

3

The spin density matrix

The state of an ensemble of particles is specified by a density matrix. In any reaction one starts with a knowledge of the density matrix of the initial system (known from its mode of preparation) and one attempts to measure the density matrix of the final system.

The properties of a density matrix are of three kinds.

- (1) Firstly, there are properties of a very general nature that follow from the very definition of a density matrix and from the basic postulates of quantum mechanics. To check that a measured density matrix conforms to those requirements is best thought of as a test of the reliability of the experimental measurements.

We shall refer to these as *basic properties*.

- (2) Secondly, there are properties of a kinematical-dynamical origin, which reflect the general properties of an interaction, for example its symmetries, but which do not depend on a detailed knowledge of the dynamics. It is important to check that the density matrix measured in a particular reaction does satisfy these properties.

We shall refer to these as *general kinematical-dynamical properties*.

- (3) Thirdly, there are properties which depend upon the specific dynamical mechanism in a reaction and which can therefore be used either to learn about these mechanisms or to test dynamical models.

We shall refer to these as *model-dependent properties*.

In this chapter we shall discuss only the *basic properties* of the density matrix. It will turn out that *all* the basic properties of the non-relativistic case hold also for the helicity density matrix provided care is taken with the physical interpretation of the latter.

We discuss, amongst other things, the expression of the density matrix in terms of multipole parameters or statistical tensors, the concept of 'degree-of-rank- L polarization' and the transformation properties of the density matrix under rotations and Lorentz transformations.

We also give a detailed discussion of the density matrix for spin-1 particles, bearing in mind that polarized deuteron beams are already in use and will become more commonly available in the near future.

The treatment of the *general kinematical-dynamical properties* will follow in Chapter 4 after the discussion of scattering amplitudes. Some *model-dependent properties* will be found in the discussion of specific dynamical models.

In all experiments involving the use of polarized targets or polarized beams we are dealing with a system of quantum mechanical particles that is not in a definite, pure quantum state. Rather we have an incoherent mixture or statistical ensemble of particles about which our knowledge is limited to the average of certain dynamical variables, an average, that is, for the whole ensemble. Strictly speaking this ought to apply also to variables such as momentum, but the averaging processes involved therein are usually quite uninteresting for hadron physics and therefore we shall ignore them, adopting the fiction that each particle in the beam emerging from an accelerator has precisely the same momentum. Our sole concern will be with the spin properties of these ensembles.

We review the main properties of the density matrix in the next section. A more general exposition can be found in the review article of Fano (Fano, 1957).

3.1 The non-relativistic density matrix

3.1.1 Definition

For a particle of spin s , a *pure* quantum mechanical spin state $|\psi\rangle$ is defined and identified by the coefficients c_m involved in its expansion into a sum of basic states $|s; m\rangle$; these are usually taken as eigenstates of \hat{s}_z , the z -component of the spin operator, i.e. one has

$$|\psi\rangle = \sum_{m=-s}^s c_m |sm\rangle. \quad (3.1.1)$$

For an arbitrary operator \hat{O} with matrix elements

$$O_{mm'} = \langle sm | \hat{O} | sm' \rangle \quad (3.1.2)$$

the mean value in the state $|\psi\rangle$, normalized to unity, is given by

$$\langle \hat{O} \rangle_\psi \equiv \langle \psi | \hat{O} | \psi \rangle = \sum_{m,m'} c_m^* O_{m'm} c_m. \quad (3.1.3)$$

For a non-pure state we might have an incoherent mixture of a number of pure states $|\psi^{(i)}\rangle$, each occurring in an ensemble with probability or

statistical weight $p^{(i)}$ with $\sum_i p^{(i)} = 1$. For each state the operator \hat{O} will have a mean value

$$\langle \hat{O} \rangle_{\psi^{(i)}} = \sum_{m,m'} c_{m'}^{(i)*} O_{m'm} c_m^{(i)}$$

and therefore its mean value over the whole ensemble will be

$$\langle \hat{O} \rangle = \sum_i p^{(i)} \langle \hat{O} \rangle_{\psi^{(i)}} = \sum_{m,m'} O_{m'm} \sum_i p^{(i)} c_{m'}^{(i)*} c_m^{(i)}. \tag{3.1.4}$$

The spin density matrix in the basis $|s; s_z = m\rangle$ is now defined by

$$\rho_{mm'} = \sum_i p^{(i)} c_m^{(i)} c_{m'}^{(i)*} \tag{3.1.5}$$

so that equation (3.1.4) becomes

$$\langle \hat{O} \rangle = \sum_{m,m'} O_{m'm} \rho_{mm'} = \text{Tr} (O\rho) \tag{3.1.6}$$

where O and ρ are the matrices whose elements are $O_{m'm}$ and $\rho_{mm'}$.

Equation (3.1.6) allows us to calculate the mean value for the ensemble of every physical operator once we know the density matrix ρ .

Conversely, and of most interest in hadron physics, a knowledge of the mean values for the ensemble of a sufficiently large number of physical observables will enable the inversion of eqn (3.1.6) and thus determination of the density matrix.

3.1.2 Some general properties of $\rho_{mm'}$

The $\rho_{mm'}$ are the elements of the density matrix referred to a particular choice of basis states $|s; s_z = m\rangle$. We can also give the density matrix in any other basis unitarily related to $|s; s_z = m\rangle$. If T is any unitary $(2s + 1) \times (2s + 1)$ matrix we can take as basis states

$$|n'\rangle = \sum_m T_{mn} |s; s_z = m\rangle$$

and if we label the density elements in the new basis as $\rho'_{nn'}$ we will have

$$\rho'_{nn'} = \sum_{m,m'} T_{nm}^{-1} \rho_{mm'} T_{m'n'} \tag{3.1.7}$$

or in matrix notation

$$\rho' = T^{-1} \rho T. \tag{3.1.8}$$

We note the important property that the trace is invariant under change of basis:

$$\text{Tr} \rho' = \text{Tr} \rho. \tag{3.1.9}$$

From the definition (3.1.5) and the condition $\sum_i p^{(i)} = 1$ the following properties can easily be read off.

(1) The trace of ρ is unity, i.e.

$$\text{Tr } \rho = 1. \quad (3.1.10)$$

(2) ρ is a hermitian matrix, i.e.

$$\rho_{mm'}^* = \rho_{m'm}. \quad (3.1.11)$$

(3) For each m , the diagonal elements are positive semi-definite, i.e.

$$\rho_{mm} \geq 0 \quad (3.1.12)$$

and this holds in any unitarily related basis.

(4) The hermitian properties of ρ guarantee the existence of a unitary matrix U that will diagonalize ρ , i.e. we have

$$U^{-1} \rho U = \rho^D \quad (3.1.13)$$

where ρ^D is the diagonal matrix

$$\left(\rho^D\right)_{mn} = \lambda_m \delta_{mn}, \quad (3.1.14)$$

with $\lambda_m \geq 0$.

(5) From (3.1.9), (3.1.12), (3.1.13), (3.1.14)

$$\text{Tr } \rho^2 = \text{Tr } \left(\rho^D\right)^2 = \sum_m \lambda_m^2 \leq \left(\sum_m \lambda_m\right)^2 = (\text{Tr } \rho)^2 = 1.$$

Thus

$$\text{Tr } \rho^2 = \sum_{m,m'} |\rho_{mm'}|^2 \leq 1. \quad (3.1.15)$$

(6) If it happens that all members of an ensemble are in a single pure quantum state, then all except one $p^{(i)}$ will be zero, and the non-zero one $p^{(j)}$ say, will be equal to unity. In this case ρ will be a rank-1 matrix¹ and it will have one eigenvalue equal to unity and all the rest equal to zero. It can then be written in a factorized form, e.g.

$$\rho_{mm'} = c_m c_{m'}^*. \quad (3.1.16)$$

For this case the equality holds in (3.1.15).

¹ The rank of ρ is the dimension of the largest non-zero determinant that can be formed from the rows and columns of ρ . Equivalently, the rank of an $n \times n$ matrix ρ is $r = n - k$, where k is the dimension of the eigenspace corresponding to the eigenvalue zero of ρ .

