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INVARIANCE OF THE COEFFICIENTS OF STRONGLY CONVEX FUNCTIONS

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Abstract

Let the function f be analytic in $\mathbb{D} = \{z : |z| < 1\}$ and given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For $0 < \beta \le 1$, denote by $C(\beta)$ the class of strongly convex functions. We give sharp bounds for the initial coefficients of the inverse function of $f \in C(\beta)$, showing that these estimates are the same as those for functions in $C(\beta)$, thus extending a classical result for convex functions. We also give invariance results for the second Hankel determinant $H_2 = |a_2a_4 - a_3^2|$, the first three coefficients of $\log(f(z)/z)$ and Fekete–Szegö theorems.

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1. Introduction and definitions

Let $\mathcal{S}$ be the class of analytic normalised univalent functions f defined on the unit disc $\mathbb{D} = \{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $0 < \beta \le 1$. We say that $f \in \mathcal{S}$ is respectively strongly starlike and strongly convex of order β in $\mathbb{D}$ if and only if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi\beta}{2}$$

and

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi\beta}{2}. \tag{1.2}$$

We denote these classes by $\mathcal{S}^*(\beta)$ and $C(\beta)$, respectively, noting that $\beta = 1$ corresponds to the well-known classes of starlike and convex functions. It is clear that $f \in C(\beta)$ if and only if $zf' \in \mathcal{S}^*(\beta)$ and that both $\mathcal{S}^*(\beta)$ and $C(\beta)$ are subsets of $\mathcal{S}$.

An early paper of Brannan *et al.* [3] established sharp upper bounds for $|a_2|$ and $|a_3|$ for $f \in \mathcal{S}^*(\beta)$, and more recently Ali and Singh [2] found sharp upper bounds for $|a_4|$.

Since $f \in C(\beta)$ if and only if $zf' \in \mathcal{S}^*(\beta)$, these results provide immediate sharp upper bounds for these coefficients when $f \in C(\beta)$.

For any univalent function f, there exists an inverse function f^{-1} defined on some disc $|\omega| < r_0(f)$, with Taylor expansion

$$f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \cdots . \tag{1.3}$$

A classical theorem of Löwner [7] established sharp bounds for the inverse coefficients A_n for all $n \geq 2$ when $f \in \mathcal{S}$, which solves this problem for functions in $\mathcal{S}^*(1)$.

For $\mathcal{S}^*(\beta)$ and $0 < \beta < 1$, the problem of finding bounds for the inverse coefficients seems far from simple, the only sharp results to date being those found by Ali [1] for A_n when $n = 2, 3$ and 4. The difficulty arises since the analysis necessarily involves raising the Taylor series of a function of positive real part to a power. Ali [1] used a technical and well-known property of functions of positive real part, giving useful inequalities for the initial coefficients of the Taylor expansion for functions of positive real part. However, finding similar useful inequalities for subsequent coefficients is much more complicated. Thus, obtaining sharp bounds for the coefficients of functions in $\mathcal{S}^*(\beta)$ for $n \geq 5$ (and $C(\beta)$) becomes more difficult.

For the convex functions $C(1)$, Libera and Zlotkiewicz [6] showed that the sharp classical inequality $|a_n| \leq 1$ remains valid for the inverse coefficients A_n for $2 \leq n \leq 7$. The primary purpose of this paper is to establish sharp inequalities for $|A_n|$ for $n = 2, 3$ and 4 when $f \in C(\beta)$, showing that these sharp upper bounds are also the same as those for the coefficients a_n of functions in $C(\beta)$.

