A Maass space in higher genus

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Abstract

We show that for arbitrary even genus $2n$ with $n \equiv 0, 1 \pmod 4$ the subspace of Siegel cusp forms of weight $k+n$ generated by the Ikeda lifts of elliptic cusp forms of weight $2k$ can be characterized by certain simple relations among the Fourier coefficients. These generalize the classical Maass relations in genus 2.

1. Introduction

For positive integers n and k, let $S_k(\Gamma_n)$ be the space of Siegel cusp forms of weight k with respect to the full Siegel modular group $\Gamma_n := Sp_n(\mathbb{Z}) \subset GL_{2n}(\mathbb{Z})$ of genus n.

Suppose that $n = 2$ and k is even. Then as is well known there exists a certain Hecke-invariant subspace $S_k^*(\Gamma_2) \subset S_k(\Gamma_2)$ called the Maass space which is Hecke-equivariantly isomorphic to $S_{2k-2}(\Gamma_1)$ ('Saito–Kurokawa lift'). If $F \in S_k^*(\Gamma_2)$ is a Hecke eigenform, then its spinor zeta function is equal to

$$\zeta(s - k + 1)\zeta(s - k + 2)L(f, s),$$

where $L(f, s)$ is the Hecke L-function of a normalized Hecke eigenform $f \in S_{2k-2}(\Gamma_1)$. As is easy to see by a formal calculation and is well known, the standard zeta function of F then equals

$$\zeta(s)L(f, s + k - 1)L(f, s + k - 2).$$

The space $S_k^*(\Gamma_2)$ can be explicitly described as the image of a certain linear map from the 'plus subspace' $S_{k-\frac{1}{2}}^+$ of the space of cusp forms of weight $k - \frac{1}{2}$ and level 4 to $S_k(\Gamma_2)$ and can also be characterized in terms of certain linear relations among the Fourier coefficients of the forms ('Maass relations'). This was intensively studied in the late 1970s and early 1980s by Andrianov, Eichler and Zagier, Maass, Saito and Kurokawa, Zagier and others; for a survey of some of the results, see e.g. [EZ85, ch. II, § 6].

Recently, in a significant breakthrough, Ikeda [Ike01] generalized the Saito–Kurokawa lift to higher genus. More precisely, for $f \in S_{2k}(\Gamma_1)$ a normalized Hecke eigenform and $n \equiv k \pmod 2$, he explicitly constructed a Hecke eigenform $F \in S_{k+n}(\Gamma_{2n})$ such that its standard zeta function is equal to

$$\zeta(s) \prod_{j=1}^{2n} L(f, s + k + n - j),$$

where again $L(f, s)$ is the Hecke L-function of f. The existence of such an F was previously conjectured by Duke and Imamoglu.

In [Koh02], generalizing the corresponding result in genus 2, one of the authors gave an explicit linear version of Ikeda's lifting map as a map from $S_{k+\frac{1}{2}}^+$ to $S_{k+n}(\Gamma_{2n})$ where $S_{k+\frac{1}{2}}^+$ again denotes the 'plus subspace' of forms of weight $k + \frac{1}{2}$ and level 4.

In [Koh02] a conjectural description of the image of this lifting map in terms of conditions on the Fourier coefficients was also given. In the case $n = 1$, this condition comes down to saying that the Tth Fourier coefficients of a form in the Maass space, where T is a rational, half-integral, symmetric, positive definite $(2, 2)$-matrix, for T primitive depend only on the discriminant of T (see [Koh02, § 8, Proposition 4]). Note that in the latter case this description holds true and in fact is a well-known classical result of Zagier [Zag81].

The main purpose of this paper is to prove the conjecture of [Koh02] for all n with $n \equiv 0, 1$ (mod 4) (Theorem 3) and as a consequence in these cases give a characterization of the image of the lifting in terms of certain linear relations among the Fourier coefficients, generalizing the classical Maass relations (Corollary 4).

The proof proceeds by using certain specific results of [Koh02] on the Fourier coefficients of Ikeda-lifted forms combined with projecting a Siegel cusp form to one of its Fourier–Jacobi coefficients ϕ_{T_0} of index T_0, where T_0 is a certain special matrix of size $2n - 1$ and producing forms in fact in $S^+_{k+\frac{1}{2}}$ in this way, analogously to the case of genus 2. Note that such a proof was partly suggested in [Koh02].

