ON THE MONOTONICITY OF CERTAIN FUNCTIONALS IN THE THEORY OF ANALYTIC FUNCTIONS

Q.I. Rahman

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Let $f(z)$ be a function regular for $|z| < R$, and denote by $L(r)$ the length of the curve Γ onto which the circle $|z| = r$ is mapped by $f(z)$, i.e.

$$L(r) = r \int_0^{2\pi} |f'(re^{i\theta})| d\theta.$$

If D_r is the image of $|z| \le r$ on the Riemann surface of $f(z)$ then its area $S(r)$ is given by

$$S(r) = \int_0^r \rho \, d\rho \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 d\theta .$$

It has been conjectured by M. Biernacki that $L^2(r)/S(r)$ increases with r. This means that with increasing r the shape of the map of the circle $|z| = r$ deviates monotonically from a circle. The conjecture is still open but we are able to prove the weaker statement that $\delta(r, f', 1) = L^2(r) - 4\pi S(r)$ is strictly increasing for $r \in (0, R)$, unless $f(z) = \frac{az + b}{cz + d}$ $(ad - bc \neq 0)$, when $\delta(r, f', 1) \equiv 0$. This was proved by Krzyz [1, Theorem 4] under the assumption that $f'(z) \neq 0$ in $|z| < R$. We remove this restriction.

What Krzyż proved is clearly equivalent to the following:

THEOREM A. _If_ $f(z)$ _is regular for_ $|z| < R$ _and_ $f(z) \neq 0$ _for_ $|z| < R$, _then the function_

$$\delta(r, f, 1) = r^2 \{ \int_0^{2\pi} |f(re^{i\theta})| d\theta \}^2 - 4\pi \int_0^r \rho \, d\rho \int_0^{2\pi} |f(\rho e^{i\theta})|^2 d\theta$$

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increases with r in $(0, R)$, unless $f(z) = (az+b)^{-2}$, *when* $\delta(r, f, 1) \equiv 0$.

We wish to show that the restriction $f(z) \neq 0$ in $|z| < R$ can be dropped.

If $f(z)$ has no zeros in $|z| < R$, any branch of $\{f(z)\}^p$ where $p > 0$ is also regular for $|z| < R$ and Theorem A can therefore be stated in the following more general form:

THEOREM A'. *Under the conditions of Theorem A and* $p > 0$, *the function*

$$\delta(r, f, p) = r^2 \{ \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\}^2 - 4\pi \int_0^r \rho \, d\rho \int_0^{2\pi} |f(\rho e^{i\theta})|^{2p} d\theta$$

increases with r *in* $(0, R)$, *unless* $f(z) = (az+b)^{-2/p}$, *when* $\delta(r, f, p) \equiv 0$.

We observe that the restriction $f(z) \neq 0$ in $|z| < R$ is unnecessary even for this generalized version of Theorem A.

So we let $f(z)$ have zeros in $|z| < R$, and prove that if $0 \le r_1 < r_2 < R$, $p > 0$, then $\delta(r_1, f, p) < \delta(r_2, f, p)$. For this we distinguish two different cases.

Case (i). Let $f(z)$ have zeros $z_1, z_2, \ldots, z_m$ in $|z| < r_2$ but no zero on $|z| = r_2$. Then the function

$$\phi(z) = \prod_{j=1}^{m} \frac{r_2(z - z_j)}{r_2^2 - \bar{z}_j z}$$

is analytic in the circle $|z| \le r_2$ and $|\phi(z)| = 1$ on $|z| = r_2$. By the maximum modulus principle $|\phi(z)| < 1$ for $|z| < r_2$. There exists a positive number ε such that $f(z)/\phi(z) \neq 0$ for $|z| < r_2 + \varepsilon$. Hence, if $p > 0$, then from Theorem A applied to $\{f(z)/\phi(z)\}^p$ with $R = r_2 + \varepsilon$ it follows that

$$\delta(r_1, \frac{f}{\phi}, p) < \delta(r_2, \frac{f}{\phi}, p),$$

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i.e.

