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Osculating Varieties of Veronese Varieties and Their Higher Secant Varieties

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Abstract. We consider the k-osculating varieties $O_{k,n.d}$ to the (Veronese) d-uple embeddings of $\mathbb{P}^n$. We study the dimension of their higher secant varieties via inverse systems (apolarity). By associating certain 0-dimensional schemes $Y \subset \mathbb{P}^n$ to $O^s_{k,n,d}$ and by studying their Hilbert functions, we are able, in several cases, to determine whether those secant varieties are defective or not.

1 Introduction

Let us consider the following case of a quite classical problem: given a generic form f of degree d in $R := K[x_0, \dots, x_n]$, what is the minimum s for which it is possible to write $f = L_1^{d-k}F_1 + \cdots + L_s^{d-k}F_s$, where $L_i \in R_1$ and $F_i \in R_k$? When $k = 0$ this is known as the "Waring problem for forms" (the original Waring problem is for integers), and it has been solved via results in [AH], (see also [IK, Ge]).

In this generality, the problem is part of what was classically called "finding canonical forms for an $(n+1)$-ary d-ic" [W]. The following examples illustrate cases where the answer to the problem is not the expected one.

Example 1 One would expect that a generic $f \in (K[x_0, \dots, x_4])_3$ could be written as $f = L_1F_1 + L_2F_2$ with $L_i \in R_1$ and $F_i \in R_2$ (by a dimension count), but actually we need three addenda: $f = L_1F_1 + L_2F_2 + L_3F_3$.

Example 2 We cannot write a generic $f \in (K[x_0, \dots, x_5])_3$ as $f = L_1F_1 + L_2F_2 + L_3F_3$, but only as $f = L_1F_1 + \cdots + L_4F_4$.

Example 3 One would expect that a generic $f \in (K[x_0, \dots, x_6])_4$ could be written as $f = L_1F_1 + L_2F_2 + L_3F_3$, with $L_i \in R_1$ and $F_i \in R_3$, but not only is it impossible to write f as a sum of three addenda, but is it not even possible to write it as a sum of four. In fact f can only be written as $f = L_1F_1 + \cdots + L_5F_5$.

All the examples above comes from Proposition 3.4.

Our approach to the problem is via the study of the dimension of higher secant varieties $O^s_{k,n.d}$ to $O_{k,n.d}$, the k-th osculating variety to the (Veronese) d-uple embeddings of $\mathbb{P}^n$, since giving an answer to this geometrical problem implies getting the solution to the problem on forms.

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We would like to point out that those secant varieties can reach a very high defectiveness (see Example 4 after Proposition 4.4), a phenomenon that does not happen for smooth varieties.

We use inverse system (apolarity) to reduce this problem to the study of the postulation of certain 0-dimensional schemes $Y \subset \mathbb{P}^n$; namely we reduce the evaluation of $\dim O^s_{k,n,d}$ to the evaluation of $\dim |\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Y|$ where $Y = Z_1 + \cdots + Z_s$ is a 0-dimensional subscheme of $\mathbb{P}^n$ such that, for each $i = 1, \ldots, s$, $(k+1)P_i \subset Z_i \subset (k+2)P_i$ and $l(Z_i) = \binom{k+n}{n} + n$.

We conjecture that the "bad behavior" of Y is always related to the scheme given by the fat points $(k+1)P_i$ or $Z_i \subset (k+2)P_i$ not being regular (Conjecture 2). By using this idea, we are able to describe the behavior of the s-th secant variety of $O_{k,n.d}$ for many values of (k, n, d).

In the case of $\mathbb{P}^2$, using known results on fat points, we are able to classify all the defective $O^s_{k,2.d}$ for small values of s ($s \leq 6$ and $s = 9$, see Corollary 4.15).

2 Preliminaries

Notation 2.1

(i) In the following, we set $R := K[x_0, \ldots, x_n]$, where $K = \bar{K}$ and $\operatorname{char} K = 0$, hence R_d will denote the forms of degree d on $\mathbb{P}^n$.

(ii) If $X \subseteq \mathbb{P}^N$ is an irreducible projective variety, an m-fat point on X is the $(m-1)$-th infinitesimal neighborhood of a smooth point P in X, and it will be denoted by mP (*i.e.*, the scheme mP is defined by the ideal sheaf $\mathcal{I}^m_{P,X} \subset \mathcal{O}_X$).
Let $\dim X = n$; then mP is a 0-dimensional scheme of length $\binom{m-1+n}{n}$.
If Z is the union of the $(m-1)$-th infinitesimal neighborhoods in X of s generic points of X, we shall say for short that Z is union of s generic m-fat points on X.

(iii) If $X \subseteq \mathbb{P}^N$ is a variety and P is a smooth point on it, the projectivized tangent space to X at P is denoted by $T_{X,P}$.

(iv) We denote by $\langle U, V \rangle$ both the linear span in a vector space or in a projective space of two linear subspaces U, V.

(v) If X is a 0-dimensional scheme, we denote by $l(X)$ its length, while its support is denoted by $\operatorname{supp} X$.

Definition 2.2 Let $X \subseteq \mathbb{P}^N$ be a closed irreducible projective variety; the $(s-1)$-th *higher secant variety* of X is the closure of the union of all linear spaces spanned by s points of X, and it will be denoted by X^s.

Let $\dim X = n$; the *expected dimension* for X^s is

$$\operatorname{expdim} X^s := \min\{N, sn + s - 1\}$$

where the number $sn + s - 1$ corresponds to ∞^{sn} choices of s points on X, plus ∞^{s-1} choices of a point on the $\mathbb{P}^{s-1}$ spanned by the s points. When this number is too big, we expect that $X^s = \mathbb{P}^N$. Since it is not always the case that X^s has the expected dimension, when $\dim X^s < \min\{N, sn + s - 1\}$, X^s is said to be *defective*.

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A classical result about secant varieties is Terracini's Lemma (see [Te, A]) which we give here in the following form:

Terracini's Lemma *Let X be an irreducible variety in $\mathbb{P}^N$, and let $P_1, \dots, P_s$ be s generic points on X. Then, the projectivised tangent space to X^s at a generic point $Q \in \langle P_1, \dots, P_s \rangle$ is the linear span in $\mathbb{P}^N$ of the tangent spaces T_{X,P_i} to X at P_i, $i = 1, \dots, s$, hence*

$$\dim X^s = \dim\langle T_{X,P_1}, \dots, T_{X,P_s} \rangle.$$

Corollary 2.3 *Let $(X, \mathcal{L})$ be an integral, polarized scheme. If $\mathcal{L}$ embeds X as a closed scheme in $\mathbb{P}^N$, then*

$$\dim X^s = N - \dim h^0(\mathcal{I}_{Z,X} \otimes \mathcal{L})$$

where Z is union of s generic 2-fat points in X.