3.1.3 Combined systems of several particle types

If the overall system is a mixture of several systems of different particles then it can be described by a joint density matrix. For example for two types of particles A, B one would have $\rho(A, B)$ with matrix elements $\rho(A, B)_{mn;m'n'}$, the labels m, n referring to the eigenstates

$$|s^A, s_z^A = m; s^B, s_z^B = n\rangle$$

of the system of two particles A and B .

The mean value for the whole system of an arbitrary operator \hat{O} is again given by an equation like (3.1.6):

$$\langle \hat{O} \rangle = \text{Tr} [O\rho(A, B)] \tag{3.1.17}$$

where now the trace is used in the generalized sense

$$\text{Tr} [O\rho(A, B)] = \sum_{m,n} [O\rho(A, B)]_{mn;mn}$$

with

$$[O\rho(A, B)]_{mn;mn} = \sum_{m',n'} O_{mn;m'n'} \rho(A, B)_{m'n';mn}$$

If on the one hand we wish to calculate the joint expectation values of an observable $\hat{O}^{(A)}$ of the particle A and an observable $\hat{O}^{(B)}$ of the particle B then we must take the expectation value of the operator product $\hat{O}^{(A)} \otimes \hat{O}^{(B)}$ defined in such a way that

$$\left(\hat{O}^{(A)} \otimes \hat{O}^{(B)} \right)_{mn;m'n'} \equiv O_{mm'}^{(A)} O_{nn'}^{(B)}. \tag{3.1.18}$$

If on the other hand we wish to calculate the expectation value for the measurement of a physical observable of just *one* type, A , then if $\hat{O}^{(A)}$ is the operator corresponding to this observable we get the mean value of $\hat{O}^{(A)}$ by calculating the mean value of $\hat{O}^{(A)} \otimes \hat{1}^{(B)}$, where $\hat{1}^{(B)}$ is the unit operator in the space of the labels referring to particle B .

Thus

$$\begin{aligned} \langle \hat{O}^{(A)} \rangle &= \text{Tr} \left[\hat{O}^{(A)} \otimes \hat{1}^{(B)} \rho(A, B) \right] \\ &= \sum_{\substack{m,m' \\ n,n'}} O_{mm'}^{(A)} \delta_{nn'} \rho(A, B)_{m'n';mn} \\ &= \sum_{m,m'} O_{mm'}^{(A)} \sum_n \rho(A, B)_{m'n';mn} \\ &= \text{Tr}_A \left[O^{(A)} \rho(A) \right] \end{aligned} \tag{3.1.19}$$

where $\rho(A)$ is the $(2s_A + 1) \times (2s_A + 1)$ effective density matrix for type- A particles, defined by

$$\rho(A)_{mm'} \equiv \sum_n \rho(A, B)_{mn; m'n}. \quad (3.1.20)$$

We note that if the rank of $\rho(A, B)$ is r , then for the rank of $\rho(A)$ one has

$$\text{rank } \rho(A) \leq (2s_B + 1)r.$$

Of course a similar result holds for an observable of the particles of type B .

If the state of the combined system is *uncorrelated* then the mean values of all measurements carried out on particles of type A and B must factorize into mean values over the separate ensembles of A and B , i.e.

$$\langle \hat{O}^{(A)} \otimes \hat{O}^{(B)} \rangle = \langle \hat{O}^{(A)} \rangle \langle \hat{O}^{(B)} \rangle$$

must hold for every observable $\hat{O}^{(A)}$, $\hat{O}^{(B)}$. This is only possible if the joint density matrix itself factorizes. We thus have the important result that

$$\rho(A, B)_{mn; m'n'} = \rho(A)_{mm'} \rho(B)_{nn'} \quad (3.1.21)$$

if and only if the ensemble of particles A and particles B is uncorrelated. An example of such ensembles is the incoming beam and the target in a scattering experiment *prior* to interaction.

In general, if several spinning particles C, D, E, \dots are produced in a reaction then the full density matrix for the final state is a joint matrix $\rho(C, D, E, \dots)$ with matrix elements

$$\rho(C, D, E, \dots)_{c,d,e,\dots; c',d',e',\dots}$$

If, as often happens in practice, the properties of only *one* of the particles are measured, say those of type C , then the mean values are to be calculated using the *effective density matrix* $\rho(C)$ where

$$\rho(C)_{c;c'} = \sum_{d,e,\dots} \rho(C, D, E, \dots)_{c,d,e,\dots; c',d,e,\dots}. \quad (3.1.22)$$

Usually one refers to this simply as ‘the density matrix for C ’.

3.1.4 The independent parameters specifying ρ

We saw in Chapter 1 that a *pure state* for a particle of spin s can be specified by $2(2s + 1) - 2 = 4s$ real parameters. For an incoherent mixture made up of particles of spin s the ensemble is completely characterized by the $(2s + 1) \times (2s + 1)$ hermitian matrix ρ . Taking into account the normalization condition (3.1.10) one requires $(2s + 1)^2 - 1$ independent real parameters to specify ρ fully. The direct listing of the individual elements

of ρ could then be limited to $2s$ of the $2s + 1$ (real) diagonal elements, and the real and imaginary parts of the elements above the diagonal. This is not always the most convenient set of numbers to deal with, from the point of view of either experiment or theory.

Various ‘representations’ of ρ can be introduced, expressing ρ as a sum over certain standard matrices, the properties of a particular ρ being then specified by the coefficients in the expansion.

The best known of these is the density matrix for spin-1/2 particles. Since ρ is now a 2×2 matrix, it can always be written, see (1.1.22), as

$$\rho = \frac{1}{2}(I + \mathcal{P} \cdot \sigma) \tag{3.1.23}$$

where \mathcal{P} is now the spin-polarization vector for the ensemble,

$$\mathcal{P} = \langle \sigma \rangle = \text{Tr } \rho \sigma. \tag{3.1.24}$$

Thus the three real numbers \mathcal{P} can be used to specify ρ . We note that whereas for a *pure* state $\mathcal{P}^2 = 1$, in general for an ensemble we have

$$\mathcal{P}^2 \leq 1 \tag{3.1.25}$$

as follows from (3.1.15).

What is the generalization of (3.1.23) for spins $s > 1/2$? Clearly it is not sufficient to replace σ by the set of three hermitian matrices $\mathbf{S} = \{S_j\}$ that represents the spin operator \hat{s} . We need to construct many more basis matrices and this can be done in principle by using products of the S_j . (It must of course be remembered that results like $\sigma_1\sigma_2 = i\sigma_3$ are specific to spin 1/2; higher-spin products, such as S_1S_2 , are independent and cannot be expressed in terms of S_3 .)

3.1.5 The multipole parameters

A very useful and convenient set of basis matrices can be obtained by forming sets of products of the spin operators that transform very simply under rotations. These so-called *spherical tensor operators* \hat{T}_M^L , $0 \leq L \leq 2s$, $-L \leq M \leq L$, and the matrices T_M^L that represent them can be chosen in such a way that the elements of these matrices are given by *vector-addition coefficients* (Edmonds, 1957 and Appel, 1968). Thus

$$\left(T_M^L\right)_{mm'} \equiv \langle sm | \hat{T}_M^L | sm' \rangle \equiv \langle sm | sm' ; LM \rangle ; \tag{3.1.26}$$

L is called the *rank* of the tensor operator. Some examples are as follows
Scalar:

$$\hat{T}_0^0 = 1.$$

Vector or rank 1 tensor:

$$\begin{aligned} \hat{T}_0^1 &= \frac{1}{\sqrt{s(s+1)}} \hat{s}_z \\ \hat{T}_1^1 &= -\frac{1}{\sqrt{2s(s+1)}} (\hat{s}_x + i\hat{s}_y). \\ \hat{T}_{-1}^1 &= \frac{1}{\sqrt{2s(s+1)}} (\hat{s}_x - i\hat{s}_y) \end{aligned} \tag{3.1.27}$$

Further examples may be found in the review article by Jackson (Jackson, 1965).

