Extensions of Vittas' Theorem

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1. *Introduction*

The Greek architect Kostas Vittas published in 2006 a beautiful theorem ([1]) on the cyclic quadrilateral as follows:

Theorem 1 (Kostas Vittas, 2006): If $ABCD$ is a cyclic quadrilateral with P being the intersection of two diagonals AC and BD, then the four Euler lines of the triangles PAB, PBC, PCD and PDA are concurrent.

A proof of Vittas' theorem and its converse using geometric transformations can be found in [2]. Theorem 1 also has an interesting converse as follows:

Theorem 2 (Converse of Theorem 1): If $ABCD$ is a quadrilateral with P being the intersection of two diagonals AC and BD and the angle between them being different from 60° and 90°, then, if the four Euler lines of the triangles PAB, PBC, PCD and PDA are concurrent, $ABCD$ is a cyclic quadrilateral.

In this Article, we present a new proof to Vittas' theorem. Simultaneously, we establish two important extensions for this theorem. Finally, we introduce a theorem that is general to both Vittas' theorem and its converse. In all the proofs, we use complex coordinates.

We now introduce the first extension of Theorem 1, which is more detailed about the parallel case of Euler lines.

Theorem 3 (More details for Theorem 1): If $ABCD$ is a cyclic quadrilateral with P being the intersection of two diagonals AC and BD, then

(i) the Euler lines of the triangles PAB, PBC, PCD and PDA are concurrent at a point Q;

(ii) the concurrency point Q is at infinity if, and only if, $\angle APB = 60°$ or $\angle APB = 120°$.

Along with that, we give another extension about the locus of the point of concurrency where the two diagonals of the quadrilateral always rotate at a constant angle around a constant point inside a fixed circle.

Theorem 4 (An extension of Theorem 1): If diagonals AC and BD of a cyclic quadrilateral $ABCD$ in a fixed circle are met at the constant point P and the angle $\angle APB = \varphi$ is a constant, then the locus of point Q, common point of the Euler lines of triangles PAB, PBC, PCD and PDA proved previously in Theorem 3, is a circle with centre that lies on the line OP.

https://doi.org/10.1017/mag.2024.9 Published online by Cambridge University Press

Finally, we propose a further generalisation of Vittas' theorem and its converse as follows:

Theorem 5 (Further generalisation of Vittas' theorem and its converse): Let ABC be a triangle and BAC be different from 60°, 90° and 120°. Take two arbitrary points B_1 and C_1 on the lines CA and AB, respectively. If the Euler line of triangle AB_1C_1 passes through a fixed point Q of the Euler line of ABC, then the perpendicular bisector of B_1C_1 passes through a constant point P of the perpendicular bisector of BC and conversely.

Remark: If $ABCD$ is a cyclic quadrilateral with P being the intersection of two diagonals AC and BD, we consider triangle PAB (points C and D lie on the lines PA and PB, respectively). Since perpendicular bisectors of AD, DC and CB pass through a point (circumcentre of $ABCD$), the Euler lines of triangles PAD, PDC and PCB must go through a point lying on the Euler line of triangle PAB. This means that the four Euler lines of triangles PAD, PDC, PCB and PAB are concurrent.

Conversely, consider quadrilateral $ABCD$ with P being the intersection of two diagonals AC and BD (we further assume that the angle formed by the two lines AC and BD is different from 60° and 90°). We also consider triangle PAB (points C and D lie on the lines PA and PB, respectively). If the Euler lines of triangles PAB, PBC and PCD are concurrent, according to the converse part of Theorem 5, the perpendicular bisectors of BC and CD must have a common point lying on the perpendicular bisector of AB, meaning that $ABCD$ is a cyclic quadrilateral.

We now see that Theorem 5 is a generalisation of Vittas' theorem and its converse.

2. *Proofs of theorems*

In this section, we shall introduce proofs of the theorems in the above section using complex numbers. First, we introduce three lemmas about complex numbers; some proofs of these lemmas are already in the references, so we shall not repeat them.

Lemma 1: The intersection P of two chords AC and BD on the unit complex circle is given for their affixes by

$$p = \frac{ac(b + d) - bd(a + c)}{ac - bd}$$

and

$$\bar{p} = \frac{bd(b + d) - ac(a + c)}{bd - ac}.$$

For a proof, see [3].

Without loss of generality, we use the conventions that the counter-clockwise direction is positive, that quadrilaterals and triangles on the unit circle always have a positive direction and that all angles are directional.