The assumption $n \equiv 0, 1$ (mod 4) seems to be essential to our arguments, since the existence of T_0 can be guaranteed only in the latter cases. Of course, this does not mean that our proof cannot be generalized to the cases $n \equiv 2, 3$ (mod 4) by modifying the arguments slightly in a somewhat different direction.

In § 2, we will recall several facts on certain polynomials attached to rational, symmetric matrices of even size that enter in connection with Ikeda's lifting map. In § 3, we will recall Ikeda's lifting theorem and its linear version given in [Koh02]. Section 4 contains the statement of our results, while their proofs are given in § 5.

Notation. If A and B are matrices of appropriate sizes over a commutative ring R, we put $A[B] := B'AB$ where B' is the transpose of B.

If A and B are square matrices over R, we often write $A \oplus B$ for the diagonal block matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

If A is a square matrix over R, we call A even if all its diagonal elements are in $2R$. If R is an integral domain of characteristic different from 2, the matrix $\frac{1}{2}A$ then will be called half-integral.

We denote by $R^{(n,1)}$ the set of column vectors of length n with components in R.

If a real, symmetric matrix T is positive definite, we write $T > 0$.

For a positive integer k, we let $S^+_{k+\frac{1}{2}}$ be the space of cusp forms of weight $k + \frac{1}{2}$ and level 4 having a Fourier expansion of the form

$$\sum_{m \geqslant 1, (-1)^k m \equiv 0, 1 \pmod{4}} c(m)e^{2\pi i m z}$$

(where $z \in \mathcal{H}$ = the complex upper half-plane) [Koh80], [Shi73].

If $F \in S_k(\Gamma_n)$ is a Siegel cusp form of weight k and genus n, we write its Fourier expansion in the form

$$F(Z) = \sum_{T > 0} A(T)e^{2\pi i \operatorname{tr}(TZ)}$$

(where $Z \in \mathcal{H}_n$ = the Siegel upper half-space of genus n), where T runs over all rational, half-integral, symmetric, positive definite matrices of size n.

2. Preliminaries

Let $T \in M_{2n}(\mathbb{Q})$ be a rational, half-integral, symmetric, non-degenerate matrix of size $2n$. We denote by

$$D_T := (-1)^n \det(2T)$$

the discriminant of T. Then $D_T \equiv 0, 1 \pmod 4$ and we write $D_T = D_{T,0} f_T^2$ with $D_{T,0}$ the corresponding fundamental discriminant and $f_T \in \mathbb{N}$.

Let us fix a prime p. Recall that one defines the local singular series of T at p by

$$b_p(T; s) := \sum_R \nu_p(R)^{-s} e_p(\operatorname{tr}(TR)) \quad (s \in \mathbb{C})$$

where R runs over all symmetric $(2n, 2n)$-matrices with entries in $\mathbb{Q}_p/\mathbb{Z}_p$ and $\nu_p(R)$ is a power of p equal to the product of denominators of elementary divisors of R. Furthermore, for $x \in \mathbb{Q}_p$ we have put $e_p(x) := e^{2\pi i x'}$ where x' denotes the fractional part of x.

As is well known, $b_p(T; s)$ is a product of two polynomials in p^{-s} with coefficients in $\mathbb{Z}$. More precisely, one has

$$b_p(T; s) = \gamma_p(T; p^{-s}) F_p(T; p^{-s}),$$

where

$$\gamma_p(T; X) := (1 - X)\left(1 - \left(\frac{D_{T,0}}{p}\right) p^n X\right)^{-1} \prod_{j=1}^n (1 - p^{2j} X^2)$$

and $F_p(T; X) \in \mathbb{Z}[X]$ has constant term 1.

A fundamental result of Katsurada [Kat99] states that the Laurent polynomial

$$\tilde{F}_p(T; X) := X^{-\operatorname{ord}_p f_T} F_p(T; p^{-n-1/2} X)$$

is symmetric, i.e.

$$\tilde{F}_p(T; X) = \tilde{F}_p(T; X^{-1}).$$

If p does not divide f_T, then $F_p(T; X) = \tilde{F}_p(T; X) = 1$.