$$r_1^2\{\int_0^{2\pi}|f(r_1e^{i\theta})/\phi(r_1e^{i\theta})|^p\,d\theta\}^2 - 4\pi\int_0^{r_1}\rho\,d\rho\int_0^{2\pi}|f(\rho e^{i\theta})/\phi(\rho e^{i\theta})|^{2p}\,d\theta$$

$$< r_2^2\{\int_0^{2\pi}|f(r_2e^{i\theta})/\phi(r_2e^{i\theta})|^p\,d\theta\}^2 - 4\pi\int_0^{r_2}\rho\,d\rho\int_0^{2\pi}|f(\rho e^{i\theta})/\phi(\rho e^{i\theta})|^{2p}\,d\theta$$

$$= r_2^2\{\int_0^{2\pi}|f(r_2e^{i\theta})|^p\,d\theta\}^2 - 4\pi\int_0^{r_2}\rho\,d\rho\int_0^{2\pi}|f(\rho e^{i\theta})/\phi(\rho e^{i\theta})|^{2p}\,d\theta.$$

Or

$$r_1^2\{\int_0^{2\pi}|f(r_1e^{i\theta})/\phi(r_1e^{i\theta})|^p\,d\theta\}^2 + 4\pi\int_{r_1}^{r_2}\rho\,d\rho\int_0^{2\pi}|f(\rho e^{i\theta})/\phi(\rho e^{i\theta})|^{2p}\,d\theta$$

$$< r_2^2\{\int_0^{2\pi}|f(r_2e^{i\theta})|^p\,d\theta\}^2.$$

Since $|\phi(\rho e^{i\theta})| < 1$ for $0 < \rho < r_2$, $0 \leq \theta < 2\pi$ we get

$$r_1^2\{\int_0^{2\pi}|f(r_1e^{i\theta})|^p\,d\theta\}^2 + 4\pi\int_{r_1}^{r_2}\rho\,d\rho\int_0^{2\pi}|f(\rho e^{i\theta})|^{2p}\,d\theta$$

$$< r_2^2\{\int_0^{2\pi}|f(r_2e^{i\theta})|^p\,d\theta\}^2.$$

But

$$4\pi\int_{r_1}^{r_2}\rho\,d\rho\int_0^{2\pi}|f(\rho e^{i\theta})|^{2p}\,d\theta = -4\pi\int_0^{r_1}\rho\,d\rho\int_0^{2\pi}|f(\rho e^{i\theta})|^{2p}\,d\theta$$

$$+ 4\pi\int_0^{r_2}\rho\,d\rho\int_0^{2\pi}|f(\rho e^{i\theta})|^{2p}\,d\theta.$$

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Therefore

$$r_1^2\{\int_0^{2\pi}|f(r_1e^{i\theta})|^p\,d\theta\}^2 - 4\pi\int_0^{r_1}\rho\,d\rho\int_0^{2\pi}|f(\rho e^{i\theta})|^{2p}\,d\theta$$

$$< r_2^2\{\int_0^{2\pi}|f(r_2\,e^{i\theta})|^p\,d\theta\}^2 - 4\pi\int_0^{r_2}\rho\,d\rho\int_0^{2\pi}|f(\rho e^{i\theta})|^{2p}\,d\theta,$$

i.e.

$$\delta(r_1, f, p) < \delta(r_2, f, p).$$

Case (ii). If $f(z)$ has zeros on $|z| = r_2$, choose r_3 such that $r_1 < r_3 < r_2$ and $f(z)$ has no zero on $|z| = r_3$. From the preceding case it follows that

$$\delta(r_3, f, p) < \delta(r_2 + \varepsilon, f, p)$$

if ε is a sufficiently small positive number. Letting $\varepsilon \to 0$ we get

$$\delta(r_3, f, p) \leq \delta(r_2, f, p).$$

But $\delta(r_1, f, p) < \delta(r_3, f, p)$. Therefore

$$\delta(r_1, f, p) < \delta(r_2, f, p)$$

in this case as well.