For our purposes it is not necessary to know the precise form of the operator \hat{T}_M^L . The crucial information is contained in equation (3.1.26). We note, incidentally, that there are $(2s+1)^2$ different T_M^L but from the properties of the vector-addition coefficients it can be shown that

$$T_{-M}^L = (-1)^M T_M^{L\dagger}. \tag{3.1.28}$$

We now proceed to derive the expansion of the density matrix ρ in terms of the matrices T_M^L .

Let us define the set of complex parameters t_M^L ($0 \leq L \leq 2s$) by

$$t_M^{L*} = \sum_{m,m'} \langle sm|sm'; LM \rangle \rho_{mm'}. \tag{3.1.29}$$

The inverse of this is

$$\rho_{mm'} = \frac{1}{2s+1} \sum_{L,M} (2L+1) \langle sm|sm'; LM \rangle t_M^{L*} \tag{3.1.30}$$

and, using (3.1.26),

$$\rho_{mm'} = \frac{1}{2s+1} \sum_{L,M} (2L+1) t_M^{L*} (T_M^L)_{mm'}. \tag{3.1.31}$$

Thus the matrix ρ is expanded in terms of the matrix set T_M^L as

$$\rho = \frac{1}{2s+1} \sum_{L,M} (2L+1) t_M^{L*} T_M^L. \tag{3.1.32}$$

This is the desired generalization of eqn (3.1.23). Now from the definition (3.1.26) it follows that

$$\text{Tr} \left(T_M^{L'} T_M^{L\dagger} \right) = \frac{2s+1}{2L+1} \delta_{LL'} \delta_{MM'} \tag{3.1.33}$$

and hence that

$$\text{Tr} \left(\rho T_M^{L\dagger} \right) = t_M^{L*}$$

or, since ρ is hermitian,

$$t_M^L = \text{Tr} (\rho T_M^L). \tag{3.1.34}$$

Thus the t_M^L , which are called either *multipole parameters* (of rank L) or *statistical tensors*, are a generalization of the spin-polarization vector. Indeed for the lowest-rank multipole parameters one has

$$\begin{aligned} t_0^0 &= \text{Tr} \rho = 1 \\ t_0^1 &= \mathcal{P}_z \sqrt{\frac{s}{s+1}} \\ t_1^1 &= -(\mathcal{P}_x + i\mathcal{P}_y) \sqrt{\frac{s}{2(s+1)}} \\ t_{-1}^1 &= (\mathcal{P}_x - i\mathcal{P}_y) \sqrt{\frac{s}{2(s+1)}} \end{aligned} \tag{3.1.35}$$

where the spin-polarization vector \mathcal{P} is defined in eqn (1.1.27).

We note that, from eqn (3.1.29) and the properties of the vector-addition coefficients, one has

$$t_{-M}^L = (-1)^M t_M^{L*}. \tag{3.1.36}$$

In particular the t_0^L are real.

Thus the set of t_M^L is actually specified by $(2s+1)^2$ real numbers. Bearing in mind that $t_0^0 = 1$ we see that the $(2s+1)^2 - 1$ remaining real parameters are just the right number to specify ρ completely.

The condition (3.1.15) leads to the inequality

$$\frac{1}{2s+1} \sum_{L,M} (2L+1) |t_M^L|^2 \leq 1. \tag{3.1.37}$$

We stress the fact that whether we choose to specify the set of numbers $\rho_{mm'}$ or the set of numbers t_M^L is merely a question of convenience. They are directly related by (3.1.29) or (3.1.30).

3.1.6 Multipole parameters for combined systems of particles

In the case of a combined system of different particles A, B, \dots , in analogy with the discussion in subsection 3.1.3 the joint density matrix $\rho(A, B, \dots)$ will be expanded in terms of the direct product of matrices $T(A)_M^L \otimes T(B)_{M'}^{L'} \otimes \dots$ with coefficients $t_{MM' \dots}^{LL' \dots}(A, B, \dots)$, the *joint multipole parameters*.

(Note that $T_M^L(A)$ is a matrix of dimension $2s_A + 1$. We will usually leave out the particle label on the T_M^L .)

If the different types of particles are uncorrelated, we will have

$$t_{MM' \dots}^{LL' \dots} = t_M^L t_{M'}^{L'} \dots \tag{3.1.38}$$

The *effective multipole parameters* $t_M^L(A)$ for particles of type A , say, when no spin measurement is carried out on the other particles, will be

$$t_M^L(A) = t_{M00\dots 0}^{L00\dots 0}. \tag{3.1.39}$$

3.1.7 Even and odd polarization

It sometimes happens that *only* the even-rank multipoles or *only* the odd-rank multipoles are non-zero. We refer to such states of polarizations as ‘even’ or ‘odd’ (Doncel *et al.*, 1970). When this happens the density matrix has a special symmetry, namely

$$\rho_{-\mu-\lambda} = \pm(-1)^{\lambda-\mu} \rho_{\lambda\mu}, \tag{3.1.40}$$

the (\pm) corresponding to an $\binom{\text{even}}{\text{odd}}$ state of polarization. In fact, it is sometimes convenient to break ρ up into its even and odd parts for a general state of polarization. Thus we write

$$\rho = \rho_+ + \rho_- \tag{3.1.41}$$

and

$$\begin{aligned} \rho_+ &= \frac{1}{2s+1} \sum_{L \substack{\text{even} \\ M}} (2L+1) t_M^L{}^* T_M^L \\ \rho_- &= \frac{1}{2s+1} \sum_{L \substack{\text{odd} \\ M}} (2L+1) t_M^L{}^* T_M^L. \end{aligned} \tag{3.1.42}$$

Equivalently

$$(\rho_{\pm})_{\lambda\mu} = \frac{1}{2} \left[\rho_{\lambda\mu} \pm (-1)^{\lambda-\mu} \rho_{-\lambda-\mu}^* \right]. \tag{3.1.43}$$

Thus, in general,

$$(\rho_{\pm})_{-\mu-\lambda} = \pm(-1)^{\lambda-\mu} \rho_{\lambda\mu}. \tag{3.1.44}$$

It will be seen later that it is usually easier to measure the elements of ρ_+ than those of ρ_- .

We note that if rank $\rho = r$ then

$$\text{rank } \rho_{\pm} \leq 2r. \tag{3.1.45}$$

3.1.8 The effect of rotations on the density matrix

Since the density matrix elements $\rho_{mm'}$ are given in a basis specified by the spin states $|s; s_z = m\rangle$, they are implicitly dependent on the choice of axis system.

We denote by $\rho_{mm'}^S$ the elements of the density matrix defined using as basis states the $|sm\rangle$ appropriate to the reference frame S and by $\rho_{mm'}^{S'}$ the

elements of the density matrix defined using as basis states the rotated states $|sm\rangle^r = U(r)|sm\rangle$, see (1.1.18), appropriate to S^r . Then, similarly to eqn (3.1.8), we have

$$\rho^{S^r} = \mathcal{D}^{(s)\dagger}(r)\rho^S\mathcal{D}^{(s)}(r) \quad (3.1.46)$$

or

$$\rho_{mm'}^{S^r} = \mathcal{D}_{nm}^{(s)*}(r)\rho_{nn'}^S\mathcal{D}_{n'm'}^{(s)}(r)$$

It is clear that ρ^{S^r} thus defined is the correct density matrix to use when evaluating expectation values as seen in reference frame S^r . For this reason we shall refer to ρ^{S^r} as the *density matrix in the frame S^r* .

The relationship between $\rho_{mm'}^{S^r}$ and $\rho_{mm'}^S$ is rather complicated. The formula can be simplified a little using the rules for the reduction of products of rotation matrices. One finds

$$\begin{aligned} \rho_{mm'}^{S^r} = & \sum_{J=0}^{2s} \sum_{n,n'=-s}^s (-1)^{m'-n'} \langle s, m; s, -m' | J, m - m' \rangle \\ & \times \langle s, n; s, -n' | J, M \rangle \mathcal{D}_{m-m', M}^{(J)}(r^{-1}) \rho_{nn'}^S. \end{aligned} \quad (3.1.47)$$

The multipole parameters t_M^L transform very simply, however. If $(t_M^L)_S$ and $(t_M^L)_{S^r}$ denote the components of the statistical tensors in the frames S and S^r then, from (3.1.46) and (3.1.29), one finds the simple result

$$(t_M^L)_{S^r} = \sum_{M'} \mathcal{D}_{M'M}^{(L)}(r)(t_{M'}^L)_S, \quad (3.1.48)$$

which is the usual rule relating the components of a spherical tensor in different reference frames.