Denote by $V = (\mathbb{F}_p^{2n}, q)$ the quadratic space over $\mathbb{F}_p$ where q is the quadratic form obtained from the quadratic form $x \mapsto T[x]$ $(x \in \mathbb{Z}_p^{2n})$ by reducing modulo p. We denote by $\langle x, y \rangle := q(x+y) - q(x) - q(y)$ $(x, y \in \mathbb{F}_p^{2n})$ the associated bilinear form and let

$$R(V) := \{x \in \mathbb{F}_p^{2n} \mid \langle x, y \rangle = 0 \; \forall y \in \mathbb{F}_p^{2n}, \; q(x) = 0\}$$

be the radical of V. We put $s_p := s_p(T) = \dim R(V)$ and denote by W an orthogonal complementary subspace of $R(V)$.

According to [Kit84], one defines a polynomial by

$$H_{n,p}(T; X) := \begin{cases} 1, & \text{if } s_p = 0, \\ \prod_{j=1}^{[(s_p-1)/2]} (1 - p^{2j-1} X^2), & \text{if } s_p > 0, \; s_p \text{ odd}, \\ (1 + \lambda_p(T) p^{(s_p-1)/2} X) \prod_{j=1}^{[(s_p-1)/2]} (1 - p^{2j-1} X^2), & \text{if } s_p > 0, \; s_p \text{ even}, \end{cases}$$

where for s_p even we have put

$$\lambda_p(T) := \begin{cases} 1, & \text{if } W \text{ is a hyperbolic subspace or } s_p = 2n, \\ -1, & \text{otherwise}. \end{cases}$$

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According to [Koh02], for $\mu \in \mathbb{Z}$, $\mu \geqslant 0$, define $\rho_T(p^\mu)$ by

$$\sum_{\mu \geqslant 0} \rho_T(p^\mu) X^\mu := \begin{cases} (1 - X^2) H_{n,p}(T; X), & \text{if } p \mid f_T, \\ 1, & \text{otherwise.} \end{cases}$$

We extend the function ρ_T multiplicatively to the whole of $\mathbb{N}$ by defining

$$\sum_{a \geqslant 1} \rho_T(a) a^{-s} := \prod_{p \mid f_T} ((1 - p^{-2s}) H_{n,p}(T; p^{-s})).$$

It follows from the definitions that $\sqrt{a}\,\rho_T(a)$ is an integer.

Finally, let

$$\mathcal{D}(T) := GL_{2n}(\mathbb{Z}) \backslash \{G \in M_{2n}(\mathbb{Z}) \cap GL_{2n}(\mathbb{Q}) \mid T[G^{-1}] \text{ half-integral}\},$$

where $GL_{2n}(\mathbb{Z})$ operates by left-multiplication. Then $\mathcal{D}(T)$ is finite as is easy to see. For $a \in \mathbb{N}$ with $a \mid f_T$ put

$$\phi(a; T) := \sqrt{a} \sum_{d^2 \mid a} \sum_{G \in \mathcal{D}(T), \, |\det(G)| = d} \rho_{T[G^{-1}]} \left(\frac{a}{d^2} \right). \tag{1}$$

Note that on the right-hand side of (1) we have $(a/d^2) \mid f_{T[G^{-1}]}$ and that $\phi(a; T) \in \mathbb{Z}$ for all a.

3. Ikeda's lifting map

Let f be a normalized Hecke eigenform in $S_{2k}(\Gamma_1)$. For a prime p, let $\lambda(p)$ and α_p be the pth Fourier coefficient and the Satake p-parameter of f, respectively. Thus

$$1 - \lambda(p) X + p^{2k-1} X^2 = (1 - p^{k-1/2} \alpha_p X)(1 - p^{k-1/2} \alpha_p^{-1} X).$$

Note that α_p is determined only up to inversion.

Let

$$g = \sum_{m \geqslant 1, (-1)^k m \equiv 0, 1 \pmod 4} c(m) e^{2\pi i m z} \qquad (z \in \mathcal{H})$$

be a Hecke eigenform in $S_{k+\frac{1}{2}}^+$ which corresponds to f under the Shimura correspondence [Koh80], [Shi73].

Let $n \in \mathbb{N}$ with $n \equiv k \pmod 2$. For T a rational, half-integral, symmetric, positive definite matrix of size $2n$, define

$$a_f(T) := c(|D_{T,0}|) f_T^{k-1/2} \prod_{p \mid f_T} \tilde{F}_p(T; \alpha_p),$$

where we have used the notation explained in § 2. Note that for n and k of the same parity, $(-1)^k D_{T,0} > 0$.