Proof By Terracini's Lemma, $\dim X^s = \dim\langle T_{X,P_1}, \dots, T_{X,P_s} \rangle$, with $P_1, \dots, P_s$ generic points on X. Since X is embedded in $\mathbb{P}^N = \mathbb{P}(H^0(X, \mathcal{L})^*)$, we can view the elements of $H^0(X, \mathcal{L})$ as hyperplanes in $\mathbb{P}^N$; the hyperplanes which contain a space T_{X,P_i} correspond to elements in $H^0(\mathcal{I}_{2P_i,X} \otimes \mathcal{L})$, since they intersect X in a subscheme containing the first infinitesimal neighborhood of P_i. Hence the hyperplanes of $\mathbb{P}^N$ containing the subspace $\langle T_{X,P_1}, \dots, T_{X,P_s} \rangle$ are the sections of $H^0(\mathcal{I}_{Z,X} \otimes \mathcal{L})$, where Z is the scheme union of the first infinitesimal neighborhoods in X of the points P_i's. ∎

Definition 2.4 Let $X \subset \mathbb{P}^N$ be a variety, and let $P \in X$ be a smooth point. We define the k-th *osculating space to X at P* as the linear space generated by $(k+1)P$, and we denote it by $O_{k,X,P}$; hence $O_{0,X,P} = \{P\}$, and $O_{1,X,P} = T_{X,P}$, the projectivised tangent space to X at P.

Let $X_0 \subset X$ be the dense set of the smooth points where $O_{k,X,P}$ has maximal dimension. The k-th *osculating variety to X* is defined as

$$O_{k,X} = \overline{\bigcup_{P \in X_0} O_{k,X,P}}.$$

3 Osculating Varieties to Veronese Varieties, and Their Higher Secant Varieties

Notation 3.1

(i) We will consider here Veronese varieties, *i.e.*, embeddings of $\mathbb{P}^n$ defined by the linear system of all forms of a given degree d: $\nu_d \colon \mathbb{P}^n \to \mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$. The d-ple Veronese embedding of $\mathbb{P}^n$, *i.e.*, $\operatorname{Im} \nu_d$, will be denoted by $X_{n,d}$.

(ii) In the following, we set $O_{k,n,d} := O_{k,X_{n,d}}$, so that the $(s-1)$-th higher secant variety to the k-th osculating variety to the Veronese variety $X_{n,d}$ will be denoted by $O^s_{k,n,d}$.

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Remark 3.2 From now on $\mathbb{P}^N = \mathbb{P}(R_d)$, and a form M will denote, depending on the situation, a vector in R_d or a point in $\mathbb{P}^N$.

We can view $X_{n,d}$ as given by the map $(\mathbb{P}^n)^* \to \mathbb{P}^N$, where $L \to L^d$, $L \in R_1$. Hence

$$X_{n,d} = \{L^d,\ L \in R_1\}.$$

Let us assume (and from now on this assumption will be implicit) that $d \geq k$; at the point $P = L^d$ we have (see [Se], [CGG, §1], [BF, §2]:

$$(*) \qquad O_{k,X_{n,d},P} = \{L^{d-k}F,\ F \in R_k\}.$$

Notice that $O_{k,X_{n,d},P}$ has maximal dimension $\dim R_k - 1 = \binom{k+n}{n} - 1$ for all $P \in X_{n,d}$. This can also be seen in the following way: the fat point $(k+1)P$ on $X_{n,d}$ gives independent conditions to the hyperplanes of $\mathbb{P}^N$, since it gives independent conditions to the forms of degree d in $\mathbb{P}^n$. Hence, $O_{k,n,d} = \bigcup_{P \in X_{n,d}} O_{k,X_{n,d},P}$.

As we have already noted for $k = 0$, $(*)$ gives $O_{k,X_{n,d},P} = \{P\} = \{L^d\}$, and for $k = 1$, it becomes $O_{k,X_{n,d},P} = T_{X_{n,d},P} = \{L^{d-1}F, \quad F \in R_1\}$. In general, we have:

$$O_{k,n,d} = \{L^{d-k}F,\ L \in R_1,\ F \in R_k\}.$$

Hence,

$$O^s_{k,n,d} = \{L_1^{d-k}F_1 + \cdots + L_s^{d-k}F_s,\ L_i \in R_1,\ F_i \in R_k,\ i = 1, \ldots, s\}.$$

In the following we also need to know the tangent space $T_{O_{k,n,d},Q}$ of $O_{k,n,d}$ at the generic point $Q = L^{d-k}F$ (with $L \in R_1, F \in R_k$); one has that the affine cone over $T_{O_{k,n,d},Q}$ is $W = W(L,F) = \langle L^{d-k}R_k, L^{d-k-1}FR_1\rangle$ (see [CGG, §1], [BF, §2]).

Lemma 3.3 *The dimension of $O_{k,n,d}$ is always the expected one, that is,*

$$\dim O_{k,n,d} = \min\left\{N,\ n + \binom{k+n}{n} - 1\right\}.$$

Proof By Remark 3.2, $\dim O_{k,n,d} = \dim W(L,F) - 1$, for a generic choice of L, F, so that we can assume that L does not divide F. When $\mathbb{P}(W) \neq \mathbb{P}^N$, we have

$$\begin{aligned}\dim W &= \dim L^{d-k}R_k + \dim L^{d-k-1}FR_1 - \dim L^{d-k}R_k \cap L^{d-k-1}FR_1\\ &= \binom{k+n}{n} + (n+1) - 1 = \binom{k+n}{n} + n,\end{aligned}$$

since there is only the obvious relation between LR_k and FR_1, namely $LF - FL = 0$. ∎

Consider the classic Waring problem for forms, *i.e.*, "if we want to write a generic form of degree d as a sum of powers of linear forms, how many of them are necessary?" The problem is completely solved. In fact, $X^s_{n,d} = \{L_1^d + \cdots + L_s^d, L_i \in R_1\}$ (see Remark 3.2), hence the Waring problem is equivalent to the problem of computing $\dim X^s_{n,d}$. By Corollary 2.3 we have that $\dim X^s_{n,d} = N - \dim H^0(\mathcal{I}_{Z,\mathbb{P}^n} \otimes \mathcal{O}(d)) = H(Z,d) - 1$, where Z is a scheme of s generic 2-fat points in $\mathbb{P}^n$, and $H(Z,d)$ is the Hilbert function of Z in degree d. Since $H(Z,d)$ is completely known [AH], we are done.

More generally, one could ask which is the least s such that a form of degree d can be written as $L_1^{d-k}F_1 + \cdots + L_s^{d-k}F_s$, with $L_i \in R_1$ and $F_i \in R_k$ for $i = 1, \dots, s$. Since by Remark 3.2 the variety $O^s_{k,n,d}$ parameterizes exactly the forms in R_d which can be written in this way, this is equivalent to answering the following question for each k, n, d: *Find the least s, for each k, n, d, for which* $O^s_{k,n,d} = \mathbb{P}^N$.

We are interested in a more complete description of the stratification of the forms of degree d parameterized by those varieties. Namely: *Describe all s for which* $O^s_{k,n,d}$ *is defective, i.e. for which*

$$\dim O^s_{k,n,d} < \operatorname{expdim} O^s_{k,n,d}.$$

Notice that, since $d \geq k$, one has $\dim O_{k,n,d} = N$ if and only if $\binom{d+n}{n} \leq n + \binom{k+n}{n}$, hence for all such k, n, d and for any s we have $\dim O^s_{k,n,d} = \operatorname{expdim} O^s_{k,n,d} = N$.