Thus we have proved the following:

THEOREM 1. *If* $f(z)$ *is regular for* $|z| < R$ *then for every* $p > 0$, *the function*

$$\delta(r, f, p) = r^2\{\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\}^2 - 4\pi\int_0^{r}\rho\,d\rho\int_0^{2\pi}|f(\rho e^{i\theta})|^{2p}\,d\theta$$

increases with r *in* $(0, R)$, *unless* $f(z) = (az+b)^{-2/p}$, *when* $\delta(r, f, p) \equiv 0$.

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COROLLARY 1. *If* $f(z)$ *is regular for* $|z| < R$ *then*

$$\delta(r, f', 1) = L^2(r) - 4\pi S(r)$$

is strictly increasing for $r \in (0, R)$, *unless* $f(z) = \frac{az + b}{cz + d}$ $(ad - bc \neq 0)$, *when* $\delta(r, f', 1) \equiv 0$.

From the above theorem it follows that for $p > 0$ the derivative of $\delta(r, f, p)$ with respect to r is positive, i.e.

$$2r\left\{\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right\}^2 + 2r^2\left\{\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right\} \frac{d}{dr}\left\{\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right\}$$

$$- 4\pi r \int_0^{2\pi} |f(re^{i\theta})|^{2p} \, d\theta > 0.$$

Hence with

$$I_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta$$

we have the following:

COROLLARY 2. *If* $f(z)$ *is regular for* $|z| < R$, *then for every* $p > 0$, $r \in (0, R)$

$$\frac{I_{2p}(r, f)}{\{I_p(r, f)\}^2} \leq 1 + r \frac{I_p'(r, f)}{I_p(r, f)}.$$

Note that in Corollary 2 equality holds if $f(z)$ is a constant.

The paper of Biernacki and Krzyż (loc. cit.) also contains the following

THEOREM B. *If* $f(z)$ *is regular for* $|z| < R$, $f(z) \neq 0$, *then the quotient*

$$\frac{r^2 I_2(r, f')}{I_2(r, f)} = \frac{\int_0^{2\pi} |rf'(re^{i\theta})|^2 \, d\theta}{\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta}$$

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is a strictly increasing function of $r \in (0, R)$, unless $f(z) = a_n z^n$ ($a_n \neq 0$, n is a non-negative integer), in which case the quotient is constant.

We prove the following stronger

THEOREM 2. If $f(z)$ is regular for $|z| < R$, $f(z) \not\equiv 0$, then the quotient

$$\frac{rI_2(r, f')}{I_2'(r, f)}$$

is a strictly increasing function of $r \in (0, R)$, unless $f(z) = a_o + a_n z_n$ ($a_n \neq 0$, n is a non-negative integer), in which case the quotient is constant.

Proof of Theorem 2. It is well known (see for example [2, pp. 173-174]) that $\log I_2(r, f)$ is a convex function of $\log r$. Noting the case of equality in Schwarz's inequality one can immediately conclude from the proof [2, p. 174] that

$$\frac{d}{d \log r} \log I_2(r, f) \equiv \frac{rI_2'(r, f)}{I_2(r, f)}$$

is strictly increasing unless $f(z)$ is a constant multiple of z^n, when it is, of course, constant.

Now, let $f(z)$ have the representation

$$f(z) = \sum_{n=o}^{\infty} a_n z^n$$

in $|z| < R$. Then the series

$$\sum_{n=o}^{\infty} \sqrt{n}\, a_n z^n$$

also represents a function $F(z)$ regular in $|z| < R$, and hence

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$$\frac{rI_2'(r,F)}{I_2(r,F)}$$

is strictly increasing unless $F(z)$ reduces to $\sqrt{n}\, a_n z^n$, i.e. $f(z) = a_o + a_n z^n$.

But

$$\frac{rI_2'(r,F)}{I_2(r,F)} \equiv \frac{4rI_2(r,f')}{I_2'(r,f)},$$

and therefore the theorem follows.

That Theorem 2 is really stronger than Theorem B follows from the fact that

$$\frac{rI_2(r,f')}{I_2'(r,f)} = \frac{r^2 I_2(r,f')}{I_2(r,f)} \div \frac{rI_2'(r,f)}{I_2(r,f)},$$

and $\frac{rI_2'(r,f)}{I_2(r,f)}$ is non-decreasing.

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Université de Montréal

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