We will also give similar invariance results for the second Hankel determinant $H_2(2)$ of functions in $C(\beta)$. For $q \geq 1$ and $n \geq 1$, the qth Hankel determinant $H_q(n)$ of f is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} \ldots & a_{n+q} \\ \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} \ldots & a_{n+2q-2} \end{vmatrix}.$$

Much attention has been given to finding upper bounds for Hankel determinants whose elements are the coefficients of univalent or multivalent functions (see [4, 5, 8, 9]). The correct order of growth for $H_q(n)$ when $f \in \mathcal{S}$ is as yet unknown [9], whereas exact bounds have been obtained in the case $q = 2$ and $n = 2$ for a variety of subclasses of S, most of these stemming from the method used in [5]. In this paper, we will establish sharp bounds for $H_2(2) = |a_2a_4 - a_3^2|$ for $f \in C(\beta)$ and show that the same result holds for the second Hankel determinant $|A_2A_4 - A_3^2|$ of the inverse coefficients.

The logarithmic coefficients γ_n of a function f given by (1.1) are defined in $\mathbb{D}$ by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n. \tag{1.4}$$

The coefficients γ_n play a central role in the theory of univalent functions. Milin conjectured that for $f \in \mathcal{S}$ and $n \geq 2$,

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \Big(k|\gamma_k|^2 - \frac{1}{k}\Big) \leq 0,$$

and it is not difficult to see that this implies the Bieberbach conjecture. It was this inequality that De Branges established in order to prove the conjecture. Few exact upper bounds for $|\gamma_n|$ have been established, with more attention being given to results in an average sense [2]. Differentiating (1.4) and equating coefficients shows that $2\gamma_1 = a_2$ and so for $f \in \mathcal{S}$ the sharp inequality $|\gamma_1| \leq 1$ follows at once. For starlike functions the sharp estimate $n|\gamma_n| \leq 1$ for $n \geq 2$ is again an immediate consequence of differentiating (1.4).

We will establish sharp coefficient estimates for $|\gamma_n|$ when $n = 1, 2$ and 3 for $f \in C(\beta)$ and show again that these results remain the same for the inverse coefficients. We end the paper with similar invariance results for Fekete–Szegö theorems for functions in $C(\beta)$.

We shall use the following lemmas from [1, 2, 6].

Lemma 1.1. *Suppose that $p \in \mathcal{P}$, the class of functions satisfying Re $p(z) > 0$ for $z \in \mathbb{D}$, with coefficients given by*

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Then, for some complex-valued x, ζ with $|x| \leq 1$, $|\zeta| \leq 1$,

$$\begin{aligned} 2p_2 &= p_1^2 + x(4 - p_1^2), \\ 4p_3 &= p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)\zeta. \end{aligned}$$

Also, $|p_n| \leq 2$ for $n \geq 1$.

Lemma 1.2. *If $p \in \mathcal{P}$, then*

$$\Big|p_2 - \frac{\mu}{2}p_1^2\Big| \leq \max\{2,\ 2|\mu - 1|\} = \begin{cases} 2 & 0 \leq \mu \leq 2, \\ 2|\mu - 1| & \textit{elsewhere.} \end{cases}$$

Also,

$$\big|p_2 - \tfrac{1}{2}p_1^2\big| \leq 2 - \tfrac{1}{2}|p_1^2|.$$

Lemma 1.3. *Let $p \in \mathcal{P}$. If $0 \leq B \leq 1$ and $B(2B - 1) \leq D \leq B$, then*

$$\big|p_3 - 2Bp_1p_2 + Dp_1^3\big| \leq 2.$$

Lemma 1.4. *If $p \in \mathcal{P}$ and $0 \leq B \leq 1$, then*

$$\big|p_3 - 2Bp_1p_2 + Bp_1^3\big| \leq 2.$$

Lemma 1.5. *If $p \in \mathcal{P}$, then*

$$\big|p_3 - (\mu + 1)p_1p_2 + \mu p_1^3\big| \leq \max\{2,\ 2|2\mu - 1|\} = \begin{cases} 2 & 0 \leq \mu \leq 1, \\ 2|2\mu - 1| & \textit{elsewhere.} \end{cases}$$

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2. Coefficients

We begin by stating the sharp results of Brannan *et al.* [3] and Ali and Singh [2] for the coefficients of functions in $S^*(\beta)$. Since $f \in C(\beta)$ if and only if $zf' \in \mathcal{S}^*(\beta)$, these results also give the corresponding sharp inequalities for $C(\beta)$.

