing the work problem for free ortholattices to the word problem for free lattices as solved by Whitman [**6**; **7**]. Familiarity with these papers is assumed throughout.

Using Whitman's results we construct in Section 1 a free ortholattice on a set X and we use this construction in Section 2 to give a solution of the word problem. In Section 3 we draw some conclusions from this. Among other things we show that the free ortholattice on two generators contains the free ortholattice on countably many generators as a subalgebra. In the last section we discuss some related results, mostly known (for example, among the lattice theoretists at the University of Massachusetts) and we mention some problems.

J. Schulte-Mönting in his doctoral dissertation, "*Die algebraische Bedeutung der Schnittelimination mit Anwendungen auf Wortprobleme*", Tübingen 1973, has developed another simple algorithm to solve the word problem for free ortholattices. It seems that our result (3.5) is also a consequence of his algorithm, but that our result (3.4) can not be obtained from his without further non-trivial considerations.

My thanks go to Gudrun Kalmbach for many discussions and helpful suggestions during the preparation of this paper.

I am grateful to the referee for suggesting several simplifications of the original version of this paper, in particular the simple proof of (3.2). It was also pointed out to me by the referee that my proof is the application of a proof that is utilized in several papers by G. Grätzer on reduced products, the idea going back to Dilworth (1945).

1. Construction of the free ortholattice. An ortholattice, abbreviated OL, is an algebra L with two binary operations $\vee$, $\wedge$, one unary operation $'$

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and two nullary operations 0, 1, where $(L; \vee, \wedge)$ is a lattice, 0, 1 are the lower and upper bound of L, respectively, and $'$ is an anti-monotone and idempotent complementation. We assume the basic facts about OLs to be known; see, for example, Birkhoff [**2**, p. 52ff],.

Let X be a set. Let X' be a set disjoint from and equipotent with X and let $x \rightsquigarrow x'$ be a one-one map of X onto X'. Extend this map to a one-one map of $X \cup X'$ onto itself by putting $y' = x$ if $y = x'$. Then $x \rightsquigarrow x'$ is idempotent, that is, $x'' = x$ for all $x \in X \cup X'$. Let $F(X \cup X')$ be an algebra with two binary operations $\vee$, $\wedge$, absolutely freely generated by $X \cup X'$. Following Whitman [**6**], define a quasi-ordering $\leqq$ of $F(X \cup X')$ recursively to be the smallest relation satisfying:

(1) $a \leqq a$ if $a \in X \cup X'$,

(2) $a \vee b \leqq c$ if $a \leqq c$ and $b \leqq c$,

(3) $a \leqq b \wedge c$ if $a \leqq b$ and $a \leqq c$,

(4) $a \leqq b \vee c$ if $a \leqq b$ or $a \leqq c$,

(5) $a \wedge b \leqq c$ if $a \leqq c$ or $b \leqq c$.

We adjoin two new elements 0, 1 to $F(X \cup X')$ to obtain a set $\bar{F}(X \cup X') = F(X \cup X') \cup \{0, 1\}$ and we extend the quasi-ordering and the operations $\vee$, $\wedge$ to $\bar{F}(X \cup X')$ by putting, for all $a \in \bar{F}(X \cup X')$:

$$\begin{aligned} &0 \leqq a \leqq 1, \\ &a \vee 0 = 0 \vee a = a \wedge 1 = 1 \wedge a = a, \\ &a \wedge 0 = 0 \wedge a = 0, \\ &a \vee 1 = 1 \vee a = 1. \end{aligned}$$

Since $F(X \cup X')$ is absolutely freely generated by $X \cup X'$ the map $x \rightsquigarrow x'$ of $X \cup X'$ into itself extends to a homomorphism (denoted by the same symbol) of $F(X \cup X')$ into its dual, that is, satisfying

$$(a \vee b)' = a' \wedge b' \quad \text{and} \quad (a \wedge b)' = a' \vee b'.$$

Since the map is idempotent on $X \cup X'$ it is idempotent on all of $F(X \cup X')$. It is furthermore anti-monotone, that is, satisfying:

$$\text{if } a \leqq b, \text{ then } b' \leqq a'.$$

If we put $0' = 1$ and $1' = 0$ it becomes a map with the same properties of $\bar{F}(X \cup X')$ into itself.