So we have to study this problem when $\binom{d+n}{n} > n + \binom{k+n}{n}$, $s \geq 2$. It is easy to check that whenever $n \geq 2$ this condition is equivalent to $d \geq k+1$. On the other hand, the case $n = 1$ (osculating varieties of rational normal curves) can be easily described (all the $O^s_{k,1,d}$'s have the expected dimension, see next section), so the question becomes:

Question Q(k,n,d): For all k, n, d such that $d \geq k+1$, $n \geq 2$, describe all s for which

$$\dim O^s_{k,n,d} < \min\Big\{ N, s(n + \binom{k+n}{n} - 1) + s - 1 \Big\}$$
$$= \min\Big\{ \binom{d+n}{n} - 1, s\binom{k+n}{n} + sn - 1 \Big\}.$$

Remark 3.4 Terracini's lemma says that $\dim O^s_{k,n,d} = N - h^0(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^N}(1))$, where X is a generic union of 2-fat points on $O_{k,n,d}$. We are not able to handle directly the study of $h^0(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^N}(1))$, nevertheless, Terracini's lemma says that the tangent space of $O^s_{k,n,d}$ at a generic point of $\langle P_1, \dots, P_s \rangle$, $P_i \in O_{k,n,d}$, is the span of the tangent spaces of $O_{k,n,d}$ at P_i. If $T_{O_{k,n,d},P_i} = \mathbb{P}(W_i)$, then

$$\dim O^s_{k,n,d} = \dim \langle T_{O_{k,n,d},P_1}, \dots, T_{O_{k,n,d},P_s} \rangle = \dim \langle W_1, \dots, W_s \rangle - 1.$$

We want to prove, via Macaulay's theory of "inverse systems" [I, IK, Ge, CGG, BF], that for a single W_i, $\dim W_i = N + 1 - h^0(\mathbb{P}^n, \mathcal{I}_Z(d))$, where $Z = Z(k,n)$ is a certain 0-dimensional scheme which we will analyze further, and $\dim\langle W_1, \dots, W_s \rangle = N + 1 - h^0(\mathbb{P}^n, \mathcal{I}_Y(d))$, where $Y = Y(k,n,s)$ is a generic union in $\mathbb{P}^n$ of s 0-dimensional schemes isomorphic to Z. Hence,

$$\dim O^s_{k,n,d} = \dim\langle W_1, \dots, W_s \rangle - 1 = N - h^0(\mathbb{P}^n, \mathcal{I}_Y(d)).$$

So, one strategy in order to answer to the question $Q(k,n,d)$ for a given (k,n,d) is the following:

Step 1: Try to compute directly $\dim\langle W_1,\dots,W_s\rangle$. If this is not possible, then

Step 2: Use the theory of inverse systems (classically *apolarity*): Compute $W^\perp \subset R_d$, with respect to the perfect pairing $\phi : R_d \times R_d \to K$, where:

- W is a vector subspace of R_d,
- $\phi(f,g) := \Sigma_{I\in A_{n,d}} f_I g_I$, where $A_{n,d} := \{(i_0,\dots,i_n) \in \mathbb{N}^{n+1}, \Sigma_j i_j = d\}$, with any fixed ordering; this gives a monomial basis $\{x_0^{i_0}\cdots x_n^{i_n}\}$ for the vector space R_d; if $f \in R_d$, $f = \Sigma_{i_0,\dots,i_n\in A_{n,d}} f_{i_0,\dots,i_n} x_0^{i_0}\cdots x_n^{i_n}$, we write for short $f = \Sigma f_I \mathbf{x}^I$, with $I = (i_0,\dots,i_n)$.

Then, consider $I_d := W^\perp \subset R_d$. It generates an ideal $(I_d) \subset R$. In this way we define the scheme $Z(k,n,d) \subset \mathbb{P}^n$ by setting: $I_{Z(k,n,d)} := (I_d)^{sat}$. We will show that these schemes do not depend on d.

Step 3: Compute the postulation for a generic union of s schemes $Z(k,n,d)$ in $\mathbb{P}^n$.

Recall that $[\langle W_1,\dots,W_s\rangle]^\perp = W_1^\perp \cap\cdots\cap W_s^\perp$.

Lemma 3.5 *For all k, n and $d \geq k+2$, we have:*

$$(k+1)O \subset Z(k,n,d) \subset (k+2)O,$$

where $Z(k,n,d)$ was defined in Remark 3.4*, and $O = \operatorname{supp} Z(k,n,d) \in \mathbb{P}^n$.*

Proof Let $W = \langle L^{d-k}R_k, L^{d-k-1}FR_1\rangle \subset R_d$ be the affine cone over $T_{O_{k,n,d},Q}$ at a generic point $Q = L^{d-k}F$, with $L \in R_1$, $F \in R_k$. Without loss of generality we can choose $L = x_0$, so that $W = x_0^{d-k-1}(x_0R_k + FR_1)$, hence $x_0^{d-k}R_k \subset W \subset x_0^{d-k-1}R_{k+1}$. So, for any (k,n,d),

$$(**)\qquad (x_0^{d-k-1}R_{k+1})^\perp \subset W^\perp \subset (x_0^{d-k}R_k)^\perp.$$

Now, denoting by $\mathfrak{p}$ the ideal $(x_1,\dots,x_n)$, we have:

$$\begin{aligned}(x_0^{d-t}R_t)^\perp &= \langle\{x_0^{i_0}\cdots x_n^{i_n} \mid \Sigma_j i_j = d, i_0 \leq d-t-1\}\rangle\\ &= \langle(\mathfrak{p}^d)_d, x_0(\mathfrak{p}^{d-1})_{d-1},\dots,x_0^{d-t-1}(\mathfrak{p}^{t+1})_{t+1}\rangle = (\mathfrak{p}^{t+1})_d.\end{aligned}$$

Now let us view everything in $(**)$ as the degree d part of a homogeneous ideal; we get:

$$(\mathfrak{p}^{k+2})_d \subset (I_{Z(k,n,d)})_d \subset (\mathfrak{p}^{k+1})_d.$$

Let $(x_1,\dots,x_n)$ be local coordinates in $\mathbb{P}^n$ around the point $O = (1,0,\dots,0)$. The above inclusions give, in terms of 0-dimensional schemes in $\mathbb{P}^n$:

$$(k+1)O \subset Z(k,n,d) \subset (k+2)O. \qquad ■$$

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Lemma 3.6 *For any k, n, d with $d \geq k+2$, the length of $Z = Z(k,n,d)$ is:*

$$l(Z) = \dim W = \binom{k+n}{n} + n.$$

Proof One $(k+2)$-fat point always imposes independent conditions to the forms of degree $d \geq k+1$. Since $Z \subset (k+2)O$, then $h^1(\mathcal{I}_Z(d)) = 0$ immediately follows. ■

Now we have seen that our problem can be translated into a problem of studying certain schemes $Z(k,n,d) \subset \mathbb{P}^n$. We want to check that these schemes are actually the same for all $d \geq k+2$, say $Z(k,n,d) = Z(k,n)$.