THEOREM 2.1. *Suppose that $f \in \mathcal{S}^*(\beta)$ is given by (1.1). Then*

$$|a_2| \le 2\beta, \quad |a_3| \le \begin{cases} \beta & 0 < \beta \le \dfrac{1}{3}, \\ 3\beta^2 & \dfrac{1}{3} \le \beta \le 1, \end{cases} \quad |a_4| \le \begin{cases} \dfrac{2\beta}{3} & 0 < \beta \le \sqrt{\dfrac{2}{17}}, \\ \dfrac{2\beta}{9}(1+17\beta^2) & \sqrt{\dfrac{2}{17}} \le \beta \le 1. \end{cases}$$

THEOREM 2.2. *Suppose that $f \in C(\beta)$ is given by (1.1). Then*

$$|a_2| \le \beta, \quad |a_3| \le \begin{cases} \dfrac{\beta}{3} & 0 < \beta \le \dfrac{1}{3}, \\ \beta^2 & \dfrac{1}{3} \le \beta \le 1. \end{cases} \quad |a_4| \le \begin{cases} \dfrac{\beta}{6} & 0 < \beta \le \sqrt{\dfrac{2}{17}}, \\ \dfrac{\beta}{18}(1+17\beta^2) & \sqrt{\dfrac{2}{17}} \le \beta \le 1. \end{cases}$$

3. Inverse coefficients

We now show that the inequalities for $f \in C(\beta)$ in Theorem 2.2 remain valid for the inverse coefficients A_2, A_3 and A_4.

THEOREM 3.1. *Suppose that $f \in C(\beta)$ with inverse coefficients given by (1.3). Then*

$$|A_2| \le \beta, \quad |A_3| \le \begin{cases} \dfrac{\beta}{3} & 0 < \beta \le \dfrac{1}{3}, \\ \beta^2 & \dfrac{1}{3} \le \beta \le 1, \end{cases} \quad |A_4| \le \begin{cases} \dfrac{\beta}{6} & 0 < \beta \le \sqrt{\dfrac{2}{17}}, \\ \dfrac{\beta}{18}(1+17\beta^2) & \sqrt{\dfrac{2}{17}} \le \beta \le 1. \end{cases}$$

All the inequalities are sharp.

PROOF. Since $f \in C(\beta)$, it follows from (1.2) that there exists $p \in \mathcal{P}$ such that

$$1 + \frac{zf''(z)}{f'(z)} = p(z)^\beta$$

and so, equating coefficients,

$$\begin{aligned} a_2 &= \frac{\beta p_1}{2}, \\ a_3 &= \tfrac{1}{12}(-\beta p_1^2 + 3\beta^2 p_1^2 + 2\beta p_2), \\ a_4 &= \tfrac{1}{144}\beta((4 - 15\beta + 17\beta^2)p_1^3 + 6(-2+5\beta)p_1 p_2 + 12 p_3). \end{aligned} \tag{3.1}$$

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Since $f(f^{-1}(\omega)) = \omega$, comparing coefficients in (1.3) gives

$$\begin{aligned} A_2 &= -a_2 \\ A_3 &= 2{a_2}^2 - a_3 \\ A_4 &= -5{a_2}^3 + 5a_2a_3 - a_4, \end{aligned}$$

which on substituting from (3.1) produces

$$\begin{aligned} A_2 &= -\frac{\beta p_1}{2}, \\ A_3 &= \tfrac{1}{12}\beta((1+3\beta)p_1^2 - 2p_2), \\ A_4 &= -\tfrac{1}{144}\beta((4+15\beta+17\beta^2)p_1^3 - 6(2+5\beta)p_1p_2 + 12p_3). \end{aligned} \tag{3.2}$$

Since $|p_1| \leq 2$, the first inequality in Theorem 3.1 follows at once.