Now let $FOL(X)$ be an OL freely generated by X. Since $F(X \cup X')$ is absolutely freely generated by $X \cup X'$ with respect to the operations $\vee$, $\wedge$, the identity map of $X \cup X'$ extends to a $(\vee, \wedge)$-homomorphism α of $F(X \cup X')$ into $FOL(X)$, which may be extended to $\bar{F}(X \cup X')$ by putting

$$\alpha(0) = 0 \quad \text{and} \quad \alpha(1) = 1.$$

It is now easy to see by induction that α also preserves the operation $'$ and is monotone. This means that α is a homomorphism of the full type of OLs of $\bar{F}(X \cup X')$ onto $FOL(X)$.

As the next step to construct a free OL we define an element $a \in F(X \cup X')$ to be reduced recursively as follows:

(i) every element $x \in X \cup X'$ is reduced,
(ii) $a \vee b$ is reduced if and only if a and b are reduced and $a', b' \nleqq a \vee b$,
(iii) $a \wedge b$ is reduced if and only if a and b are reduced and $a', b' \ngeqq a \wedge b$.

Let $R(X \cup X')$ be the set of reduced elements of $F(X \cup X')$ and define $\bar{R}(X \cup X') = R(X \cup X') \cup \{0, 1\}$. Note that an element $a \in F(X \cup X')$ is reduced if and only if a' is reduced, so that $R(X \cup X')$ is closed under $'$. The crucial observation now is the following:

(1.1) *If a is reduced then $a' \nleqq a$.*

Proof (by induction). If $a \in X \cup X'$ the claim is obvious. Assume $a = b \vee c$ and $a' \leqq a$, that is, $b' \wedge c' \leqq b \vee c$. It would follow from conditions (1)–(5) that $b' \leqq b \vee c$ or $c' \leqq b \vee c$, contradicting the reducedness of a. If, finally, $a = b \wedge c$ then $a' \leqq a$ would imply $b' \vee c' \leqq b \wedge c$, hence $b' \leqq b$, contradicting the inductive hypothesis.

(1.1) has the following consequence:

(1.2) *If $a, b \in F(X \cup X')$ and if b is reduced, then $a \nleqq b$ or $a' \nleqq b$.*

Proof. $a \leqq b$ and $a' \leqq b$ would imply $b' \leqq a \leqq b$, contradicting (1.1).

Define now a relation Φ on $\bar{R}(X \cup X')$ by

$$a \,\Phi\, b \text{ if and only if } a \leqq b \text{ and } b \leqq a.$$

It is clear that Φ is an equivalence relation on $\bar{R}(X \cup X')$ and that the quotient $\bar{R}(X \cup X')/\Phi$ with the relation $\leqq$ defined by

$$a/\Phi \leqq b/\Phi \text{ if and only if } a \leqq b,$$

is a partially ordered set with smallest element $0/\Phi$ and largest element $1/\Phi$. Furthermore, for any $a, b \in R(X \cup X')$ we have: If $a \vee b$ is reduced then $(a \vee b)/\Phi$ is the least upper bound of a/Φ and b/Φ in $\bar{R}(X)/\Phi$. If, on the other hand, for elements $a, b \in R(X \cup X')$, $a \vee b$ is not reduced, then $a' \leqq a \vee b$ or $b' \leqq a \vee b$ and for any $c \in R(X \cup X')$, $a, b \leqq c$ would imply $a, a' \leqq c$ or $b, b' \leqq c$, contradicting (1.2). Thus $1/\Phi$ is the least upper bound of a/Φ and b/Φ in this case. For meets the situation is dual. Since for any $a \in R(X \cup X')$, $a \vee a'$ and $a \wedge a'$ are obviously not reduced, it follows in particular that a'/Φ is a complement of a/Φ in $\bar{R}(X \cup X')/\Phi$ and that the map $a/\Phi \rightsquigarrow a'/\Phi$ is an orthocomplementation in the lattice $\bar{R}(X \cup X')/\Phi$. Thus $\bar{R}(X \cup X')/\Phi$ becomes an OL.