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Lemma 2: If the cyclic quadrilateral $ABCD$ is on the unit complex circle and the intersection of the diagonals is the point P, then, writing $\angle APB = \varphi$,

$$bd = ac\left(\cos 2\varphi + i \sin 2\varphi\right).$$

Proof (See Figure 1): It is obvious that $\varphi = \varphi_1 + \varphi_2$, $\angle AOB = 2\varphi_1$, $\angle COD = 2\varphi_2$ and

$$\frac{b}{a} = z_1 = \cos 2\varphi_1 + i \sin 2\varphi_1$$

$$\frac{d}{c} = z_2 = \cos 2\varphi_2 + i \sin 2\varphi_2.$$

Hence $\dfrac{bd}{ac} = z_1 z_2$ or $bd = ac\left(\cos 2\varphi + i \sin 2\varphi\right)$.

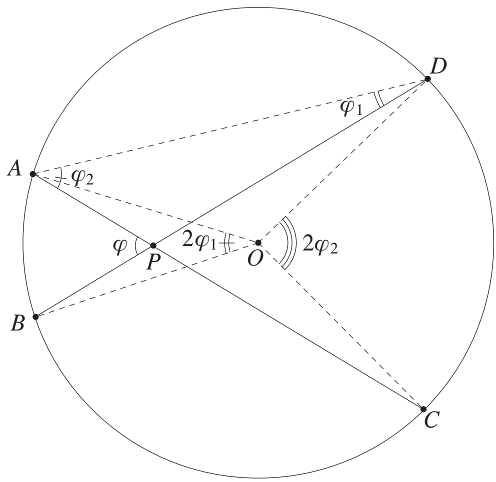

FIGURE 1: Proof of Lemma 2

Lemma 3: The circumcentre o of triangle abc in the complex plane is given by

$$o = -\frac{\begin{vmatrix} a\bar{a} & a & 1 \\ b\bar{b} & b & 1 \\ c\bar{c} & c & 1 \end{vmatrix}}{\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}}.$$

For a proof, see [4].

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Proofs of Theorem 3 *and Theorem* 4 (See Figure 2): Let the circle satisfying the conditions be the unit complex circle and the line OP be the real axis.

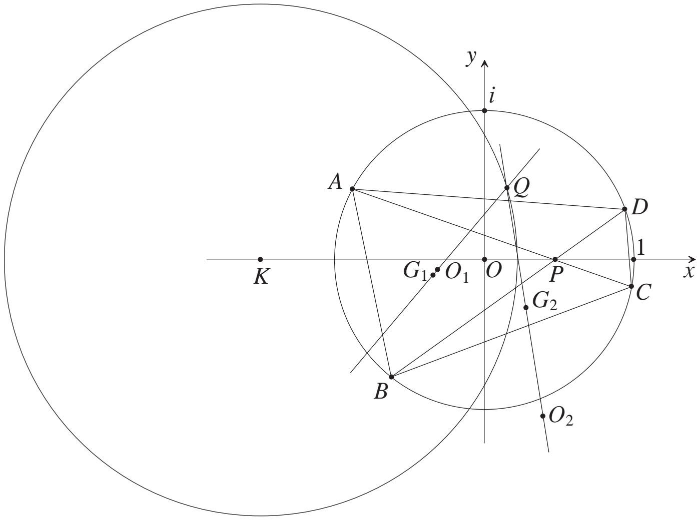

FIGURE 2: Proofs of Theorem 3 and Theorem 4

Let $w = \cos\varphi + i\sin\varphi$; since $\varphi = \angle APB$ from Lemma 2, we have

$$bd = ac\left(\cos 2\varphi + i\sin 2\varphi\right) = acw^2.$$

Using Lemma 1, we have

$$p = \frac{ac(b+d) - bd(c+a)}{ac - bd} \text{ and } \bar{p} = \frac{bd(b+d) - ac(c+a)}{bd - ac}.$$

Using Lemma 3, the circumcentre of triangle PAB is

$$o_1 = -\frac{\begin{vmatrix} p\bar{p} & p & 1 \\ a\bar{a} & a & 1 \\ b\bar{b} & b & 1 \end{vmatrix}}{\begin{vmatrix} p & \bar{p} & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix}} = \frac{\begin{vmatrix} \frac{ac(b+d)-bd(c+a)}{ac-bd}\cdot\frac{bd(b+d)-ac(c+a)}{bd-ac} & \frac{ac(b+d)-bd(c+a)}{ac-bd} & 1 \\ 1 & a & 1 \\ 1 & b & 1 \end{vmatrix}}{\begin{vmatrix} \frac{ac(b+d)-bd(c+a)}{ac-bd} & \frac{bd(b+d)-ac(c+a)}{bd-ac} & 1 \\ a & \frac{1}{a} & 1 \\ b & \frac{1}{b} & 1 \end{vmatrix}}$$

$$= \frac{ab(c - d)}{ac - bd}.$$

Similarly, the circumcentre of PBC is $o_2 = \dfrac{bc(a - d)}{ac - bd}$.