For $|A_3|$, we use Lemma 1.2 as follows. Write

$$|A_3| = \frac{\beta}{6}\left|p_2 - \frac{(1+3\beta)}{2}p_1^2\right|$$

and let $\mu = 1 + 3\beta$ in Lemma 1.2. Then

$$\left|p_2 - \frac{1+3\beta}{2}p_1^2\right| \leq \begin{cases} 2 & 0 < \beta \leq \frac{1}{3}, \\ 6\beta & \frac{1}{3} \leq \beta \leq 1, \end{cases}$$

which gives the inequality for $|A_3|$.

For $|A_4|$, write

$$|A_4| = \frac{\beta}{12}\left|\frac{(4+15\beta+17\beta^2)}{12}p_1^3 - \frac{(2+5\beta)}{2}p_1p_2 + p_3\right|.$$

We apply Lemma 1.3 with $2B = \frac{1}{2}(2+5\beta)$ and $D = \frac{1}{12}(4+15\beta+17\beta^2)$, which gives

$$|A_4| \leq \frac{\beta}{6} \quad \text{provided} \quad 0 < \beta \leq \min\left\{\frac{2}{5}, \sqrt{\frac{8}{41}}, \sqrt{\frac{2}{17}}\right\} = \sqrt{\frac{2}{17}}.$$

For $\sqrt{\frac{2}{17}} \leq \beta \leq \frac{2}{5}$, we apply Lemma 1.4 with $2B = \frac{1}{2}(2+5\beta)$, together with the inequality $|p_1| \leq 2$, to obtain

$$\begin{aligned} |A_4| &\leq \frac{1}{12}\beta\left(\left|p_3 - 2Bp_1p_2 + Bp_1^3\right| + \frac{(17\beta^2-2)}{12}|p_1|^3\right) \\ &\leq \frac{1}{12}\beta\left(2 + \frac{2(17\beta^2-2)}{3}\right) = \frac{\beta}{18}(1+17\beta^2). \end{aligned}$$

Finally, for $\frac{2}{5} \leq \beta \leq 1$, we apply Lemma 1.5 with $\mu + 1 = \frac{1}{2}(2+5\beta)$, so that

$$\begin{aligned} |A_4| &\leq \frac{1}{12}\beta\left(\left|p_3 - (\mu+1)p_1p_2 + \mu p_1^3\right| + \frac{(4-15\beta+17\beta^2)}{12}|p_1|^3\right) \\ &\leq \frac{1}{12}\beta\left(2(5\beta-1) + \frac{2(4-15\beta+17\beta^2)}{3}\right) = \frac{\beta}{18}(1+17\beta^2). \end{aligned}$$

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Clearly, $|A_2|$ is sharp when $p_1 = 2$. The first inequality for $|A_3|$ is sharp when $p_1 = 0$ and $p_2 = 2$, and the second inequality is sharp when $p_1 = 2$ and $p_2 = 2$. Choosing $p_1 = p_2 = 0$ and $p_3 = 2$ shows that the first inequality for $|A_4|$ is sharp, and $p_1 = 2(1+5\beta)/(2+5\beta)$, $p_2 = 2$ and $p_3 = 0$ shows that the second inequality is sharp.

This completes the proof of Theorem 3.1. □

4. Hankel determinants

We next find sharp bounds for the second Hankel determinant of functions in $C(\beta)$.

THEOREM 4.1. *Let $f \in C(\beta)$. Then*

$$H_2(2) \le \begin{cases} \dfrac{\beta^2}{9} & 0 < \beta \le \dfrac{1}{3}, \\ \dfrac{\beta(1+\beta)(1+17\beta)}{72(3+\beta)} & \dfrac{1}{3} \le \beta \le 1. \end{cases}$$

The inequalities are sharp.