Define now a map β: $\bar{F}(X \cup X') \to \bar{R}(X \cup X')$ recursively by $\beta(a) = a$ if $a \in X \cup X' \cup \{0, 1\}$.

$$\beta(a \vee b) = \begin{cases} \beta(a) \vee \beta(b) & \text{if } \beta(a) \vee \beta(b) \text{ is reduced, or } \beta(a) = 0, \text{ or} \\ \beta(b) = 0 & \\ 1 & \text{otherwise} \end{cases}$$

$$\beta(a \wedge b) = \begin{cases} \beta(a) \wedge \beta(b) & \text{if } \beta(a) \wedge \beta(b) \text{ is reduced, or } \beta(a) = 1, \text{ or} \\ \beta(b) = 1 & \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to prove by induction that $\alpha(\beta(a)) = \alpha(a)$ holds for all $a \in \bar{F}(X \cup \bar{X})$. Furthermore, it follows immediately from the definition of β and the above comments concerning joins and meets in $\bar{R}(X \cup X')/\Phi$ that the map $\bar{\beta}$: $\bar{F}(X \cup X') \to \bar{R}(X \cup X')/\Phi$ defined by $\bar{\beta}(a) = \beta(a)/\Phi$ is a homomorphism. As a consequence of this the kernel of $\bar{\beta}$ is contained in the kernel of α and hence there exists a homomorphism γ: $\bar{R}(X \cup X')/\Phi \to FOL(X)$ satisfying $\gamma \circ \bar{\beta} = \alpha$. Since $R'(X \cup X')/\Phi$ is an OL and since $FOL(X)$ is free, it follows that γ is an isomorphism and we obtain:

(1.3) *The OL* $\bar{R}(X \cup X')/\Phi$ *is freely generated by* $\{x/\Phi | x \in X\}$.

2. The word problem for free ortholattices. We use the result (1.3) to give a solution of the word problem for free OLs.

Let again X be an arbitrary set and let $W(X)$ be an algebra of the type of OLs, which is absolutely freely generated by X. Let $\kappa : W(X) \to FOL(X)$ be the homomorphism extending the identity map of X. The word problem for free OLs then is to find an algorithm to decide whether $\kappa(a) = \kappa(b)$ holds for any pair of elements $a, b \in W(X)$. In order not to confuse the notation we let $^\#$ be the unary operation in $W(X)$.

Define $X' = \{x^\# | x \in X\}$ and define in $X \cup X'$ a unary operation $x \rightsquigarrow x'$ by $x' = x^\#$ if $x \in X$ and $x' = y$ if $y \in X$ and $y' = x$. Then the subalgebra of $W(X)$ with respect to the operations $\vee$, $\wedge$ generated by $X \cup X'$ is absolutely freely generated by $X \cup X'$ and hence may be taken as the algebra $F(X \cup X')$ of the last section. Since $W(X)$ is absolutely freely generated by X the identity map of X extends to a homomorphism (with respect to all operations in $W(X)$) $\delta : W(X) \to F(X \cup X')$. Since $\alpha(\delta(x)) = x = \kappa(x)$ holds for all $x \in X$ it follows that $\alpha \circ \delta = \kappa$. Since $\gamma \circ \bar{\beta} = \alpha$ was proved earlier it follows that $\gamma \circ \beta \circ \delta = \kappa$. We thus obtain:

(2.1) *For elements* $a, b \in W(X)$:

$\kappa(a) \leqq \kappa(b)$ *if and only if* $\beta(\delta(a)) \leqq \beta(\delta(b))$.

Note that for $a \in W(X)$, $\delta(a)$ was obtained in a constructive fashion via the homomorphism conditions

$$\delta(x) = x \quad \text{if } x \in X \cup \{0, 1\},$$
$$\delta(a \vee b) = \delta(a) \vee \delta(b),$$
$$\delta(a \wedge b) = \delta(a) \wedge \delta(b),$$
$$\delta(a^{\#}) = \delta(a)'.$$

The conditions (1)–(5) of the last section give a constructive procedure to decide whether $a \leqq b$ holds in $\bar{F}(X \cup X')$ and thus whether an element of $F(X \cup X')$ is reduced. This means that for any $a \in \bar{F}(X \cup X')$, $\beta(a)$ can also be found in a constructive fashion. Finally, it can again be decided constructively whether $a \leqq b$ holds for elements $a, b \in \bar{R}(X \cup X')$. Thus (2.1) gives a solution of the word problem for free *OL*s.

3. Subalgebras of free ortholattices. Let $F(X \cup X')$ have the same meaning as in Section 1. Define a set $Y \subseteq F(X \cup X')$ to be *independent* if and only if $Y \cap Y' = \emptyset$ and the set $Y \cup Y'$ is free in the sense of Whitman [7, Definition 3.1]. In our special case this means that for every finite non-empty subset S of Y and for every element $y \in Y$ the conditions

$$(*) \quad \begin{cases} y \nleqq \bigvee (S - \{y\}) \vee \bigvee S' & \text{and} \\ y \ngeqq \bigwedge (S - \{y\}) \wedge \bigwedge S' \end{cases}$$

hold. (Strictly speaking the right hand sides of these relations are not well-defined since the operations in $F(X \cup X')$ are not associative. But it follows from Whitman [**6**] that the validity of the relations is independent of any bracketing.)