3.1.9 Diagonalization of ρ . The quantization axis

Although it is always possible to diagonalize ρ it is *not* always possible to do so by means of an actual physical rotation of axes. If, however, the ensemble consists of a mixture of magnetic substates, i.e. eigenstates $|sm\rangle$ where m is the projection of \hat{s} along the quantization direction, then in a frame that has OZ along the quantization direction clearly ρ will be diagonal, and all multipole parameters t_M^L with $M \neq 0$ will be zero. We shall refer to the direction OZ that makes ρ diagonal as the *quantization axis*. For spin-1/2 particles the quantization axis coincides with the polarization vector \mathcal{P} , but the quantization axis is a somewhat more general concept since for higher spins one can easily have the vector polarization zero yet still have some 'alignment' along the quantization axis (see subsection 3.1.12 below).

3.1.10 Other choices of basis matrices

The case we know best, namely spin 1/2, is misleadingly simple. Here we succeed in expanding the hermitian ρ in terms of the *hermitian* Pauli matrices σ_j with real coefficients of direct observable relevance, and at the same time we enjoy the very simple properties of the σ_i under rotations.

The T_M^L used for the general case have simple rotation properties but are not hermitian. As a result the t_M^L are complex and are not so closely related to what is actually measured.

In fact, it is very easy for arbitrary spin s to introduce a set of *hermitian* basis matrices (Doncel *et al.*, 1970) Q_M^L , defined as follows:

$$\begin{aligned} M \geq 1 & \quad Q_M^L = \frac{(-1)^M}{2} \sqrt{\frac{2L+1}{s}} \{T_M^L + T_M^{L\dagger}\} \\ M = 0 & \quad Q_0^L = \sqrt{\frac{2L+1}{2s}} T_0^L \\ M \leq -1 & \quad Q_M^L = \frac{(-1)^M}{2i} \sqrt{\frac{2L+1}{s}} \{T_{-M}^L + T_{-M}^{L\dagger}\} \end{aligned} \quad (3.1.49)$$

with corresponding *real* multipole parameters r_M^L given by

$$\begin{aligned} M \geq 1 & \quad r_M^L = (-1)^M \sqrt{\frac{2L+1}{s}} \operatorname{Re} t_M^L \\ M = 0 & \quad r_0^L = \sqrt{\frac{2L+1}{2s}} t_0^L \\ M \leq -1 & \quad r_M^L = (-1)^M \sqrt{\frac{2L+1}{s}} \operatorname{Im} t_{-M}^L \end{aligned} \quad (3.1.50)$$

and the density matrix expansion

$$\rho = \frac{1}{2s+1} \left(I + 2s \sum_{L=1}^{2s} \sum_{M=-L}^L r_M^L Q_M^L \right). \quad (3.1.51)$$

This approach is especially useful for discussing the ‘domain’ of the density matrix, i.e. the range of permitted values for the parameters specifying ρ . However, the price one pays is that the rotational properties of the Q_M^L and hence of the r_M^L are more complicated.

For this reason, we have chosen to develop our general treatment of reactions in terms of the usual T_M^L .

3.1.11 Invariant characterization of the state of polarization of an ensemble

Full information about the state of an ensemble requires a knowledge of the whole density matrix. It is useful, however, to have a simple, invariant, albeit cruder, characterization of the ensemble. Thus for spin 1/2 we talk of an *unpolarized* ensemble or a polarized ensemble with *degree of polarization* $\mathcal{P} = \sqrt{\mathcal{P}^2}$. We wish to generalize these concepts to arbitrary spin.

An unpolarized or *isotropic* ensemble of spin- s particles has equal probabilities $p^{(i)} = 1/(2s + 1)$ of being in any pure state $|\psi^{(i)}\rangle$ and is therefore given by the density matrix

$$\rho_{\text{iso}} = \frac{1}{2s + 1} I \quad (3.1.52)$$

in any basis.

Therefore the matrix

$$\rho - \rho_{\text{iso}} = \frac{1}{2s + 1} \sum_{\substack{L \geq 1 \\ M}} (2L + 1) t_M^L * T_M^L \quad (3.1.53)$$

measures the *departure from isotropy* (Doncel *et al.*, 1972).

To characterize this difference in a rotationally invariant fashion we have to introduce some measure of the ‘difference’ between two matrices, or, as it is often described, the ‘distance’ between them.

A suitable invariant measure is

$$\text{Tr} (\rho - \rho_{\text{iso}})^2 = \text{Tr} \rho^2 - \text{Tr} \rho_{\text{iso}}^2 = \text{Tr} \rho^2 - \frac{1}{2s + 1} \quad (3.1.54)$$

by (3.1.10) and (3.1.52). In fact, the ratio $(\text{Tr} \rho^2 - \text{Tr} \rho_{\text{iso}}^2) / \text{Tr} \rho_{\text{iso}}^2$ takes the value zero for an unpolarized ensemble and the value $2s$ for a pure state. We thus define the *overall degree of polarization*

$$d \equiv \frac{1}{\sqrt{2s}} \left[(2s + 1) \text{Tr} \rho^2 - 1 \right]^{1/2} \quad (3.1.55)$$

so that

$$0 \leq d \leq 1.$$

For spin 1/2, as expected,

$$d = \mathcal{P} = \sqrt{\mathcal{P}^2} \quad (3.1.56)$$

but this case is misleadingly simple. For higher spin we can have vector polarization, rank-2 tensor polarization etc. and the magnitude of the vector polarization is no longer the overall degree of polarization.

The representation in terms of Cartesian spin matrices gets very clumsy for higher spin so we restrict ourselves to the multipole parameter expansion (3.1.32). One can define a measure of rank- L polarization ($L \geq 1$) by

$$d_L = \sqrt{\frac{2L+1}{2s}} \left(\sum_M |t_M^L|^2 \right)^{1/2} \quad (3.1.57)$$

and the overall degree of polarization is then

$$d = \left\{ \sum_{L \geq 1} d_L^2 \right\}^{1/2}. \quad (3.1.58)$$

However, the d_L can be a little misleading since the individual d_L cannot usually attain the value 1, although d itself can. (For example, for spin-1 particles $(d_1)_{\max} = \sqrt{3}/2$.)

3.1.12 Spin-1 particles and photons

(i) Massive particles

With the production and general use of polarized deuteron beams this case has become of great interest and we therefore treat it in some detail.

The density matrix can either be written in the standard form (3.1.32) involving multipole parameters or it can be given in a Cartesian form as follows:

$$\rho = \frac{1}{3} \left[1 + \frac{3}{2} \mathcal{P} \cdot \mathbf{S} + \sqrt{\frac{3}{2}} T_{ij} (S_i S_j + S_j S_i) \right] \quad (3.1.59)$$

with T_{ij} real and symmetric, and traceless: $\sum_i T_{ii} = 0$. Here \mathbf{S} stands for the 3×3 traceless matrices S_j representing the spin operators \hat{s}_j for spin 1:

$$\begin{aligned} S_x &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & S_y &= \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\ S_z &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{aligned} \quad (3.1.60)$$

The three real parameters \mathcal{P}_j and the five independent T_{ij} are all independent of each other.