PROOF. From the equations (3.1),

$$H_2(2) = |a_2 a_4 - a_3^2|$$
$$= \left| \frac{1}{288}\beta^2(\beta^2 + 3\beta - 2)p_1{}^4 + \frac{1}{144}\beta^2(2-3\beta)p_1{}^2 p_2 + \frac{\beta^2 p_2{}^2}{36} + \frac{1}{24}\beta^2 p_1 p_3 \right|. \quad (4.1)$$

We now use Lemma 1.1 to express p_2 and p_3 in terms of p_1 and, without loss of generality, normalise p_1 so that $p_1 = p$, where $0 \le p \le 2$. From the triangle inequality,

$$H_2(2) \le \tfrac{1}{288}\beta^2(1-\beta^2)p^4 + \tfrac{1}{96}\beta^3 p^2(4-p^2)|y| + \tfrac{1}{96}\beta^2 p^2(4-p^2)|y|^2$$
$$+ \tfrac{1}{144}\beta^2(4-p^2)^2|y|^2 + \tfrac{1}{48}\beta^2 p(4-p^2)(1-|y|^2) := \phi(|y|).$$

Elementary calculus shows that $\phi'(|y|) \ge 0$ when $0 \le |y| \le 1$ and $0 \le p \le 2$. Thus, $\phi(|y|) \le \phi(1)$, and simplifying gives

$$H_2(2) \le \tfrac{1}{288}\beta^2(32 + 4(3\beta - 1)p^2 - \beta(3+\beta)p^4).$$

Again elementary calculus now shows that this expression has a maximum value of $\frac{1}{9}\beta^2$ when $0 < \beta \le \frac{1}{3}$, and a maximum value of

$$\frac{\beta(1+\beta)(1+17\beta)}{72(3+\beta)}$$

when $\frac{1}{3} \le \beta \le 1$. Choosing $p_1 = p_3 = 0$ and $p_2 = 2$ in (4.1) shows that the first inequality is sharp. To show that the second inequality is sharp, choose $p_1 = 2$, $p_3 = 0$ and

$$p_2 = \frac{1}{2\beta(3+\beta)}\left(\beta(3+\beta)(3\beta-2) + \sqrt{\beta(3+\beta)(2+96\beta-54\beta^2-33\beta^3+\beta^4)}\right),$$

noting that $|p_2| \le 2$ for $0 < \beta \le 1$. This completes the proof of Theorem 4.1. □

We now show that the same result holds for the Hankel determinant of the inverse coefficients.

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THEOREM 4.2. *Let $f \in C(\beta)$ and the coefficients of $f^{-1}(\omega)$ be given by (3.2). Then*

$$H_2(2) = |A_2A_4 - A_3^2| \leq \begin{cases} \dfrac{\beta^2}{9} & 0 < \beta \leq \dfrac{1}{3}, \\ \dfrac{\beta(1+\beta)(1+17\beta)}{72(3+\beta)} & \dfrac{1}{3} \leq \beta \leq 1. \end{cases}$$

The inequalities are sharp.

PROOF. The method used in Theorem 4.1 gives the same expression to maximise and so the result follows at once. The first inequality is sharp when $p_1 = 0$ and $p_2 = 2$. The second inequality is sharp when $p_1 = 2$, $p_3 = 0$ and again

$$p_2 = \frac{1}{2\beta(3+\beta)}\Big(\beta(3+\beta)(3\beta-2) + \sqrt{\beta(3+\beta)(2+96\beta-54\beta^2-33\beta^3+\beta^4)}\Big). \qquad \square$$

5. Logarithmic coefficients

We now find sharp estimates for the initial coefficients of $\log(f(z)/z)$ for $f \in C(\beta)$, and for the inverse coefficients of $\log(f(z)/z)$.