The centroids of these triangles are $g_1 = \frac{1}{3}(p + a + b)$ and $g_2 = \frac{1}{3}(p + b + c)$.

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Therefore the intersection q of two Euler lines g_1o_1 and g_2o_2 is the solution of system

$$\begin{cases} \frac{q - g_1}{q - o_1} = \overline{\left(\frac{q - g_1}{q - o_1}\right)} \\ \frac{q - g_2}{q - o_2} = \overline{\left(\frac{q - g_2}{q - o_2}\right)}. \end{cases}$$

Solving this system, we obtain

$$q = \frac{abcd\left(ac\left(a + c\right) - bd\left(b + d\right)\right)}{(bd)^3 - (ac)^3} \tag{1}$$

or since $bd = acw^2$, we get

$$q = \frac{w^2\left(a + c - w^2\left(b + d\right)\right)}{w^6 - 1}. \tag{2}$$

Theorem 3

(i) From (1) if we substitute $PAB \leftrightarrow PCD$ and $PBC \leftrightarrow PDA$, we conclude that the point q lies also on the Euler lines g_3o_3 and g_4o_4 of triangles PCD and PDA, respectively.

(ii) From (2), the concurrency point q is at infinity (the Euler lines are parallel) if, and only if,

$$w^6 = 1 \Leftrightarrow \cos 6\varphi + i \sin 6\varphi = 1 \Leftrightarrow \varphi = 60° \text{ or } \varphi = 120°.$$

This completes the proof of Theorem 3.

Theorem 4

Since P, A and C are collinear, we have

$$\frac{c - p}{c - a} = \overline{\left(\frac{c - p}{c - a}\right)} = \frac{\frac{1}{c} - p}{\frac{1}{c} - \frac{1}{a}}.$$

This implies $c = \dfrac{p - a}{1 - ap}$ so that $a + c = \dfrac{p\left(1 - a^2\right)}{1 - ap}$.

Since P, B and D are collinear, we have

$$\frac{b - p}{b - d} = \overline{\left(\frac{b - p}{b - d}\right)} = \frac{\frac{1}{b} - p}{\frac{1}{b} - \frac{1}{d}}.$$

This implies $d = \dfrac{p - b}{1 - bp}$ so that

$$b + d = p + pbd = p + pacw^2 = \frac{p - ap^2 + pa(p - a)w^2}{1 - ap}.$$

From this, using (2), we have

$$q = \frac{w^2\left(\dfrac{p\left(1 - a^2\right)}{1 - ap} - w^2\dfrac{p - ap^2 + pa(p - a)w^2}{1 - ap}\right)}{w^6 - 1} = \frac{pw^2\left(a^2 - 1 + a^2w^2 - apw^2\right)}{(1 - ap)\left(w^4 + w^2 + 1\right)}.$$

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Consider the point $k = \frac{-pw^2}{w^4 + w^2 + 1}$, which is a real constant point. Its squared distance from q is

$$|q - k|^2 = (q - k)(\bar{q} - \bar{k}) = \frac{p^2w^2(1 + w^2)^2}{(w^4 + w^2 + 1)^2},$$

which is also a constant. So the locus of Q is a circle with centre $K\left(\frac{-pw^2}{w^4 + w^2 + 1}\right)$ and radius $r^2 = \frac{p^2w^2(1 + w^2)^2}{(w^4 + w^2 + 1)^2}$. Since k is a real number, K lies on the line OP. These complete the proof of Theorem 4.

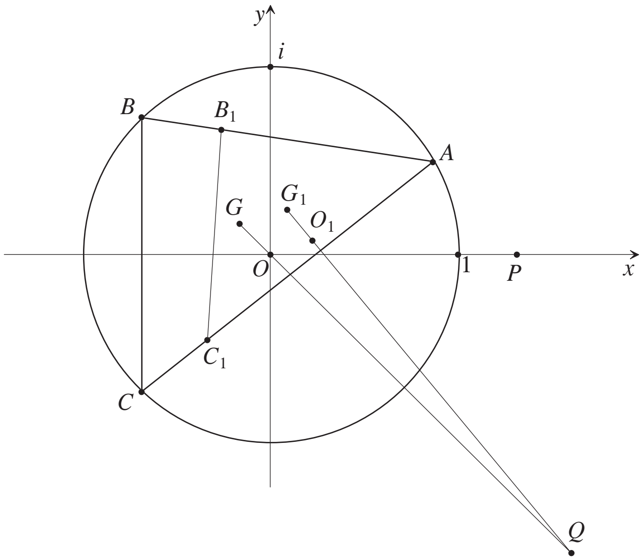

FIGURE 3: Proof of Theorem 5

Proof of Theorem 5 (See Figure 3):