Other definitions of \mathcal{P} and T_{ij} are sometimes given in the literature (Werle, 1966) but ours are designed to have the simplest physical interpretation. One finds that \mathcal{P} is the spin-polarization vector

$$\mathcal{P} = \langle \hat{\mathbf{s}} \rangle. \quad (3.1.61)$$

in agreement with (1.1.27) for the case $s = 1$ and T_{ij} measures the rank-2 spin tensor

$$T_{ij} = \frac{1}{2} \sqrt{\frac{3}{2}} \left(\langle \hat{s}_i \hat{s}_j + \hat{s}_j \hat{s}_i \rangle - \frac{4}{3} \delta_{ij} \right). \quad (3.1.62)$$

The degrees of vector polarization \mathcal{P} and of tensor polarization T are

$$\mathcal{P} = \sqrt{\mathcal{P}^2} \quad 0 \leq \mathcal{P} \leq 1 \quad (3.1.63)$$

and

$$T = \sqrt{\sum_{ij} (T_{ij})^2} \quad 0 \leq T \leq 1. \quad (3.1.64)$$

The overall degree of polarization is

$$d = \left(\frac{3}{4} \mathcal{P}^2 + T^2 \right)^{1/2}. \quad (3.1.65)$$

The multipole parameters are related to \mathcal{P}_j and T_{ij} via

$$t_0^1 = \frac{1}{\sqrt{2}} \mathcal{P}_z \quad t_{\pm 1}^1 = \mp \frac{1}{2} (\mathcal{P}_x \pm i \mathcal{P}_y) \quad (3.1.66)$$

and

$$\begin{aligned} t_0^2 &= \sqrt{\frac{3}{5}} T_{zz} & t_{\pm 1}^2 &= \mp \sqrt{\frac{2}{5}} (T_{xz} \pm i T_{yz}) \\ t_{\pm 2}^2 &= \sqrt{\frac{1}{10}} (T_{xx} - T_{yy} \pm 2i T_{xy}). \end{aligned} \quad (3.1.67)$$

They are related to the elements of the density matrix itself via

$$t_1^1 = -\frac{1}{\sqrt{2}} (\rho_{10} + \rho_{0-1})^* \quad t_0^1 = \frac{1}{\sqrt{2}} (\rho_{11} - \rho_{-1-1})^* \quad (3.1.68)$$

and

$$\begin{aligned} t_2^2 &= \sqrt{\frac{3}{5}} \rho_{1-1}^* & t_1^2 &= -\sqrt{\frac{3}{10}} (\rho_{10} - \rho_{0-1})^* \\ t_0^2 &= \sqrt{\frac{1}{10}} (\rho_{11} + \rho_{-1-1} - 2\rho_{00})^*. \end{aligned} \quad (3.1.69)$$

Often the ensemble is made up of particles whose spin is quantized along the Z -axis. Let p_+ , p_0 and p_- be the probabilities of finding a particle with spin projection 1, 0, -1 respectively along the quantization axis. Then from (3.1.61) and (3.1.62)

$$\begin{aligned} \mathcal{P}_x &= \mathcal{P}_y = 0 & \mathcal{P}_z &= (p_+ - p_-) \\ T_{ij} &= 0 \text{ if } i \neq j & T_{xx} &= T_{yy} = -\frac{1}{2} T_{zz} & T_{zz} &= \frac{1}{\sqrt{6}} (1 - 3p_0) \end{aligned} \quad (3.1.70)$$

The degree of vector and tensor polarization are then

$$\mathcal{P} = |p_+ - p_-| \quad T = \frac{1}{2} |1 - 3p_0| \quad (3.1.71)$$

and the density matrix is

$$\rho = \frac{1}{3} \begin{pmatrix} 1 + \frac{3}{2}\mathcal{P}_z + \sqrt{\frac{3}{2}}T_{zz} & 0 & 0 \\ 0 & 1 - \sqrt{6}T_{zz} & 0 \\ 0 & 0 & 1 - \frac{3}{2}\mathcal{P}_z + \sqrt{\frac{3}{2}}T_{zz} \end{pmatrix} \quad (3.1.72)$$

In this case, in the frame with OZ along the quantization axis, the multipole parameters take the simple form

$$\begin{aligned} t_0^1 &= \sqrt{\frac{1}{2}}\mathcal{P}_z & t_0^2 &= \sqrt{\frac{3}{5}}T_{zz} = \sqrt{\frac{1}{10}}\mathcal{A} \\ t_{\pm 1}^1 &= t_{\pm 1}^2 = t_{\pm 2}^2 & &= 0 \end{aligned} \quad (3.1.73)$$

where

$$\mathcal{A} = 1 - 3p_0 \quad (3.1.74)$$

is referred to as the *alignment* (Steenberg, 1953).

It should be noted that ensembles of the above type are by no means the most general ones for spin-1 particles. To discuss the general case, consider the orthonormal basis states

$$\begin{aligned} |\mathbf{e}_{(x)}\rangle &\equiv \frac{1}{\sqrt{2}} (|\lambda = -1\rangle - |\lambda = +1\rangle) \\ |\mathbf{e}_{(y)}\rangle &\equiv \frac{i}{\sqrt{2}} (|\lambda = -1\rangle + |\lambda = +1\rangle) \\ |\mathbf{e}_{(z)}\rangle &\equiv |\lambda = 0\rangle \end{aligned} \quad (3.1.75)$$

The most general normalized pure spin state for a spin-1 particle is then

$$|\boldsymbol{\varepsilon}\rangle = \epsilon_x|\mathbf{e}_{(x)}\rangle + \epsilon_y|\mathbf{e}_{(y)}\rangle + \epsilon_z|\mathbf{e}_{(z)}\rangle \quad (3.1.76)$$

where $\boldsymbol{\varepsilon} = (\epsilon_x, \epsilon_y, \epsilon_z)$, the polarization vector, is a complex vector with

$$\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon} = 1. \quad (3.1.77)$$

It is $\boldsymbol{\varepsilon}$ that is the analogue of the polarization vector in classical electrodynamics.

For a *pure* state one finds that the spin polarization vector \mathcal{P} is related to the polarization vector $\boldsymbol{\varepsilon}$ via

$$\mathcal{P} = \text{Im} (\boldsymbol{\varepsilon}^* \times \boldsymbol{\varepsilon}); \quad (3.1.78)$$

this will be given a covariant form in Section 3.4. Thus $\mathcal{P} = 0$ for any pure state with real $\boldsymbol{\varepsilon}$.

For the tensor T_{ij} one finds

$$T_{ij} = \sqrt{\frac{3}{2}} \left[\frac{1}{3}\delta_{ij} - \text{Re} (\epsilon_i^* \epsilon_j) \right]. \quad (3.1.79)$$

Note that there is no pure state for which $T_{ij} = 0$ for *all* i and j .

For example, for states with $s_z = \pm 1, 0$ we have

$$\begin{aligned}\mathbf{e}^{(\pm)} &= \frac{1}{\sqrt{2}}(\mp 1, -i, 0) \\ \mathbf{e}^{(0)} &= (0, 0, 1)\end{aligned}\tag{3.1.80}$$

so that

$$\begin{aligned}\mathcal{P}^{(+)} &= (0, 0, 1) \\ \mathcal{P}^{(-)} &= (0, 0, -1). \\ \mathcal{P}^{(0)} &= (0, 0, 0)\end{aligned}\tag{3.1.81}$$

(ii) Photons

Although intrinsically relativistic we may treat photons as above provided, as will be justified in Section 3.2, we interpret the states $|s_z = \pm 1\rangle$ as helicity states for the photons moving along the direction OZ . Of course the states $|s_z = 0\rangle$ are now absent. As a consequence, an ensemble of photons can never be isotropic. Indeed, from (3.1.73) and (3.1.74) we see that for all ensembles of photons $T_{zz} = 1/\sqrt{6}$ and therefore $t_0^1 = \sqrt{1/10}$.

(α) *Circular polarization.* A photon with helicity ± 1 is said to be circularly polarized. For a mixture of such states (3.1.72) becomes, since now $T_{zz} = 1/\sqrt{6}$,

$$\rho_\gamma^{\text{circ}} = \frac{1}{2} \begin{pmatrix} 1 + \mathcal{P}_{\text{circ}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - \mathcal{P}_{\text{circ}} \end{pmatrix}.\tag{3.1.82}$$

$\mathcal{P}_{\text{circ}}$ is conventionally referred to as the *circular polarization* of the photons. From (3.1.70) $\mathcal{P}_{\text{circ}}$ is given in terms of the probabilities for finding helicity $+1$ and helicity -1 polarized photons as

$$\mathcal{P}_{\text{circ}} = p_+ - p_-\tag{3.1.83}$$

as expected.