Let Y be an independent set of reduced elements. Let $F(Y \cup Y')$ be the $(\vee, \wedge)$-subalgebra of $F(X \cup X')$ generated by $Y \cup Y'$. By Whitman [7, theorem 5], $F(Y \cup Y')$ is absolutely freely generated by $Y \cup Y'$. We may thus apply the constructions of Section 1 replacing X by Y and $F(X \cup X')$ by $F(Y \cup Y')$. We use for the new notions obtained the symbols for the corresponding old notions and attach an index 0. Thus $\leqq_0$, $R_0(Y \cup Y')$, $\bar{R}_0(Y \cup Y')$ and Φ_0 are well defined. It is now an easy exercise to prove that $\leqq_0$ is nothing but the restrictions of $\leqq$ to $F(Y \cup Y')$. Since the elements of $Y \cup Y'$ are reduced (in the old sense) it follows that the reduced elements of $F(Y \cup Y')$ in the new and in the old sense are the same, i.e. that $R_0(Y \cup Y') = R(X \cup X') \cap F(Y \cup Y')$ holds. This in turn implies that Φ_0 is the restriction Φ to $F(Y \cup Y')$. Using (1.3) and the first isomorphism theorem we obtain:

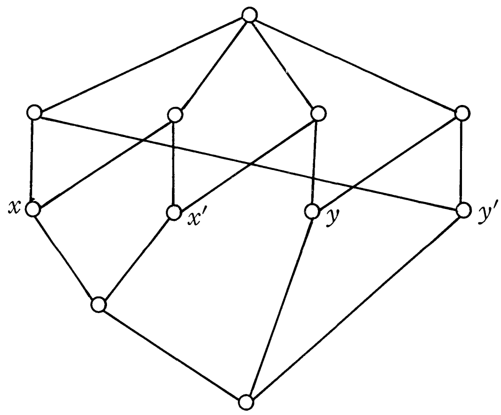
(3.1) *If Y is an independent subset of reduced elements of $F(X \cup X')$ then $\bar{R}_0(Y \cup Y')/\Phi_0$ is an OL freely generated by $\{y/\Phi_0 | y \in Y\}$ and is isomorphic (via the extension of the map $y/\Phi_0 \rightsquigarrow y/\Phi$) with the sub-OL of $\bar{R}(X \cup X')/\Phi$ generated by $\{y/\Phi | y \in Y\}$.*

(3.2) *Assume* $x, y \in X$, $x \neq y$. *Then the elements*

$$a = (x \wedge (x' \vee y)) \vee (y \wedge (x \vee y')),$$
$$b = (x \wedge (x' \vee y')) \vee (y' \wedge (x \vee y)),$$
$$c = (x' \wedge (x \vee y)) \vee (y \wedge (x' \vee y'))$$

are reduced and form an independent set.

Proof. It is easy to check that the elements are reduced. Since the quasi-ordering of $F(X \cup X')$ is preserved under every homomorphism of $F(X \cup X')$ into a lattice it is enough to exhibit a homomorphism of $F(X \cup X')$ into some lattice L such that the relations (*) hold for the images of the elements involved. Such a lattice is the following.



If the elements x, x', y, y' are mapped as indicated one obtains $b \nleq a \vee a' \vee b' \vee c \vee c'$ and $c \nleq a \vee a' \vee b \vee b' \vee c'$. The remaining relations are obtained by permuting x, x', y, y' suitably.

The statements (3.1) and (3.2) give:

(3.3) *The free OL on two generators contains a free OL on three generators as a subalgebra.*

Using Whitman [**7**, Lemma 3.2], one obtains from this:

(3.4) *The free OL on two generators contains a free OL on countably many generators as a subalgebra.*

The next result is concerned with sublattices (not sub-*OL*s) of free *OL*s.

(3.5) *The sublattice of* $FOL(X)$ *generated by* X *is freely generated by* X.