Lemma 3.7 *For any k, n and $d \geq k+2$, we have $Z(k,n,d) = Z(k,n,k+2)$. Henceforth we will denote $Z(k,n) = Z(k,n,d)$, for all $d \geq k+2$.*

Proof By the previous lemmata we already know that $Z(k,n,d)$ and $Z(k,n,k+2)$ have the same support and the same length, hence it is enough to show that $Z(k,n,d) \subset Z(k,n,k+2)$ (as schemes) in order to conclude. This will be done if we check that $I(Z(k,n,k+2))_d \subset I(Z(k,n,d))_d$. In fact, since both ideals are generated in degrees $\leq d$, this will imply that $I(Z(k,n,k+2))_j \subset I(Z(k,n,d))_j$, $\forall j \geq d$, hence the inclusion will hold also between the two saturations, implying $Z(k,n,d) \subset Z(k,n,k+2)$.

Let $f \in I(Z(k,n,k+2))_d$, then $f = h_1 g_1 + \cdots + h_r g_r$, where $h_j \in R_{d-k-2}$ and $g_j \in I(Z(k,n,k+2))_{k+2}$. Since $I(Z(k,n,d))_d$ is the perpendicular to $W = \langle L^{d-k} R_k, L^{d-k-1} F R_1 \rangle$, it is enough to check that $h_j g_j \in W^\perp$, $j = 1, \ldots, r$. Without loss of generality we can assume $L = x_0$; hence, since $g_j \in \langle L^2 R_k, L F R_1 \rangle^\perp$, $g_j = x_0 g' + g''$, with $g', g'' \in K[x_1, \ldots, x_n]$ and $g' \in (FR_1)^\perp$. It will be enough to prove $x_0^{i_0} \cdots x_n^{i_n} g_j = x_0^{i_0+1} \cdots x_n^{i_n} g' + x_0^{i_0} \cdots x_n^{i_n} g'' \in W^\perp$, $\forall i_0, \ldots, i_n$ such that $i_0 + \cdots + i_n = d-k-2$. It is clear that $x_0^{i_0} \cdots x_n^{i_n} g'' \in W^\perp$, since $i_0 \leq d-k-2$. On the other hand, $x_0^{i_0+1} \cdots x_n^{i_n} g' \in (x_0^{d-k} R_k)^\perp$ again by looking at the degree of x_0, while $x_0^{i_0+1} \cdots x_n^{i_n} g' \in (x_0^{d-k-1} F R_1)^\perp$ since $g' \in (FR_1)^\perp$. ■

Remark 3.8 From the lemmata above it follows that in order to study the dimension of $O^s_{k,n,d}$ for $d \geq k+2$, we only need to study the postulation of unions of schemes $Z(k,n)$. For $d = k+1$, we will work directly on W, see Proposition 4.4.

What we have is a sort of "generalized Terracini's lemma" for osculating varieties to Veronese varieties, since the formula $\dim O^s_{k,n,d} = N - h^0(\mathcal{I}_Y(d))$ reduces to the one in Corollary 2.3 for $k = 0$. Instead of studying 2-fat points on $O_{k,n,d}$ (see Remark 3.4), we can study the schemes $Y \subset \mathbb{P}^n$.

Notation 3.9 Let $Y \subset \mathbb{P}^n$ be a 0-dimensional scheme; we say that Y is *regular* in degree d, $d \geq 0$, if the restriction map $\rho\colon H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_Y(d))$ has maximal rank, *i.e.*, if $h^0(\mathcal{I}_Y(d)) \cdot h^1(\mathcal{I}_Y(d)) = 0$. We set $\exp h^0(\mathcal{I}_Y(d)) := \max\{0, \binom{d+n}{n} - l(Y)\}$; hence to say that Y is regular in degree d amounts to saying that $h^0(\mathcal{I}_Y(d)) = \exp h^0(\mathcal{I}_Y(d))$.

Since we always have $h^0(\mathcal{I}_Y(d)) \geq \exp h^0(\mathcal{I}_Y(d))$, we write

$$h^0(\mathcal{I}_Y(d)) = \exp h^0(\mathcal{I}_Y(d)) + \delta,$$

where $\delta = \delta(Y, d)$. Hence, whenever $\binom{d+n}{n} - l(Y) \geq 0$, we have $\delta = h^1(\mathcal{I}_Y(d))$. While if $\binom{d+n}{n} - l(Y) \leq 0$, $\delta = \binom{d+n}{n} - l(Y) + h^1(\mathcal{I}_Y(d))$. In any case, by setting $\exp h^1(\mathcal{I}_Y(d)) := \max\{0, l(Y) - \binom{d+n}{n}\}$, we get $h^1(\mathcal{I}_Y(d)) = \exp h^1(\mathcal{I}_Y(d)) + \delta$.

Remark 3.10 For any k, n, d such that $d \geq k+1$, let $Y = Y(k, n, s) \subset \mathbb{P}^n$ be the 0-dimensional scheme defined in Remark 3.4 for $Z = Z(k, n)$, and $\delta = \delta(Y, d)$. Then

$$\dim O^s_{k,n,d} = \operatorname{expdim} O^s_{k,n,d} - \delta.$$

In particular, $\dim O^s_{k,n,d} = \operatorname{expdim} O^s_{k,n,d}$ if and only if

$$h^0(\mathcal{I}_Y(d)) = \begin{cases} 0 & \text{when } \binom{d+n}{n} \leq s\binom{k+n}{n} + sn, \\ N + 1 - l(Y) = \binom{d+n}{n} - s\binom{k+n}{n} - sn^{\dagger} & \text{when } \binom{d+n}{n} \geq s\binom{k+n}{n} + sn. \end{cases}$$

†(i.e., $h^1(\mathcal{I}_Y(d)) = 0$)

4 A Few Results and a Conjecture

First let us consider the cases where the question $Q(k, n, d)$ has already been answered.

Case $\mathbf{Q(k, 1, d)}$

In this case every $O^s_{k,1,d}$, with $d \geq k+2$, has the expected dimension; in fact here $Z(k, 1) = (k+2)O$, and the scheme $Y = \{s\ (k+2)\text{-fat points}\} \subset \mathbb{P}^1$ is regular in any degree d. Notice that for $d = k+1$ we trivially have $O_{k,1,k+1} = \mathbb{P}^N$.

Case $\mathbf{Q(1, n, d)}$

Here the variety $O_{1,n,d}$ is the tangential variety to the Veronese $X_{n,d}$. It is shown in [CGG] that $Z(1, n)$ is a $(2, 3)$-scheme, *i.e.,* the intersection in $\mathbb{P}^n$ of a 3-fat point with a double line. This is easy to see, *e.g.,* by choosing coordinates so that $L = x_0, F = x_1$.

The postulation of generic unions of such schemes in $\mathbb{P}^n$, and hence the defectiveness of $O^s_{1,n,d}$, has been studied. Moreover, a conjecture regarding all defective cases is stated there:

Conjecture 1 ([CGG]) $O^s_{1,n,d}$ is not defective, except in the following cases:

(1) $d = 2$ and $n \geq 2s$, $s \geq 2$;
(2) $d = 3$ and $n = s = 2, 3, 4$.