Note that $\mathcal{P}_{\text{circ}} = +1$ corresponds to photons with *positive* helicity. In terms of the electric field vector of a classical electromagnetic wave propagating along OZ , the case $\mathcal{P}_{\text{circ}} = +1$ corresponds to the case when the electric field vector is seen to rotate *anticlockwise* when looking into the wave. In optics this is referred to as *left-circularly polarized light*.

In the case of circular polarization the spin-polarization vector and the multipole parameters are given by

$$\mathcal{P}_{\text{circ}} = (0, 0, \mathcal{P}_{\text{circ}})\tag{3.1.84}$$

$$t_0^1 = \frac{1}{\sqrt{2}}\mathcal{P}_{\text{circ}} \quad t_0^2 = \frac{1}{\sqrt{10}} \quad t_{\pm 1}^1 = t_{\pm 2}^2 = t_{\pm 1}^2 = 0$$

Note that $\mathcal{P}_{\text{circ}}$ is, in magnitude, a measure of the degree of *vector* polarization.

Because of the absence of the $|m = 0\rangle$ states it is sometimes convenient to write (3.1.82) in the form

$$\rho_\gamma^{\text{circ}} = \frac{1}{2}(I + \mathcal{P}_{\text{circ}}\sigma_z). \quad (3.1.85)$$

(β) *Linear polarization.* A photon is said to be linearly polarized along OX or OY if its state is $|\mathbf{e}_{(x)}\rangle$ or $|\mathbf{e}_{(y)}\rangle$ respectively as defined in (3.1.75).

Consider a mixture of photons linearly polarized along the directions OX' , OY' in the XY -plane, where OX' and OY' make an angle γ with OX and OY respectively. The *linear polarization along OX'* is defined by

$$\mathcal{P}_{\text{lin}} = p_{x'} - p_{y'}$$

where $p_{x'}$, $p_{y'}$ are the probabilities for finding photons linearly polarized along OX' and OY' respectively.

Using the fundamental definition (3.1.5) of the density matrix, and eqn (3.1.46), we get the density matrix for photons linearly polarized in the XY -plane at angle γ to the X -axis:

$$\rho_\gamma^{\text{lin}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -\mathcal{P}_{\text{lin}}e^{-2i\gamma} \\ 0 & 0 & 0 \\ -\mathcal{P}_{\text{lin}}e^{2i\gamma} & 0 & 1 \end{pmatrix}. \quad (3.1.86)$$

In this case the spin-polarization vector and multipole parameters are given by

$$\begin{aligned} \mathcal{P}_{\text{lin}} &= (0, 0, 0) \\ t_2^2 &= -\frac{1}{2}\sqrt{\frac{3}{5}}\mathcal{P}_{\text{lin}}e^{2i\gamma} & t_0^2 &= \frac{1}{\sqrt{10}} & t_m^1 &= t_{\pm 1}^2 = 0 \end{aligned} \quad (3.1.87)$$

and \mathcal{P}_{lin} contributes only to the tensor polarization, as follows from (3.1.67).

Again, it is sometimes useful to abbreviate (3.1.86) in the form

$$\rho_\gamma^{\text{lin}} = \frac{1}{2} [I - \mathcal{P}_{\text{lin}}(\cos 2\gamma \sigma_x + \sin 2\gamma \sigma_y)]. \quad (3.1.88)$$

The physical interpretation of (3.1.82) and (3.1.86) when the photon has momentum $\mathbf{p} = (p, \theta, \varphi)$ will be explained in subsection 3.2.1.

(γ) *Mixed polarization.* Although light sources are usually either linearly or circularly polarized, it is in principle possible to have a mixture of both.

Let f be the fraction of circularly polarized photons and $1 - f$ the

fraction linearly polarized. Then the density matrix for the mixture is just

$$\rho_\gamma = f\rho_\gamma^{\text{circ}} + (1 - f)\rho_\gamma^{\text{lin}}.$$

3.1.13 Positivity of the density matrix

The density matrix, being hermitian, can always be diagonalized. In a basis in which it is diagonal it is clear that its elements $\rho_{mm} = \lambda_m$ simply measure the probability p_m of finding the state $|m\rangle$ in the ensemble. Thus the eigenvalues of ρ are either positive or zero.

A hermitian matrix whose diagonal elements have this property is called a *positive semi-definite* matrix. When a density matrix is measured experimentally it is essential to check that the matrix so obtained is indeed positive semi-definite. If it is not, this is a sure indicator of experimental error. Unfortunately it is a non-trivial task to get enough information experimentally to allow the calculation of the eigenvalues of ρ ; it requires a knowledge of the whole matrix ρ .

Often, however, ρ is only partially known and it is important to be able to test whether this partial knowledge is compatible with the ultimate positive semi-definiteness of ρ . Thus we require criteria for the positivity of ρ that do not involve a knowledge of its eigenvalues.

The most useful result is the following. Let ρ_{ij} be the elements of ρ in any basis. Then every principal minor of the matrix must be positive semi-definite, i.e. if in some basis

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \dots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \dots & \rho_{2n} \\ \vdots & & & \vdots \\ \rho_{n1} & \rho_{n2} & \dots & \rho_{nn} \end{pmatrix} \tag{3.1.89}$$

then one must have

- (1) $\rho_{jj} \geq 0$ for every j
- (2) $\begin{vmatrix} \rho_{jj} & \rho_{jk} \\ \rho_{kj} & \rho_{kk} \end{vmatrix} \geq 0$ for every j and all $k > j$
- (3) $\begin{vmatrix} \rho_{jj} & \rho_{jk} & \rho_{jl} \\ \rho_{kj} & \rho_{kk} & \rho_{kl} \\ \rho_{lj} & \rho_{lk} & \rho_{ll} \end{vmatrix} \geq 0$ for every j and all $l > k > j$
- \vdots
- (n) $\begin{vmatrix} \rho_{11} & \rho_{12} & \dots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \dots & \rho_{2n} \\ \vdots & & & \vdots \\ \rho_{n1} & \rho_{n2} & \dots & \rho_{nn} \end{vmatrix} \geq 0.$

Failure of any one of these conditions will imply that ρ is not positive semi-definite.

Thus even partial measurements of ρ can be tested for compliance.

When ρ is diagonal it is trivial to see the consequences of positivity. For example in the case of spin 1, from (3.1.72) one has clearly

$$T_{zz} \leq 1/\sqrt{6} \quad (3.1.90)$$

and

$$-\frac{2}{3} \left(1 + \sqrt{\frac{3}{2}} T_{zz} \right) \leq \mathcal{P}_z \leq \frac{2}{3} \left(1 + \sqrt{\frac{3}{2}} T_{zz} \right)$$

which, combined with (3.1.90), gives

$$-1 \leq \mathcal{P}_z \leq 1. \quad (3.1.91)$$

For a more detailed analysis of the positivity conditions and an introduction to the concept of the *polarization domain* the reader should consult the review of Bourrely, Leader and Soffer (1980).

3.2 The relativistic case

We turn now to the relativistic case and introduce the helicity density matrix. All the properties discussed in Section 3.1 remain valid provided that care is exercised in the physical interpretation.

3.2.1 Definition of the helicity density matrix

Using as basis the helicity states discussed in Section 1.2 we can formally define the density matrix ρ in a given reference frame in which the particle is moving with momentum \mathbf{p} , in exact analogy with the non-relativistic case. If we have an ensemble of particles, all with momentum \mathbf{p} but distributed with probability $p^{(i)}$ over various states $|\psi^{(i)}; \mathbf{p}\rangle$, where

$$|\psi^{(i)}; \mathbf{p}\rangle = \sum_{\lambda=-s}^s c_{\lambda}^{(i)} |\mathbf{p}; \lambda\rangle, \quad (3.2.1)$$

then we define ρ by

$$\rho_{\lambda\lambda'} = \sum_i p^{(i)} c_{\lambda}^{(i)} c_{\lambda'}^{(i)*}. \quad (3.2.2)$$

The only question is: what is the physical meaning and use of this matrix?