Proof. Since the $(\vee, \wedge)$-subalgebra ΓX of $F(X \cup X')$ generated by X is absolutely freely generated by X it is by Whitman [**6**] and the results of Section 1 enough to show that every element in ΓX is reduced. This is by definition true for elements of X and if it is true for elements $a, b \in \Gamma X$ then $a \vee b \in \Gamma X$

and a', $b' \in \Gamma X'$, thus by Whitman [**7**, Lemma 3.2], $a', b' \nleqq a \vee b$, that is, $a \vee b$ is reduced. The dual argument holds for $a \wedge b$.

Combining (3.4) and (3.5) we obtain furthermore :

(3.6) *A free OL on two generators contains a free lattice on countably many generators as a sublattice.*

4. Related results and open problems. We mention here some results related to the word problem for free algebras in varieties of *OL*s.

(4.1) *Let L be an OL, S a subset of L which is closed under orthocomplementation and C a MacNeille completion of the partially ordered set* $S \cup \{0, 1\}$. *Then C admits an orthocomplementation which extends the orthocomplementation in S.*

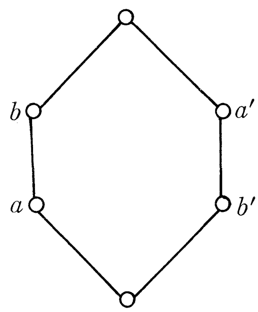
Proof. We consider the standard representation of C by the set of all normal (also, closed) ideals of $S \cup \{0, 1\}$, that is, all those subsets A of $S \cup \{0, 1\}$ which are equal to the set of all lower bounds of the set of all upper bounds of A in $S \cup \{0, 1\}$. For such an ideal A define $A^\perp$ by $A^\perp = \{x \in S \cup \{0, 1\} \mid x \leqq a'$ for all $a \in A\}$. It is easy to see that the map $A \rightsquigarrow A^\perp$ is an orthocomplementation in C with the desired property.

If the set S in (4.1) is finite then C is finite. It follows from this by a standard argument that every equation for *OL*s which is not valid in all *OL*s is not valid in some finite *OL*, hence:

(4.2) *The variety of all OLs is generated by its finite members.*

This implies that the set of all equations which are not valid in all *OL*s is recursively enumerable and again shows the decidability of the word problem for free *OL*s; but the algorithm obtained this way is of no practical use.

More difficult problems seem to be whether the variety of all orthomodular lattices is generated by its finite members and whether the word problem for free algebras in this variety is solvable. Both problems seem to be open. It is, however, easy to see that the method of (4.1) fails in this case. The *OL* is a



sub-partially ordered set of every orthomodular lattice containing a four-element chain. But the lattice is equal to its MacNeille completion and is not orthomodular.

The result (3.5) can be extended to orthomodular lattices as was first noted by G. Kalmbach.

(4.3) *The sublattice generated by X of an orthomodular lattice freely generated by X is freely generated by X.*

This is an immediate consequence of the fact, proved by G. Kalmbach [**5**], that every lattice, in particular a lattice freely generated by a set X, can be embedded as a sublattice into an orthomodular lattice. The orthomodular analogue of (3.4) is wrong since the free orthomodular lattice on two generators is finite. We suspect that the free orthomodular lattice on three generators contains the free orthomodular lattice on countably many generators as a subalgebra, but we could not prove this.

The third variety of *OL*s which is of interest in our context is the variety of all modular *OL*s. I do not know whether the word problem for free algebras in this class is solvable. The class is, however, not generated by its finite members. To see this, define for elements x, y of an *OL*:

$$c(x, y) = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y').$$

One then has:

(4.4) *The equation $c(x, c(y, z)) = 0$ holds in all finite modular OLs but is not valid in the modular OL of all subspaces of a three-dimensional vector space over the real numbers.*

Proof. Every finite, subdirectly irreducible, modular and complemented lattice in which every maximal chain has exactly four elements is a finite projective plane. An orthocomplementation in such a lattice would be a polarity without absolute point, which does not exist by Baer [**1**, Theorem 5]. Using this it follows from Bruns-Kalmbach [**4**, (3.2)], that the only finite, subdirectly irreducible, modular *OL*s are the *OL*s MOn, consisting of $2n$ ($n \geqq 0$) pairwise incomparable elements and the bounds. By Bruns-Kalmbach [**3**, Lemma 2], every such lattice satisfies the equation $c(x, c(y, z)) = 0$, which, as is easily seen, fails in the lattice of subspaces of a three-dimensional vector space over the real numbers.

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McMaster University,
Hamilton, Ontario