THEOREM 1 [Ike01]. *The function*

$$F(Z) := \sum_{T > 0} a_f(T) e^{2\pi i \, \mathrm{tr}(TZ)} \qquad (Z \in \mathcal{H}_{2n}),$$

where T runs over all rational, half-integral, symmetric, positive definite matrices of size $2n$, is a cuspidal Siegel–Hecke eigenform of weight $k + n$ with respect to Γ_{2n}.

Remarks. (a) Ikeda [Ike01] further proves that the standard zeta function of F is given by

$$\zeta(s) \prod_{j=1}^{2n} L(f, s + k + n - j),$$

where $L(f, s)$ is the Hecke L-function of f. We will not use this fact here.

(b) We note that it is implicit in Ikeda's theorem (respectively its proof) that his lift is injective. Indeed, if f_1 and f_2 are different normalized eigenforms in $S_{2k}(\Gamma_1)$, then their p-eigenvalues are different for at least one prime p. Hence their lifts are in different eigenspaces for the local Hecke algebra at p in genus $2n$ and therefore are orthogonal with respect to the Petersson scalar product.

THEOREM 2 [Koh02]. *With the notation of § 2, one has*

$$a_f(T) = \sum_{a|f_T} a^{k-1}\phi(a;T)c\left(\frac{|D_T|}{a^2}\right). \tag{2}$$

4. A characterization of the image of Ikeda's lifting map

We denote by $S_{k+n}^*(\Gamma_{2n})$ the image of the linear mapping $I_{k,n}$ from $S_{k+\frac{1}{2}}^+$ to $S_{k+n}(\Gamma_{2n})$ defined by (2). (Note that this notation differs from the one used in [Koh02, § 8, iv].)

Our main result is the following.

THEOREM 3. *Suppose that $n \equiv 0, 1 \pmod 4$ and let $k \in \mathbb{N}$ with $n \equiv k \pmod 2$. Let $F \in S_{k+n}(\Gamma_{2n})$ with Fourier coefficients $A(T)$. Then the following assertions are equivalent:*

(i) $F \in S_{k+n}^*(\Gamma_{2n})$;

(ii) *there exist complex numbers $c(m)$ (with $m \in \mathbb{N}$, and $(-1)^k m \equiv 0, 1 \pmod 4$) such that*

$$A(T) = \sum_{a|f_T} a^{k-1}\phi(a;T)c\left(\frac{|D_T|}{a^2}\right)$$

for all T.

Remarks. (a) The statement of Theorem 3 was conjectured in [Koh02, § 8, iv].

(b) According to [Koh02, § 8, iv] the condition in (ii) is also equivalent to saying that $A(T) = A(\tilde{T})$ whenever $D_T = D_{\tilde{T}}$ and $\phi(a;T) = \phi(a;\tilde{T})$ for all $a|f_T$.

To state our second result, recall [CS88, ch. 15, § 8.2, table 15.5] that for each $g \in \mathbb{N}$ there exists exactly one genus of integral, even, symmetric matrices S of size g with determinant equal to 2. A matrix in this genus is positive definite if and only if $g \equiv \pm 1 \pmod 8$, and in this case as a representative we can take

$$S_0 := \begin{cases} E_8^{\oplus(g-1)/8} \oplus 2, & \text{if } g \equiv 1 \pmod 8, \\ E_8^{\oplus(g-7)/8} \oplus E_7, & \text{if } g \equiv -1 \pmod 8, \end{cases} \tag{3}$$

where (by abuse of language) E_8 and E_7 denote the Gram matrices of the E_8- and E_7-root lattices, respectively.

Explicitly, recall that

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

and E_7 is the upper left $(7,7)$-submatrix of E_8.

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The matrix E_8 is unimodular and E_7 has determinant equal to 2.

Suppose that $n \equiv 0, 1 \pmod 4$ and define S_0 by the right-hand side of (3), with $g := 2n - 1$.

For $m \in \mathbb{N}$ with $(-1)^n m \equiv 0, 1 \pmod 4$ define a rational, half-integral, symmetric, positive definite matrix T_m of size $2n$ by

$$
T_m := \begin{cases} \begin{pmatrix} \frac{1}{2}S_0 & 0 \\ 0 & m/4 \end{pmatrix}, & \text{if } m \equiv 0 \pmod 4, \\[2mm] \begin{pmatrix} \frac{1}{2}S_0 & \frac{1}{2}e_{2n-1} \\ \frac{1}{2}e'_{2n-1} & [m + 2 + (-1)^n]/4 \end{pmatrix}, & \text{if } m \equiv (-1)^n \pmod 4, \end{cases}
\tag{4}
$$

where $e_{2n-1} = (0,\ldots,0,1)' \in \mathbb{Z}^{(2n-1,1)}$ is the usual standard column vector.