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In [CGG] the conjecture is proved for $s \le 5$ (any d, n), for $s \ge \frac{1}{3}\binom{n+2}{2} + 1$ (any d, n); for $d = 2$ (any s, n), for $d \ge 3$ and $n \ge s + 1$, for $d \ge 4$ and $s = n$. In [B], the conjecture is proved for $n = 2, 3$ (any s, d).

$\mathbf{Q(2, 2, d)}$. In [BF] it is proved that for any $(s, d) \ne (2, 4)$, $O^s_{2,2,d}$ has the expected dimension.

Now we are going to prove some other cases. The following (quite immediate) lemma describes what can be deduced about the postulation of the scheme Y from information on fat points:

Lemma 4.1 *Let $P_1, \dots, P_s$ be generic points in $\mathbb{P}^n$, and set $X := (k + 1)P_1 \cup \cdots \cup (k + 1)P_s$, $T := (k + 2)P_1 \cup \cdots \cup (k + 2)P_s$. Now let Z_i be a 0-dimensional scheme supported on P_i, $(k + 1)P_i \subset Z_i \subset (k + 2)P_i$, with $l(Z_i) = l((k + 1)P_i) + n$ for each $i = 1, \dots, s$, and set $Y := Z_1 \cup \cdots \cup Z_s$. Then*

(i) *Y is regular in degree d if one of the following holds:*
  (a) $h^1(\mathcal{I}_T(d)) = 0$ *(hence* $\binom{d+n}{n} \ge s\binom{k+n+1}{n}$*).*
  (b) $h^0(\mathcal{I}_X(d)) = 0$ *(hence* $\binom{d+n}{n} \le s\binom{k+n}{n}$*).*

(ii) *Y is not regular in degree d, with defect δ, if one of the following holds:*
  (c) $h^1(\mathcal{I}_X(d)) > \exp h^1(\mathcal{I}_Y(d)) = max\{0, l(Y) - \binom{d+n}{n}\}$*; in this case* $\delta \ge h^1(\mathcal{I}_X(d))) - \exp h^1(\mathcal{I}_Y(d)$.
  (d) $h^0(\mathcal{I}_T(d)) > \exp h^0(\mathcal{I}_Y(d)) = max\{0, \binom{d+n}{n} - l(Y)\}$*; in this case* $\delta \ge h^0(\mathcal{I}_T(d)) - \exp h^0(\mathcal{I}_Y(d))$.

Proof The statement follows by considering the cohomology of the exact sequences:

$$0 \to \mathcal{I}_T(d) \to \mathcal{I}_Y(d) \to \mathcal{I}_{Y,T}(d) \to 0,$$
$$0 \to \mathcal{I}_Y(d) \to \mathcal{I}_X(d) \to \mathcal{I}_{X,Y}(d) \to 0,$$

where we have $h^1(\mathcal{I}_{Y,T}(d)) = h^1(\mathcal{I}_{X,Y}(d)) = 0$, since those two sheaves are supported on a 0-dimensional scheme. ■

Lemma 4.2 *Let $s \ge n + 2$ and $d < k + 1 + 2\frac{k+1}{n}$. Then $O^s_{k,n,d}$ is not defective and $O^s_{k,n,d} = \mathbb{P}^N$.*

Proof Let $Y \subset \mathbb{P}^n$ be as in Remark 3.4. We have to prove that $h^0(\mathcal{I}_Y(d)) = 0$ in our hypotheses.

Let $P_1, \dots, P_s$ be the support of Y. We can always choose a rational normal curve $C \subset \mathbb{P}^n$ containing $n + 2$ of the P_i's . For any hypersurface F given by a section of $\mathcal{I}_Y(d)$, since $nd < (k+1)(n+2)$, by Bezout's theorem we get $C \subset F$. But we can always find a rational normal curve containing $n + 3$ points in $\mathbb{P}^n$, so this would imply that any $P \in \mathbb{P}^n$ is on F, *i.e.*, $\mathcal{I}_Y(d) = 0$. ■

Lemma 4.3 *Assume $s = n + 1$. If $d \le k + 1 + \frac{k+2}{n}$, then $O^s_{k,n,d} = \mathbb{P}^N$.*

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Proof Since $d \geq k+1$, we can set $d = k+j$ with $j > 0$. Let $W_i = \langle L_i^j R_k,\ L_i^{j-1} F_i R_1 \rangle$ with $F_i \in R_k$ for $i = 1, \dots, s$. Since $s = n+1$, without loss of generality we can assume that $L_1 = x_0, \dots, L_{n+1} = x_n$.

Hence $W_1 + \cdots + W_s$ contains $U := x_0^j R_k + \cdots + x_n^j R_k$. Now in U the missing monomials are $x_0^{i_0} \cdots x_n^{i_n}$ with $i_l \leq j-1$ for each $l = 0, \dots, n$, and $d = \deg(x_0^{i_0} \cdots x_n^{i_n}) \leq (n+1)(j-1)$. Hence if $d \geq (n+1)(j-1)$, *i.e.*, $d < k+1+\frac{k+1}{n}$, we get $U = R_d$.

If $d = (n+1)(j-1)$, the only missing monomial in U is $x_0^{j-1} \cdots x_n^{j-1}$, hence it is enough to choose one of the F_i's in a proper way to get $W_1 + \cdots + W_s = R_d$. If $d = (n+1)(j-1) - 1$, *i.e.*, $d = k+1+\frac{k+2}{n}$, the $n+1$ missing monomials in U are $x_0^{j-1} \cdots x_i^{j-2} \cdots x_n^{j-1}$ with $i = 0, \dots, n$ and again it is possible to choose the F_i's so that $W_1 + \cdots + W_s = R_d$. ■

$\mathbf{Q}(\mathbf{k}, \mathbf{n}, \mathbf{k}+\mathbf{1})$. The description for $k = 1$ given in [CGG], together with following proposition, describe this case completely.

Proposition 4.4 *If $s \geq 2$, $k \geq 2$ and $d = k+1$, consider the secant variety $O^s_{k,n,d} \subset \mathbb{P}^N$:*

(i) *If $s \leq n-1$ and its expected dimension is $s\binom{k+n}{n} + sn - 1$, then $O^s_{k,n,k+1}$ is defective with defect $\delta = s^2 - s + s\binom{k+n}{n} + \binom{n-s+d}{d} - N$.*

(ii) *If $s \leq n-1$ and the expected dimension is $N = \binom{d+n}{n} - 1$, then*

  (a) *$O^s_{d-1,n,d}$ is defective with defect $\delta = \binom{n-s+d}{d} - s(n-s+1)$ if $s < \frac{1}{d}\binom{n-s+d}{d-1}$;*

  (b) *$O^s_{d-1,n,d} = \mathbb{P}^N$ if $s \geq \frac{1}{d}\binom{n-s+d}{d-1}$.*

(iii) *If $s \geq n$ then $O^s_{d-1,n,d} = \mathbb{P}^N$.*

Proof (i) We have that $W = W_1 + \cdots + W_s = \langle x_0 R_k, \dots, x_{s-1} R_k; F_1 R_1, \dots, F_s R_1 \rangle$ in R_d. We can suppose that the F_i's, $i = 1, \dots, s$ are generic in $K[x_s, \dots, x_n]_d := S_d$, and we have that $\frac{R_d}{W} \cong \frac{S_d}{(F_1, \dots, F_s)_d}$. Then, since $(F_1, \dots, F_s)_d = \langle F_1 S_1, \dots, F_s S_1 \rangle$ and the F_i's are generic, $\dim(F_1, \dots, F_s)_d = \min\left\{ \binom{n-s+d}{d}, s(n-s+1) \right\}$.