Let ABC be on the unit complex circle, the perpendicular bisector of BC be the real axis, and the perpendicular bisector of B_1C_1 meet the perpendicular bisector of BC at a real point P. We have

$$c = \bar{b} = \frac{1}{b}, \tag{3}$$

$$b_1 = mb + (1 - m)a, \tag{4}$$

$$c_1 = nc + (1 - n)a = \frac{n}{b} + (1 - n)a. \tag{5}$$

Using (3), (4) and (5), we find that

$$g = \frac{a + b + c}{3}, \tag{6}$$

$$g_1 = \frac{a + b_1 + c_1}{3} = \frac{3ab - abm + b^2m + n - abn}{3b} \tag{7}$$

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and the circumcentre (again by using Lemma 3)

$$o_1 = -\frac{\begin{vmatrix} a\bar{a} & a & 1 \\ b_1\overline{b_1} & b_1 & 1 \\ c_1\overline{c_1} & c_1 & 1 \end{vmatrix}}{\begin{vmatrix} a & \bar{a} & 1 \\ b_1 & \overline{b_1} & 1 \\ c_1 & \overline{c_1} & 1 \end{vmatrix}} = \frac{a + bm + ab^2n - ab^2 - am - bn}{(1-b)(1+b)}. \tag{8}$$

Since P lies on the perpendicular bisector of B_1C_1 and using (4), (5), we have

$$|p - b_1|^2 = |p - c_1|^2,$$

which is equivalent to

$$\mathcal{M} = (a-b)^2m^2 - (a-b)(a-b-p+abp)m + (1-ab)n(1-ab-n+abn-ap+bp) = 0. \tag{9}$$

Since Q is the intersection of Euler lines of triangles ABC and AB_1C_1 or the intersection of lines OG and O_1G_1, using (6), (7) and (8), we obtain

$$\mathcal{N} = \mathcal{P}m^2 + \mathcal{Q}m + \mathcal{R} = 0, \tag{10}$$

where

$$\mathcal{P} = (a-b)^2(1+b^2)(1+ab+b^2),$$

$$\mathcal{Q} = -(a-b)\left((1+ab+b^2)(a-2b+2ab^2-b^3) - (1-ab)(1+b^2+b^4)q\right),$$

and

$$\mathcal{R} = (1-ab)n\left((1+ab+b^2)(1-2ab+2b^2-ab^3-(1-ab)(1+b^2)n) - (a-b)(1+b^2+b^4)q\right).$$

Eliminating parameter m from the trinomials (9) and (10), by considering the expression

$$(1+b^2)(1+ab+b^2)\mathcal{M} - \mathcal{N}$$

$$= (m-n)(a-b)(ab-1)\left[b(1+ab+b^2) - (1+b^2)(1+ab+b^2)p + (1+b^2+b^4)q\right]. \tag{11}$$

From (9), (10) and (11) note that with $a \neq b$, $m \neq n$ (because B_1C_1 is not parallel to BC for them to have the intersection Q), and $ab \neq 1$, we deduce that

$$b(1+ab+b^2) - (1+b^2)(1+ab+b^2)p + (1+b^2+b^4)q = 0$$

or

$$bc(a+b+c) - (b+c)(a+b+c)p + (b^2+bc+c^2)q = 0. \tag{12}$$

Since $\angle BAC$ is not 60°, 90° or 120°, so $(b+c)(a+b+c) \neq 0$ and $b^2 + bc + c^2 \neq 0$. Then from (12) we conclude that

- if q is constant, then p is also constant,
- if p is constant, then q is also constant.

This completes the proof.

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Special cases

We still assume that the circle (O) circumcircle of triangle ABC is the unit circle and O is the origin. Let $\angle BAC = \varphi$ and $u = \cos 2\varphi + i \sin 2\varphi$, then $\angle BOC = 2\varphi$; we get $c = bu$ or

$$b^2 + bc + c^2 = b^2 u\left(1 + u + \frac{1}{u}\right) = b^2 u\left(1 + 2\cos 2\varphi\right).$$

If $b + c = 0$ or $\angle BAC = 90°$, then $q \equiv A$, but p is arbitrary, except if B_1C_1 is parallel to BC, in which case p is constant, the orthogonal infinite point of line BC.

If $a + b + c = 0$ or $H = O$ or ABC is equilateral the Euler line of ABC is not defined so we exclude this case.

If $b^2 + bc + c^2 = 0$ or $1 + 2\cos 2\varphi = 0$, which is equivalent to $\varphi = 60°$ or $120°$, then $AH = |2R \cos A| = R = AO$ and $AH_1 = AO_1$. Since AH is isogonal to AO and AH_1 is isogonal to AO_1, the Euler lines are parallel to one bisector of the lines AB and AC. Hence q is constant and p will be constant only when B_1C_1 is parallel to BC.

Acknowledgement

The authors thank the reviewer for the helpful comments.

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10.1017/mag.2024.9
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