Observe that the matrices on the right-hand side of (4) are indeed positive definite. This is obvious except for the case $n \equiv 0 \pmod 4$, $m \equiv 1 \pmod 4$. In the latter case, the Jacobi decomposition shows that

$$
\det(2T_m) = 2\left(\frac{m+3}{2} - E_7^{-1}[e_7]\right)
$$
$$
= m
$$

since $E_7^{-1}[e_7] = \frac{3}{2}$. Our claim follows from this.

We note that

$$
\det(2T_m) = m
\tag{5}
$$

for all m.

COROLLARY 4 ('Maass relations'). *Under the same assumptions as in Theorem 3 and with the above notation, the following assertions are equivalent:*

(i) $F \in S^*_{k+n}(\Gamma_{2n})$;

(ii) *for all T, one has*

$$
A(T) = \sum_{a \mid f_T} a^{k-1}\phi(a;T)A(T_{|D_T|/a^2}).
$$

Remarks. (a) That (i) implies (ii), in a slightly different formulation and with a different proof, has already been shown before by one of the authors [Koh04].

(b) If $n = 1$, using the fact that

$$
\phi(a;T) = \begin{cases} a, & \text{if } a \mid \gcd(\alpha, \beta, \gamma), \\ 0, & \text{otherwise,} \end{cases}
$$

where

$$
T = \begin{pmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{pmatrix}
$$

(see [Koh02, Equation (32)]), one recovers the classical 'Maass relations' in genus 2.

(c) One can easily give a formulation of the statement of Corollary 4 which is valid for *any* matrix S in the genus of S_0, by properly modifying the definition of the matrices T_m. This, of course, is consistent with the fact that the Fourier coefficients $A(T)$ of an Ikeda-lifted form by definition depend only on the genus of T. The more general formulation is slightly more involved and is left to the reader. We have preferred to choose $S = S_0$ here, since in this case the statement for higher genus formally seems to come closest to the corresponding classical statement in genus 2.

5. Proof of Theorem 3 and Corollary 4

To prove Theorem 3, in view of Theorem 2 it is sufficient to show that (ii) implies (i).

We will use the notation explained in § 4, after the statement of Theorem 3.

For convenience, let us put

$$T_0 := \tfrac{1}{2}S_0.$$

Let $F(Z) \, (Z \in \mathcal{H}_{2n})$ be in $S_{k+n}(\Gamma_{2n})$ and denote by $\phi_{T_0}(\tau, z)$ (with $\tau \in \mathcal{H}$, and $z \in \mathbb{C}^{(2n-1,1)}$) the Fourier–Jacobi coefficient of F of index T_0, i.e. the coefficient with index T_0 in a partial Fourier expansion of $F(Z)$ with respect to $w \in \mathcal{H}_{2n-1}$ where

$$Z = \begin{pmatrix} w & z \\ z' & \tau \end{pmatrix}.$$

Then ϕ_{T_0} is a Jacobi cusp form of weight $k + n$ and index T_0 with respect to the Jacobi group $\Gamma_1 \rhd (\mathbb{Z}^{(2n-1,1)} \times \mathbb{Z}^{(2n-1,1)})$ and ϕ_{T_0} has a Fourier expansion

$$\phi_{T_0}(\tau, z) = \sum_{T = \binom{T_0 \;\; r/2}{r'/2 \;\; N} > 0} A(T) \, e^{2\pi i (N\tau + r'z)}$$

(see [EZ85], [Ike01, §§ 7 and 8], [Shi78], [Zie89]).