From this, and from our hypothesis about the expected dimension, we immediately get that $\dim W = N - \binom{n-s+d}{d} + s(n-s+1)$, and hence that the defect is $\delta = s^2 - s + s\binom{k+n}{n} + \binom{n-s+d}{d} - N$.

(ii) If $s\binom{n+d-1}{n} + ns \geq \binom{n+d}{n}$, we expect that $O^s_{d-1,n,d} = \mathbb{P}^N$. Proceeding as in the previous case, in order to compute $\dim W$ we can actually consider just the vector space $\langle F_1 S_1, \dots, F_s S_1 \rangle$ whose dimension is $\min\left\{ \binom{n-s+d}{d}, s(n-s+1) \right\}$; so we get that (a) If $s(n-s+1) < \binom{n-s+d}{d}$, then $O^s_{d-1,n,d}$ is defective. This happens if and only if $s < \frac{1}{d}\binom{n-s+d}{d-1}$, in this case the defect is $\delta = \binom{n-s+d}{d} - s(n-s+1)$. (b) If $s(n-s+1) \geq \binom{n-s+d}{d}$, then $O^s_{d-1,n,d} = \mathbb{P}^N$ (for example this is always true for $d \geq n$);

(iii) It suffices to prove that $O^s_{d-1,n,d} = \mathbb{P}^N$ for $s = n$. If $s = n$ and $d = k+1$, the subspace $W_1 + \cdots + W_s$ can be written as $\langle x_0 R_k, F_1 R_1, \dots, x_{n-1} R_k, F_n R_1 \rangle$, which turns out to be equal to $\langle x_0 R_k, \dots, x_{n-1} R_k, x_n^{k+1} \rangle = R_{k+1}$ so $O^n_{d-1,n,d} = \mathbb{P}^N$. ■

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Example 4 (The osculating fourth variety of $X_{6,5} \subset \mathbb{P}^{461}$) Let us consider the secant varieties of the fourth osculating variety $O_{4,6,5}$. We begin with $O^2_{4,6,5}$ (Proposition 4.4(i)) and we expect that $\dim O^2_{4,6,5} = 431$, but we get that the defect is $\delta = 86$ so that $\dim O^2_{4,6,5} = 345$.

When $s = 3, 4$ (Proposition 4.4(ii)), $\delta = 44$ for $O^3_{4,6,5}$, while $\delta = 9$ for $O^4_{4,6,5}$. Eventually, $O^5_{4,6,5} = \mathbb{P}^{461}$ So, even if we expect that $O^3_{4,6,5}$ should fill up $\mathbb{P}^N$, even the 4-secant variety does not.

In terms of forms we get that we can write a generic $f \in (K[x_0, \dots, x_6])_5$ neither as $f = L_1F_1 + L_2F_2 + L_3F_3$ with $L_i \in R_1$ and $F_i \in R_4$ (as we expect), nor as $f = L_1F_1 + \cdots + L_4F_4$, but we need five addenda.

Case $\mathbf{Q(k, 2, k+2)}$

Corollary 4.5 *Assume $d = k+2$ and $n = 2$. Then $O^s_{k,2,k+2}$ is not defective for $s \geq 3$ and $k \geq 1$, and $O^s_{k,2,k+2}$ is defective for $s = 2$ and $k \geq 1$.*

Proof By Lemma 4.2 and Lemma 4.3, $O^s_{k,2,k+2}$ is not defective for $s \geq 3$ and $d \geq 3$, *i.e.,* $k \geq 2$. The case $k = 1$ is already known by [B]. For $s = 2$ and $k \geq 1$, let $Y = Y(k, 2) \subset \mathbb{P}^2$ be the 0-dimensional scheme defined in Remark 3.4. It is easy to check that $\exp h^0(\mathcal{I}_Y(d)) = \exp h^0(\mathcal{I}_T(d)) = 0$, T denoting the generic union of two $(k+2)$-fat points in $\mathbb{P}^2$. Since T is not regular in degree $d = k+2$ for any $k \geq 1$, we conclude by Lemma 4.1(ii)(d) that $O^s_{k,n,k+2}$ is defective with defect $\geq h^0(\mathcal{I}_T(d)) = 1$ (the only section is given by the $(k+2)$-ple line through the two points). ■

Case $\mathbf{Q(k, 3, k+2)}$

Corollary 4.6 *Assume $d = k+2$ and $n = 3$. Then $O^s_{k,3,k+2} = \mathbb{P}^N$ for $s \geq n+1 = 4$ and $k \geq 1$, while $O^s_{k,3,k+2}$ is defective for $s \leq 3$.*

Proof The case $s \leq 3$ will be treated in Proposition 4.10.

If $s = 4$ and $k = 1$, $O^4_{1,3,3} = \mathbb{P}^N$ [CGG, (4.6)]. If $s = 4$ and $k = 2$, we have $O^4_{2,3,4} = \mathbb{P}^N$ by Lemma 4.3. If $s \geq 5$ and $k \geq 1$, or $s = 4$ and $k \geq 3$, the thesis follows by Lemmata 4.2 and 4.3, respectively. ■

Case $\mathbf{Q(k, 4, k+2)}$

Corollary 4.7 *Assume $d = k+2$ and $n = 4$. Then $O^s_{k,4,k+2} = \mathbb{P}^N$ for $s \geq 5$ and $k \geq 1$, while $O^s_{k,4,k+2}$ is defective for $s \leq 4$.*

Proof The case $s \leq 4$ will be given by Proposition 4.10.

If $s \geq 5$ and $k = 1$, $O^s_{1,4,3} = \mathbb{P}^N$ [CGG, (4.6),(4.5)]. If $s = 5$ and $k = 2, 3$, we have $O^5_{k,4,k+2} = \mathbb{P}^N$ by Lemma 4.3. If $s \geq n+2 = 6$ and $k \geq 2$, or $s = 5$ and $k \geq 4$, the thesis follows by Lemmata 4.2 and 4.3, respectively. ■

Case Q(k, 2, k + 3)

Corollary 4.8 *Assume $d = k + 3$ and $n = 2$. Then*

(i) *for $s = 2$ and $k = 1, 2$, $\dim O^2_{k,2,k+3} = s\binom{k+2}{2} + 2s - 1$ (the expected one);*