In Section 1.2 we discussed the physical interpretation of helicity states. From this it is clear that $\rho_{\lambda\lambda'}$ for a given particle A is the ordinary, non-relativistic spin density matrix for particle A if we observe A in the *helicity rest frame of A* .

Thus for any observable \hat{O} connected with particle A , $\text{Tr}(\rho O)$ is the expectation value of \hat{O} for the ensemble, in the helicity rest frame of A .

If there is a mixture of several particle types, as in the initial or final state of a reaction, then one can, as in the non-relativistic case, define a joint density matrix using helicity states as a basis. For example, for two types of particles A, B one will have $\rho(A, B)$ with matrix elements $\rho(A, B)_{\lambda, \mu; \lambda', \mu'}$ defined in terms of simple direct products of the helicity states of A and B . This density matrix thus describes the spin distributions in the respective helicity rest frames of A and B . If \hat{O} is an observable connected with both particles A and B then $\text{Tr}[\rho(A, B)O]$ gives the ensemble expectation value of \hat{O} for measurements on A carried out in the helicity rest frame of A and measurements on B carried out in the helicity rest frame of B . As in the non-relativistic case, if we measure an observable belonging only to one of the particles, say A , then we require the $(2s_A + 1) \times (2s_A + 1)$ effective density matrix for A , $\rho(A)$, where

$$\rho(A)_{\lambda\lambda'} = \sum_{\mu} \rho(A, B)_{\lambda, \mu; \lambda', \mu}; \quad (3.2.3)$$

then

$$\langle \hat{O}^{(A)} \rangle = \text{Tr}[\rho(A)O^{(A)}]. \quad (3.2.4)$$

An identical result holds for B .

In a similar fashion, for massless particles ρ gives the density matrix of a particle in the standard frame where its momentum is $p^\mu = (p, 0, 0, p)$.

3.2.2 Definition of helicity multipole parameters

Because of the simple, i.e. non-relativistic, meaning of the helicity density matrix in the respective helicity rest frames, it is clear that multipole parameters defined in terms of ρ , as in the non-relativistic case, will also enjoy the same simple rotational properties.

Thus for any particle A we define

$$t_M^L(A) = \sum_{\lambda, \lambda'} \langle s\lambda | s\lambda' ; LM \rangle \rho_{\lambda\lambda'}(A) \quad (3.2.5)$$

as the helicity-basis *multipole parameters* for A .

In A 's helicity rest frame S_A , the $t_M^L(A)$ are just the non-relativistic multipole parameters corresponding to an axis system coinciding with S_A .

Joint helicity multipole parameters are defined in terms of joint helicity density matrices, exactly as in Section 3.1, and all the properties derived there hold equally well.

3.2.3 The effect of Lorentz transformations on the helicity density matrix

(i) *Rotations of rest frame.* Let $\rho(A)$ be the helicity density matrix of A . As discussed above $\rho(A)$ is the density matrix for A in its helicity rest frame. The density matrix for A in any other rest frame S'_A is simply obtained from $\rho(A)$ by a rotation. If $S'_A = rS_A$ then by (3.1.46)

$$\rho^{S'_A}(A) = \mathcal{D}^{(s)\dagger}(r)\rho(A)\mathcal{D}^{(s)}(r). \quad (3.2.6)$$

(ii) *Lorentz transformations.* The density matrix in a Lorentz-transformed frame is obtained as follows.

Let $\rho^S(\mathbf{p})$ be the helicity density matrix in frame S where the particle has momentum \mathbf{p} . Let $\rho^{S^l}(\mathbf{p}')$ be the helicity density matrix in the frame $S^l = lS$ obtained from S by an arbitrary Lorentz transformation l and in which the particle has momentum $\mathbf{p}' = l^{-1}\mathbf{p}$.

If we think of $\rho^{S^l}_{\mu\mu'}(\mathbf{p}')$ as the matrix of an operator $\hat{\rho}'$,

$$\rho^{S^l}_{\mu\mu'}(\mathbf{p}') = \langle \mathbf{p}'; \mu | \hat{\rho}' | \mathbf{p}'; \mu' \rangle,$$

then it is clear, from the meaning of $\rho_{\lambda\lambda'}$ as a probability correlation, that we must have the numerical relation

$$\rho^{S^l}_{\lambda\lambda'}(\mathbf{p}') = {}_{S^l} \langle \mathbf{p}; \lambda | \hat{\rho}' | \mathbf{p}; \lambda' \rangle_{S^l}$$

from which, using (2.1.9), we eventually obtain

$$\rho^{S^l}(\mathbf{p}') = \mathcal{D}[r(l, \mathbf{p})]\rho^S(\mathbf{p})\mathcal{D}^\dagger[r(l, \mathbf{p})] \quad (3.2.7)$$

where $r(l, \mathbf{p})$ is the Wick helicity rotation defined in Section 2.1.

Note that if the particle is *not at rest* in S , then for any frame S^r obtained from S by a rotation r , eqn (3.2.7) reduces to

$$\rho^{S^r}_{\lambda\lambda'}(\mathbf{p}') = \rho^S_{\lambda\lambda'}(\mathbf{p})e^{i\zeta(\lambda-\lambda')} \quad (3.2.8)$$

where ζ is given in (2.2.3). If the particle *is at rest* in S then (3.2.6) holds.

3.2.4 Transformation law for multipole parameters

It was stressed earlier that the multipole parameters t^L_M transform more simply under rotations than does ρ . Because Lorentz transformations are effected ultimately by just the Wick helicity rotation, also in this case the helicity-basis multipole parameters defined in eqn (3.2.5) will enjoy simpler transformation properties. Thus the analogue of (3.2.7) is

$$({}^L t_M)_{S^l} = \mathcal{D}^{(L)\dagger}_{M'M}(r(l, \mathbf{p}))[{}^L t_{M'}]_S. \quad (3.2.9)$$

Note that for a spin-1/2 particle, if we write

$$\rho^{S^l} = \frac{1}{2}(I + \mathcal{P}' \cdot \boldsymbol{\sigma}) \quad \rho = \frac{1}{2}(I + \mathcal{P} \cdot \boldsymbol{\sigma})$$

then from (3.2.7) and (1.1.28)

$$\mathcal{P}'_i = R_{ij}(r)\mathcal{P}_j \quad (3.2.10)$$

where r is short for $r(l, \mathbf{p})$.

Analogous results will hold for \mathcal{P} and T_{ij} for spin-1 particles, etc.

In the case of a pure rotation for which ζ in (3.2.8) happens to be zero, so that $\rho^{S^r} = \rho$, putting

$$\rho^{S^r} = \frac{1}{2} (I + \mathcal{P}' \cdot \boldsymbol{\sigma})$$

implies the perhaps surprising result $\mathcal{P}' = \mathcal{P}$. It must not be forgotten that \mathcal{P}'_i and \mathcal{P}_i are the components of the spin-polarization vector in the helicity rest frames reached from S^r and S respectively and that these rest frames coincide in this particular case.

3.3 Choices of reference frame for a reaction

We consider a general reaction



taking place either in the Lab frame, corresponding to a fixed-target experiment, where B is at rest, or in a frame, corresponding to collider physics, where A and B collide head-on. In the latter case the frame may or may not be the actual CM frame.

The actual choice of axes is partly a matter of convention, partly a matter of convenience in the context of the particular experiment.

Quite generally the collision axis is taken as the Z -axis. In the Lab frame OZ is taken to lie along the incoming beam.

The choice of Y -axis depends on the kind of experiment. Much of the early work on spin-dependent reactions utilized fixed spectrometers, which therefore defined the reaction plane; the spin-polarization vectors of beam and target particles, which could be varied in the experiment, were referred to this reaction plane.

For the $2 \rightarrow 2$ reaction $A + B \rightarrow C + D$, according to the so-called Basel convention OY is defined to lie along the *normal to the reaction plane*, defined as the direction of $\mathbf{p}_A \times \mathbf{p}_C$.