As is well known, ϕ_{T_0} has an expansion in terms of Jacobi theta functions

$$\phi_{T_0}(\tau, z) = \sum_{\lambda \in \Lambda} h_\lambda(\tau) \theta_\lambda(\tau, z),$$

where $\Lambda := S_0^{-1} \mathbb{Z}^{(2n-1,1)} / \mathbb{Z}^{(2n-1,1)}$ and where for $\lambda \in \Lambda$ one sets

$$h_\lambda(\tau) := \sum_{N \in \mathbb{Z}, N - T_0[\lambda] > 0} A\left(\begin{pmatrix} T_0 & T_0\lambda \\ \lambda'T_0 & N \end{pmatrix} \right) e^{2\pi i (N - T_0[\lambda])\tau} \quad (\tau \in \mathcal{H})$$

and

$$\theta_\lambda(\tau, z) := \sum_{r \in \mathbb{Z}^{(2n-1,1)}} e^{2\pi i (T_0[r+\lambda]\tau + 2(r+\lambda)'T_0 z)} \quad (\tau \in \mathcal{H}, \, z \in \mathbb{C}^{(2n-1,1)})$$

(see [EZ85], [Ike01, §§ 7 and 8], [Shi78], [Zie89]).

We note that $|\Lambda| = 2$ and that representatives can be chosen as λ_0, λ_1 where λ_0 is the zero vector and $\lambda_1 := S_0^{-1} e_{2n-1}$. Indeed, this is trivial if $n \equiv 1 \pmod 4$ and for $n \equiv 0 \pmod 4$ follows from the well-known fact that E_7 is obtained from E_8 as the orthogonal complement of a vector of length 2 and 2-adically E_8 is the sum of four hyperbolic planes.

In the case $n \equiv 0 \pmod 4$, in more elementary terms one can also argue as follows. One has

$$E_7^{-1} \mathbb{Z}^{(7,1)} / \mathbb{Z}^{(7,1)} \simeq \prod_p E_7^{-1} \mathbb{Z}_p^{(7,1)} / \mathbb{Z}_p^{(7,1)}$$

$$\simeq E_7^{-1} \mathbb{Z}_2^{(7,1)} / \mathbb{Z}_2^{(7,1)}$$

since $\det(E_7) = 2$. Furthermore

$$E_7 = \begin{pmatrix} E_7^{(1)} & * \\ * & 2, \end{pmatrix},$$

where $E_7^{(1)}$ is of size 6 and has determinant equal to $3 \in \mathbb{Z}_2^*$; hence we see from the Jacobi decomposition that E_7 over $\mathbb{Z}_2$ is equivalent to $E_7^{(1)} \oplus \frac{2}{3}$ which implies that

$$E_7^{-1} \mathbb{Z}_2^{(7,1)} / \mathbb{Z}_2^{(7,1)} \simeq \mathbb{Z}/2\mathbb{Z}.$$

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On the other hand, a direct calculation shows that the congruence $E_7 r \equiv e_7 \pmod 2$ with $r \in \mathbb{Z}^{(7,1)}$ is not solvable hence $E_7^{-1} e_7$ is not in $\mathbb{Z}^{(7,1)}$.

With the above representatives, we now find that

$$h_0(\tau) := h_{\lambda_0}(\tau) = \sum_{N>0} A\left(\begin{pmatrix} T_0 & 0 \\ 0 & N \end{pmatrix}\right) e^{2\pi i N \tau} \quad (\tau \in \mathcal{H}) \tag{6_1}$$

and

$$h_1(\tau) := h_{\lambda_1}(\tau) = \sum_{N>[2+(-1)^n]/4} A\left(\begin{pmatrix} T_0 & \frac{1}{2} e_{2n-1} \\ \frac{1}{2} e'_{2n-1} & N \end{pmatrix}\right) e^{2\pi i (N-[2+(-1)^n]/4)\tau} \quad (\tau \in \mathcal{H}). \tag{6_2}$$

We claim that the function

$$h(\tau) := h_0(4\tau) + h_1(4\tau) \quad (\tau \in \mathcal{H})$$

is in $S^+_{k+\frac{1}{2}}$. Indeed, from [Ike01, § 9], [Shi78], [Zie89, ch. 3, Equations (1) and (2)] (in the case $n = 1$, cf. also [EZ85, ch. II, § 5]) one finds that

$$\begin{pmatrix} h_0(\tau+1) \\ h_1(\tau+1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -\epsilon_n i \end{pmatrix} \begin{pmatrix} h_0(\tau) \\ h_1(\tau) \end{pmatrix}$$

and (noting $k + n - (2n-1)/2 = k + \frac{1}{2}$) that

$$\begin{pmatrix} h_0(-1/\tau) \\ h_1(-1/\tau) \end{pmatrix} = \frac{1+\epsilon_n i}{2} \tau^{k+\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} h_0(\tau) \\ h_1(\tau) \end{pmatrix}$$

where

$$\epsilon_n := \begin{cases} 1, & \text{if } n \equiv 1 \pmod 4, \\ -1, & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

and complex powers as usual are defined by the principal value of the complex logarithm.