THEOREM 5.1. *Let $f \in C(\beta)$ and the coefficients of* $\log(f(z)/z)$ *be given by (1.4). Then*

$$|\gamma_1| \leq \frac{\beta}{2}, \quad |\gamma_2| \leq \begin{cases} \dfrac{\beta}{6} & 0 < \beta \leq \dfrac{2}{3}, \\ \dfrac{\beta^2}{4} & \dfrac{2}{3} \leq \beta \leq 1, \end{cases} \quad |\gamma_3| \leq \begin{cases} \dfrac{\beta}{12} & 0 < \beta \leq \sqrt{\dfrac{2}{5}}, \\ \dfrac{\beta}{36}(1+5\beta^2) & \sqrt{\dfrac{2}{5}} \leq \beta \leq 1. \end{cases}$$

PROOF. Differentiating (1.4) and comparing coefficients gives

$$\gamma_1 = \tfrac{1}{2}a_2, \quad \gamma_2 = \tfrac{1}{2}(a_3 - \tfrac{1}{2}a_2^2), \quad \gamma_3 = \tfrac{1}{2}(a_4 - a_2a_3 + \tfrac{1}{3}a_2^3)$$

and, using (3.1),

$$\begin{aligned} \gamma_1 &= \frac{\beta p_1}{4}, \\ \gamma_2 &= \tfrac{1}{48}\beta(4p_2 + (3\beta-2)p_1^2), \\ \gamma_3 &= \tfrac{1}{288}\beta(12p_3 + 6(3\beta-2)p_1p_2 + (4-9\beta+5\beta^2)p_1^3). \end{aligned} \tag{5.1}$$

The first inequality is trivial. For $|\gamma_2|$, by Lemma 1.2 with $\mu = \frac{1}{2}(2-3\beta)$,

$$\left|p_2 - \frac{2-3\beta}{4}p_1^2\right| \leq \begin{cases} 2 & 0 < \beta \leq \frac{2}{3}, \\ 3\beta & \frac{2}{3} \leq \beta \leq 1. \end{cases}$$

For γ_3, write

$$\gamma_3 = \frac{\beta}{24}(p_3 - 2Bp_1p_2 + Dp_1^3),$$

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where $B = \frac{1}{4}(2 - 3\beta)$ and $D = \frac{1}{12}(4 - 9\beta + 5\beta^2)$. The first inequality now follows at once from Lemma 1.3, noting that the conditions of the lemma are satisfied when $0 < \beta \le \sqrt{\frac{2}{5}}$. For the interval $\sqrt{\frac{2}{5}} \le \beta \le 1$, again write

$$p_3 - 2Bp_1p_2 + Dp_1^3 = p_3 - 2Bp_1p_2 + Dp_1^3 + (D - B)p_1^3.$$

Since $D \ge B$ provided $\sqrt{\frac{2}{5}} \le \beta \le 1$, the second inequality follows using Lemma 1.4.

Choosing $p_1 = 2$ and $p_2 = p_3 = 0$ in (5.1) shows that the inequality for $|\gamma_1|$ is sharp. The first inequality for $|\gamma_2|$ is sharp when $p_1 = 0$ and $p_2 = 2$, and the second when $p_1 = p_2 = 2$. For $|\gamma_3|$, choose $p_1 = p_2 = 0$ and $p_3 = 2$ in the first inequality, and $p_1 = p_2 = p_3 = 2$ in the second. This completes the proof of Theorem 5.1. □

We now show that the same result is true for the logarithmic coefficients of the inverse functions. Since the final expression to maximise differs from that in Theorem 5.1, we include the proof.