(ii) *for $s = 2$ and $k \geq 3$, $O^2_{k,2,k+3}$ is defective;*

(iii) *for $s \geq 3$ and $k \geq 1$, $O^s_{k,2,k+3} = \mathbb{P}^N$.*

Proof If $s \geq n+2 = 4$ and $k \geq 2$, or $s = 3$ and $k \geq 4$, the thesis follows by Lemmata 4.2 and 4.3, respectively. If $s \geq 3$ and $k = 1$, $O^s_{1,2,k+3} = \mathbb{P}^N$ [CGG, (4.5)]. If $s = 3$ and $k = 2, 3$, we have $O^2_{k,2,k+3} = \mathbb{P}^N$ by Lemma 4.3. If $s = 2$ and $k = 1$, or $s = 2$ and $k = 2$, $O^2_{k,2,k+3} \neq \mathbb{P}^N$ is not defective, by [CGG, (4.6)] and [BF, Theorem 1], respectively. If $s = 2$ and $k \geq 3$, then, in the notations of Lemma 4.1, we have for $k = 3, 4$ $\exp h^1(\mathcal{I}_X(d)) = \exp h^1(\mathcal{I}_Y(d)) = 0$, and the union X of 2 $(k+1)$-fat points is not regular in degree $d = k+3$. For $k \geq 5$ $\exp h^0(\mathcal{I}_Y(d)) = \exp h^0(\mathcal{I}_T(d)) = 0$, and the union T of 2 $(k+2)$-fat points is not regular in degree $d = k+3$. so we conclude by Lemma 4.1(c) and (d). ■

For $s \leq n+1$, we have several partial results:

Proposition 4.9 *If $s \leq n+1$, $d \geq 2k+1$ and $k \geq 2$, then $O^s_{k,n,d}$ is regular.*

Proof We have to study the dimension of the vector space $W_1 + \cdots + W_s = \langle L_1^{d-k}R_k, L_1^{d-k-1}F_1R_1, \ldots, L_s^{d-k}R_k, L_s^{d-k-1}F_sR_1\rangle$, where $L_1, \ldots, L_s$ are generic in R_1 and $F_1, \ldots, F_s$ are generic in R_k. Since $s \leq n+1$, without loss of generality we may suppose $L_i = x_{i-1}$ for $i = 1, \ldots, s$. Since $d \geq 2k+1$, for $\beta = d - k \geq 3$, the vector space $W_1 + \cdots + W_s$ can be written as $\langle x_0^{\beta}R_k, x_0^{\beta-1}F_1R_1, \ldots, x_{s-1}^{\beta}R_k, x_{s-1}^{\beta-1}F_sR_1\rangle$. If we show that for a particular choice of $F_1, \ldots, F_s \in R_k$ the dimension of $W_1 + \cdots + W_s = \operatorname{expdim}(O^s_{k,n,d}) + 1$ we can conclude by semi-continuity that $O^s_{k,n,d}$ has the expected dimension. Let us consider the case $F_i = x_i x_{i+1}\widetilde{F}_i$ for $i = 1, \ldots, s-2$, $F_{s-1} = x_{s-1}x_0\widetilde{F}_{s-1}$ and $F_s = x_0x_1\widetilde{F}_s$, where the $\widetilde{F}_j$'s are generic forms in R_{k-2}, $j = 1, \ldots, n+1$. Let $\langle x_i^{\beta}R_k\rangle =: A_i$ and $\langle x_i^{\beta-1}F_{i+1}R_1\rangle =: A'_i$, $i = 0, \ldots, s-1$; then we get $A'_i = \langle x_i^{\beta-1}x_{i+1}x_{i+2}\widetilde{F}_{i+1}R_1\rangle$, $i = 0, \ldots, s-3$; $A'_{s-2} = \langle x_{s-2}^{\beta-1}x_{s-1}x_0\widetilde{F}_{s-1}R_1\rangle$ and $A'_{s-1} = \langle x_{s-1}^{\beta-1}x_0x_1\widetilde{F}_sR_1\rangle$. Now $W_1 + \cdots + W_s = \sum_{j=0}^{s-1} A_j + \sum_{j=0}^{s-1} A'_j$. We can easily notice that $A'_i \cap (\sum_{j=0}^{s-1} A_j + \sum_{j=0, j\neq i}^{s-1} A'_j) = A_i \cap (\sum_{j=0, j\neq i}^{s-1} A_j + \sum_{j=0}^{s-1} A'_j) = A_i \cap A'_i = \langle x_i^{\beta}R_k\rangle \cap \langle x_i^{\beta-1}x_{i+1}x_{i+2}\widetilde{F}_{i+1}R_1\rangle = \langle x_i^{\beta}x_{i+1}x_{i+2}\widetilde{F}_{i+1}\rangle$ for any fixed $i = 0, \ldots, s-3$ (analogously if $i = s-2, s-1$). So we have found exactly s relations and we can conclude that $\dim(W_1 + \cdots + W_s) = \dim(\sum_{j=0}^{s-1} A_j) + \dim(\sum_{j=0}^{s-1} A'_j) - s = s\binom{k+n}{n} + s(n+1) - s$, which is exactly the expected dimension. ■

Proposition 4.10 *If $s \leq n$ and $k+2 \leq d \leq 2k$, then $O^s_{k,n,d}$ is defective with defect δ such that*

(i) $\delta \geq \binom{n-s+d}{d}$ *if the expected dimension is* $\binom{d+n}{n} - 1$;

(ii) $\delta \geq \binom{s}{2}\binom{2k-d+n}{n}$ *if the expected dimension is* $s\binom{k+n}{n} + sn - 1$.

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Proof Let $\beta := d-k \geq 2$. We can rewrite the vector space $W_1+\cdots+W_s$ as follows: $\langle x_0^\beta R_k, x_0^{\beta-1}F_1R_1,\dots,x_{s-1}^\beta R_k, x_{s-1}^{\beta-1}F_sR_1\rangle$.

(i) We can observe that $K[x_s,\dots,x_n]_d \cap (W_1+\cdots+W_s)=\{0\}$, so if we expect that $O^s_{k,n,d}=\mathbb{P}^N$ we get a defect $\delta \geq \binom{n-s+d}{d}$.

(ii) Suppose now that $s\big[\binom{k+n}{n}+n\big] < \binom{d+n}{n}$. If $O^s_{k,n,d}$ were to have the expected dimension we would not be able to find more relations among the W_i's other than $x_i^\beta F_{i+1}\in\langle x_i^\beta R_k\rangle\cap\langle x_i^{\beta-1}F_{i+1}R_1\rangle$, for $i=0,\dots,s-1$ (as it happens in Proposition 4.9). But it is easy to see that $x_i^\beta x_j^\beta F\in\langle x_i^\beta R_k\rangle\cap\langle x_j^\beta R_k\rangle$ with $i\neq j$ and $F\in R_{k-\beta}$. We have exactly $\binom{s}{2}$ such terms for any choice of $F\in R_{k-\beta}$. We can also suppose that the $F_i\in R_k$ which appear in $W_1+\cdots+W_s$ are different from $x_j^\beta F$ for any $F\in R_{k-\beta}$ and $j=0,\dots,s-1$, because $F_1,\dots,F_s$ are generic forms of R_k. Then we can be sure that the form $x_i^\beta x_j^\beta F$ belonging to $\langle x_i^\beta R_k\rangle\cap\langle x_j^\beta R_k\rangle$ is not one of the $x_i^\beta F_{i+1}$ which belong to $\langle x_i^\beta R_k\rangle\cap\langle x_i^{\beta-1}F_{i+1}R_1\rangle$. Now $\dim(R_{k-\beta})=\binom{k-\beta+n}{n}$ so we can find $\binom{s}{2}\binom{k-\beta+n}{n}$ independent forms that give defectiveness. Hence in case $s\big[\binom{k+n}{n}+n\big]<\binom{d+n}{n}$ we have $\dim(O^s_{k,n,d})\leq \operatorname{expdim}-\binom{s}{2}\binom{k-\beta+n}{n}=\operatorname{expdim}-\binom{s}{2}\binom{2k-d+n}{n}$. ■

Proposition 4.11 *If $s=n+1$, $k+2\leq d\leq 2k$ and*

$$\operatorname{expdim}(O^{n+1}_{k,n,d})=(n+1)\Big(\binom{k+n}{n}+n\Big)-1,$$

then $O^{n+1}_{k,n,d}$ is defective with defect $\delta\geq\binom{n+1}{2}\binom{2k-d+n}{n}$.