A formal computation similar to the case $n = 1$ now easily shows that

$$h(\tau+1) = h(\tau), \quad h\left(\frac{\tau}{4\tau+1}\right) = (4\tau+1)^{k+\frac{1}{2}} h(\tau).$$

Since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

generate $\Gamma_0(4)$, we conclude that $h(\tau)$ behaves like a modular form of weight $k + \frac{1}{2}$ and level 4. From the above transformation formulas one sees that $h(\tau)$ is cuspidal and Equations (6_1) and (6_2) then imply that $h(\tau)$ in fact is contained in $S^+_{k+\frac{1}{2}}$ (note that $n \equiv k \pmod 2$ by assumption).

The Fourier coefficients $\tilde{c}(m)$ (where $m \in \mathbb{N}$, and $(-1)^k m \equiv 0, 1 \pmod 4$) of h are given by the Fourier coefficients of the right-hand sides of Equations (6_1) and (6_2), with τ replaced by 4τ, and with obvious identifications of m and N. More precisely, note that for a typical matrix T with $A(T)$ appearing on the right-hand sides of Equations (6_1) and (6_2) we have

$$\det(2T) = m = \begin{cases} 4N, & \text{if } \lambda = \lambda_0, \\ 4N - (2+(-1)^n), & \text{if } \lambda = \lambda_1 \end{cases} \tag{7}$$

(compare (5)). We claim that each T appearing in this way has the property that $s_p(T) = 1$ for all primes p such that $p | f_T$.

Since E_8 is unimodular and $\det(E_7) = 2$, this readily follows if $p > 2$. If $p = 2$ and $2 | f_T$, then by (7) only the case $\lambda = \lambda_0$ matters. As was already noted before, E_7 over $\mathbb{Z}_2$ is equivalent to a matrix

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$E_7^{(1)} \oplus \frac{2}{3}$ with $E_7^{(1)} \in GL_6(\mathbb{Z}_2)$. Since $s_2(1 \oplus N) = 1$ as follows from the definitions, we therefore see that $s_2(T) = 1$ in the case under consideration. This implies our claim.

We can now finish the proof of Theorem 3. By assumption, there exist complex numbers $c(m)$ (with $m \in \mathbb{N}$, and $(-1)^k m \equiv 0, 1 \pmod 4$) such that

$$A(T) = \sum_{a|f_T} a^{k-1} \phi(a; T) c\left(\frac{|D_T|}{a^2}\right)$$

for all T. If we choose for T one of the special matrices occurring on the right-hand sides of Equations of (6_1) and (6_2), then for those T we have $s_p(T) = 1$ for all primes p with $p|f_T$ as was just proved. By a formal calculation, using only the definition of $\phi(a; T)$ in exactly the same way as in the proof of [Koh02, Proposition 2, p. 801], we therefore deduce that $A(T) = c(|D_T|)$ for those T; in other words we find that

$$\tilde{c}(m) = c(m)$$

for all $m \in \mathbb{N}$ with $(-1)^k m \equiv 0, 1 \pmod 4$.

This proves Theorem 3.

Let us now prove Corollary 4. Checking the definitions, we see that the implication (i) $\Rightarrow$ (ii) has already been proved above.

Conversely, suppose that (ii) holds. For $m \in \mathbb{N}$ with $(-1)^k m \equiv 0, 1 \pmod 4$, define

$$c(m) := A(T_m).$$

Then

$$A(T) = \sum_{a|f_T} a^{k-1} \phi(a; T) c\left(\frac{|D_T|}{a^2}\right)$$

for all T; hence by Theorem 3 we infer that $F \in S_{k+n}^*(\Gamma_{2n})$.

Remark. Note that, as a side result, we have proved that in the above notation the mapping $S_{k+n}(\Gamma_{2n}) \to S_{k+\frac{1}{2}}^+$, $F \mapsto h$ is surjective and that its restriction to $S_{k+n}^*(\Gamma_{2n})$ is bijective with inverse given by $I_{k,n}$. Thus also in higher genus one recovers an analogue of the well-known situation in genus 2.

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