Theorem 5.2. *Let $f \in C(\beta)$ and write*

$$\log \frac{f^{-1}(\omega)}{\omega} = 2\sum_{n=1}^{\infty} c_n \omega^n.$$

Then

$$|c_1| \le \frac{\beta}{2}, \quad |c_2| \le \begin{cases} \dfrac{\beta}{6} & 0 < \beta \le \dfrac{2}{3}, \\ \dfrac{\beta^2}{4} & \dfrac{2}{3} \le \beta \le 1, \end{cases} \quad |c_3| \le \begin{cases} \dfrac{\beta}{12} & 0 < \beta \le \sqrt{\dfrac{2}{5}}, \\ \dfrac{\beta}{36}(1 + 5\beta^2) & \sqrt{\dfrac{2}{5}} \le \beta \le 1. \end{cases}$$

Proof. Proceeding as in Theorem 5.1,

$$c_1 = \tfrac{1}{2}A_2, \quad c_2 = \tfrac{1}{2}(A_3 - \tfrac{1}{2}A_2^2), \quad c_3 = \tfrac{1}{2}(A_4 - A_3A_2 + \tfrac{1}{3}A_2^3),$$

which on substitution gives

$$\begin{aligned} c_1 &= -\frac{\beta p_1}{4}, \\ c_2 &= -\tfrac{1}{48}\beta(4p_2 - (2 + 3\beta)p_1^2), \\ c_3 &= -\tfrac{1}{288}\beta(12p_3 - 6(2 + 3\beta)p_1p_2 + (4 + 9\beta + 5\beta^2)p_1^3). \end{aligned} \tag{5.2}$$

Again $|c_1| \le \frac{1}{2}\beta$ is trivial. Lemma 1.2 with $\mu = (2 + 3\beta)/2$ gives

$$|c_2| = \frac{1}{12}\beta\left|p_2 - \frac{2 + 3\beta}{4}p_1^2\right|,$$

which gives the inequality for $|c_2|$. Finally, for c_3, write

$$\frac{1}{288}\beta(12p_3 - 6(2 + 3\beta)p_1p_2 + (4 + 9\beta + 5\beta^2)p_1^3) = \frac{\beta}{24}(c_3 - 2Bp_1p_2 + Dp_1^3),$$

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where $B = \frac{1}{4}(2 + 3\beta)$ and $D = \frac{1}{12}(4 + 9\beta + 5\beta^2)$. Since the conditions of Lemma 1.3 are satisfied when $0 < \beta \le \sqrt{\frac{2}{5}}$, the first inequality follows. Now write

$$p_3 - 2Bp_1p_2 + Dp_1^3 = p_3 - 2Bp_1p_2 + Bp_1^3 + (D - B)p_1^3.$$

Since $D \ge B$ on $\sqrt{\frac{2}{5}} \le \beta \le 1$, the second inequality follows from Lemma 1.4 on noting that $|p_1| \le 2$.

Choosing $p_1 = 2$ and $p_2 = p_3 = 0$ in (5.2) shows that the inequality for $|c_1|$ is sharp. The first inequality for $|c_2|$ is sharp when $p_1 = 0$ and $p_2 = 2$, and the second when $p_1 = p_2 = 2$. For $|c_3|$, choose $p_1 = p_2 = 0$ and $p_3 = 2$ in the first inequality, and $p_1 = p_2 = p_3 = 2$ in the second. This completes the proof of Theorem 5.2. □

6. Fekete–Szegö theorems

We end by stating Fekete–Szegö theorems for the coefficients of $f \in C(\beta)$ and for the inverse function f^{-1}, noting that again they give the same result. We omit the simple proofs.

THEOREM 6.1. *If $f \in C(\beta)$ is given by (1.1), then, for any complex number ν,*

$$\left|a_3 - \nu a_2^2\right| \le \max\left\{\frac{\beta}{3}, \beta^2|1 - \nu|\right\}.$$

The inequalities are sharp when either $p_1 = 0$ and $p_2 = 2$, or $p_1 = p_2 = 2$.

THEOREM 6.2. *If $f \in C(\beta)$ and f^{-1} is given by (1.4), then, for any complex number ν,*

$$\left|A_3 - \nu A_2^2\right| \le \max\left\{\frac{\beta}{3}, \beta^2|1 - \nu|\right\}.$$

The inequalities are sharp when either $p_1 = 0$ and $p_2 = 2$, or $p_1 = p_2 = 2$.

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