Proof The proof of this fact is the same as Proposition 4.10(ii). ■

Proposition 4.12 *If $s=n+1$, $n\geq\frac{k+2}{d-k-2}$, $k+2<d\leq 2k$ and $\operatorname{expdim}(O^{n+1}_{k,n,d})=N$, then $O^{n+1}_{k,n,d}$ is defective with defect $\delta\geq\binom{(n+1)(d-k-1)-(d+1)}{n}$.*

Proof If $k+2<d\leq 2k$, then $2<\beta:=d-k\leq k$ and we have to study the dimension of $W_1+\cdots+W_{n+1}=\langle x_0^\beta R_k, x_0^{\beta-1}F_1R_1,\dots,x_n^\beta R_k, x_n^{\beta-1}F_{n+1}R_1\rangle$. It is easy to see that a monomial of the form $f=x_0^{\beta_0}\cdots x_n^{\beta_n}$ with $\sum_{i=0}^n\beta_i=d$ and $0\leq\beta_i\leq\beta-2$ for all $i\in\{0,\dots,n\}$ is a form of degree d which does not belong to $W_1+\cdots+W_{n+1}$. In fact f can be written as $x_0^{d-(\gamma_0+k+2)}\cdots x_n^{d-(\gamma_n+k+2)}$ with $\sum_{i=0}^n\gamma_i=nd-(n+1)(k+2)$ and $\gamma_i\geq 0$ for all $i=0,\dots,n$ and these forms are exactly $\binom{n+(n+1)(d-k-2)-d}{n}=\binom{(n+1)(d-k-1)-(d+1)}{n}$. In order for these forms to exist, one needs that $(n+1)(d-k-2)-d\geq 0$, *i.e.*, that $n\geq\frac{k+2}{d-k-2}$. This is sufficient to show that if we expect that $O^{n+1}_{k,n,d}=\mathbb{P}^N$, and if $n\geq\frac{k+2}{d-k-2}$ and $k+2<d\leq 2k$, then $O^{n+1}_{k,n,d}$ is defective.

Let us note that what we just saw is not sufficient to say that the defect δ is exactly equal to $\binom{(n+1)(d-k-1)-(d+1)}{n}$, because in $R_d\backslash\langle W_1+\cdots W_{n+1}\rangle$ we can find also monomials like $x_0^{\beta_0}\cdots x_n^{\beta_n}$ with $\sum_{i=0}^n\beta_i=d$, at least one $\beta_i=\beta-1$ and each of the others $\beta_j\leq\beta-2$. Hence $\delta\geq\binom{(n+1)(d-k-1)-(d+1)}{n}$. ■

All the results on defectiveness lead us to formulate the following:

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Conjecture 2 $O^s_{k,n,d}$ is defective only if Y is as in Lemma 4.1(c) or (d).

The conjecture amounts to saying that the defect of Y can only occur if defect of the fat points schemes X or T imposes it.

Remark 4.13 In many examples the defect of Y is exactly the one imposed by X or by T, *i.e.,* the inequalities on δ in Lemma 4.1 are equalities. But this is not always the case. For example if we consider the variety $O^2_{4,5,6}$ (see Example 4) here, we get that the corresponding scheme Y has defect 86 in degree 5. Here we have that X is given by two 5-fat points in $\mathbb{P}^6$, and it is easy to check that $h^0(\mathcal{I}_X(5)) = 126$ (all quintics through X can be viewed as cones over a quintic of a $\mathbb{P}^4$), so that its defect is 84. Hence, even if Y is "forced" to be defective by X, its defect is bigger, *i.e.,* Y should impose on quintics 12 conditions more than X does, but it imposes only ten conditions more.

It is easy to find similar behavior if $d = k + 1$, for instance for $n = 8$, $s = 3$, $d = k + 1 = 2$ or $n = 10$, $s = 3$, $d = k + 1 = 2$.

In the case of $\mathbb{P}^2$, we are able to prove our conjecture for small values of s:

Theorem 4.14 *Let X, Y be as above, $n = 2$ and $s = 3, 4, 5, 6$ or 9. then*

$$H(Y, d) = \min\left\{ H(X, d) + 2s, \binom{d+2}{2} \right\}.$$

The proof uses mainly the method of Horace on the scheme Y [Hi]. For a detailed proof, see [Be, BC].

Notice that this result implies that Y can be defective only when X is.

In general, it is quite a hard problem to determine, and even to formulate a conjecture upon, the postulation for a union of s m-fat points in $\mathbb{P}^n$.

For what concerns $\mathbb{P}^2$, there is a conjecture for the postulation of a generic union of fat points, [Ha]. For a generic union $A \subset \mathbb{P}^2$ of s m-fat points with $s \geq 10$, the conjecture says that A is regular in any degree d. This has been proved for $m \leq 20$ [CCMO]. For $s \leq 9$ all the defective cases are known (see [Ha] or [CCMO] for a complete list).

This allows us to list all the defective cases for some values of s (for related results see also [BF2]):

Corollary 4.15 *Let $n = 2$, $s \leq 6$ or $s = 9$. Then $O^s_{k,2,d}$ is defective if and only if*

(i) $s = 2$, $k = 1$ *and* $d = 3$, *or* $k \geq 2$ *and* $k + 2 \leq d \leq 2k$,

(ii) $s = 3$, $\frac{3k+5}{2} \leq d \leq 2k$,

(iii) $s = 5$, $2k + 4 \leq d \leq \frac{5k+3}{2}$,

(iv) $s = 6$, $k \equiv 2 \pmod 5$ *and* $\frac{12(k+1)}{5} \leq d \leq \frac{5k+3}{2}$, *or* $k \not\equiv 2 \pmod 5$ *and* $\frac{12(k+1)}{5} + 1 \leq d \leq \frac{5k+3}{2}$.

The case $s = 2$ is given by Corollary 4.8 and Propositions 4.4, 4.9 and 4.10, while the other cases follow from Theorem 4.14 and the classification in [CCMO]. Notice that there are no defective cases for $s = 4$ or $s = 9$. In case $s = 2$ defectiveness is forced exactly by defectiveness of X or T.

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