In more modern experiments, where the collision axis is surrounded by detectors, there is no obviously preferred fixed reaction plane and OY is then chosen arbitrarily, according to convenience. This is particularly important when a reaction is being used as an analysing reaction, i.e. to measure the direction and magnitude of the spin-polarization vectors of a beam and/or target; it may then be necessary to perform weighted integrals over the ϕ -dependent angular distributions in some fixed reference frame in order to determine the components of the spin-polarization vectors in that frame.

In doing calculations it is generally simplest to work with the CM helicity amplitudes, which are directly related to the CM helicity density matrix (see Section 5.3). Thus it is helpful to specify the states of the particles and to carry out analysing measurements on the particles, in frames that are related to the CM simply by a Lorentz transformation.

3.3.1 Density matrix for the initial particles

It is simplest to give the density matrices or multipole parameters of A and B in their helicity rest frames S_A , S_B , reached from the CM of the reactions as shown in Fig. 3.1. (Note that Y_A and Y_B are in opposite directions.) These are then the correct parameters to use in specifying the initial state in the reaction CM.

The laboratory (Lab) frame will always be taken to have the same orientation of axes as the CM frame and is to be thought of as reached in the limit as we boost along the negative Z -axis until B is just barely at rest. The helicity rest frames for the initial particles A , B reached from the Lab frame will then coincide with those shown in Fig. 3.1.

Note that for the target the axes of S_B do not point in the same direction as the Lab axes. One has

$$S_B = r_{y'}(\pi)r_z(\pi)S_L$$

and spin information about the target, if specified in the Lab frame, must always be transformed into S_B .

Often, however, for magnetically prepared beam and target, the polarization information is given in a rest frame whose Z -axis (\hat{Z}) lies along the quantization axis. Let this axis have polar angles $\theta = \beta$, $\phi = \gamma$ relative to the axes of the CM frame (or of the Lab frame) as shown in Fig. 3.2.

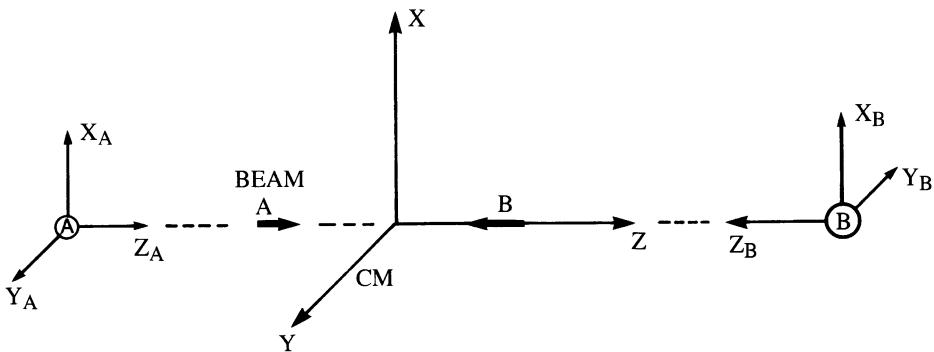


Fig. 3.1. Helicity rest frames for beam and target reached from the CM.

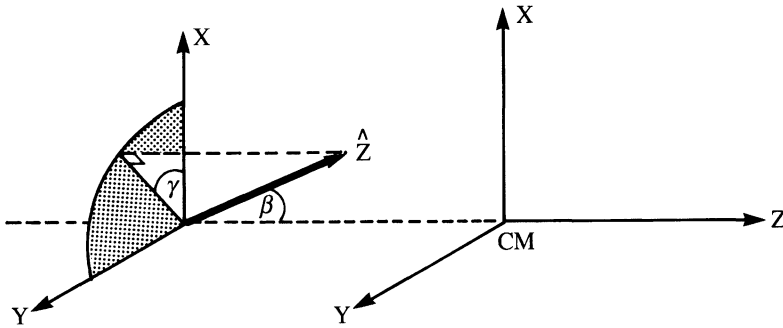


Fig. 3.2. Rest frame with $O\hat{Z}$ along quantization axis.

In this rest frame ρ_{ij} is diagonal, its elements being just the probability of the various magnetic substates. Equivalently, the multipole parameters are such that $t_m^l = 0$ for $m \neq 0$. Let us label the non-zero multipole parameters in this frame by \hat{t}_0^l .

Then for particle A , coming in along OZ , the CM helicity multipole parameters are

$$t_m^l(A) = e^{i\gamma_A m} d_{m0}^l(\beta_A) \hat{t}_0^l(A) \quad (3.3.1)$$

as follows from eqn (3.1.48).

For particle B , moving in the negative OZ direction, the CM helicity multipole parameters are

$$t_m^l(B) = e^{-i\gamma_B m} d_{m0}^l(\pi - \beta_B) \hat{t}_0^l(B) \quad (3.3.2)$$

wherein account has been taken of the fact that Y_B in Fig. 3.1 is opposite in direction to Y_{CM} .

As an example consider an electron or nucleon with spin-polarization vector $(0, 0, \mathcal{P}_z)$ along or opposite to $O\hat{Z}$. From (3.1.35)

$$\hat{t}_0^1 = \sqrt{\frac{1}{3}} \mathcal{P}_z. \quad (3.3.3)$$

For the case of *longitudinal* polarization, $O\hat{Z}$ lies along OZ or opposite to it. For particle A , (3.3.3) then gives, for degree of polarization \mathcal{P}_A ,

$$t_0^1(A) = \pm \sqrt{\frac{1}{3}} \mathcal{P}_A \quad t_{\pm 1}^1 = 0 \quad (3.3.4)$$

for longitudinal polarization along or opposite to A 's motion.

Similarly

$$t_0^1(B) = \mp \sqrt{\frac{1}{3}} \mathcal{P}_B \quad t_{\pm 1}^1 = 0 \quad (3.3.5)$$

for longitudinal polarization along or opposite to B 's motion.

For the case of *transverse* polarization, say perpendicular to the plane XZ , we get

$$t_0^1(A) = 0 \quad t_1^1(A) = t_{-1}^1(A) = \mp \sqrt{\frac{1}{6}} i \mathcal{P}_A \quad (3.3.6)$$

where \mathcal{P}_A is the degree of polarization along or opposite to OY_{CM} .

For particle B one has

$$t_0^1(B) = 0 \quad t_1^1(B) = t_{-1}^1(B) = \pm \sqrt{\frac{1}{6}} i \mathcal{P}_B \quad (3.3.7)$$

where \mathcal{P}_B is the degree of polarization along or opposite to OY_{CM} .

As a second example consider a beam of photons incident along OZ_{CM} . Here we may directly use the results (3.1.82), (3.1.83) and (3.1.86) for the density matrix. For the case of circular polarization one has from (3.1.82), (3.1.68) and (3.1.69)

$$t_0^1 = \frac{1}{\sqrt{2}} \mathcal{P}_{\text{circ}} \quad t_0^2 = \frac{1}{\sqrt{10}} \quad (3.3.8)$$

with all other $t_m^l = 0$ for $l \neq 0$.

For the case of linear polarization in the XY -plane at angle γ to the X -axis, from (3.1.86), (3.1.68) and (3.1.69),

$$t_2^2 = -\frac{1}{2} \sqrt{\frac{3}{5}} \mathcal{P}_{\text{lin}} e^{2i\gamma} \quad t_0^2 = \frac{1}{\sqrt{10}} \quad (3.3.9)$$

with all other $t_m^l = 0$ for $l \neq 0$.

If the linear polarization of the photon is specified with respect to the CM X - and Y -axes and if the photon is incident in the *negative* Z -direction, then, bearing in mind Fig. 3.1, one must take $\gamma \rightarrow -\gamma$ in (3.3.9).

3.3.2 Density matrix of final state particles

The density matrix of a produced particle may be obtained experimentally from studying the decay of the particle or by letting it undergo a secondary analysing reaction.

Unfortunately many conventions exist and many different frames have been used in the past for this analysis. A comprehensive discussion of the Adair, Gottfried–Jackson, and transversity frames can be found in Bourrely, Leader and Soffer (1980).

The frame in which one wishes to know the density matrix is dictated by the kind of reaction under study. There are basically two situations:

- (i) reactions in which a resonance is produced and its decay studied;
- (ii) reactions in which a stable final state particle undergoes a secondary, analysing reaction.

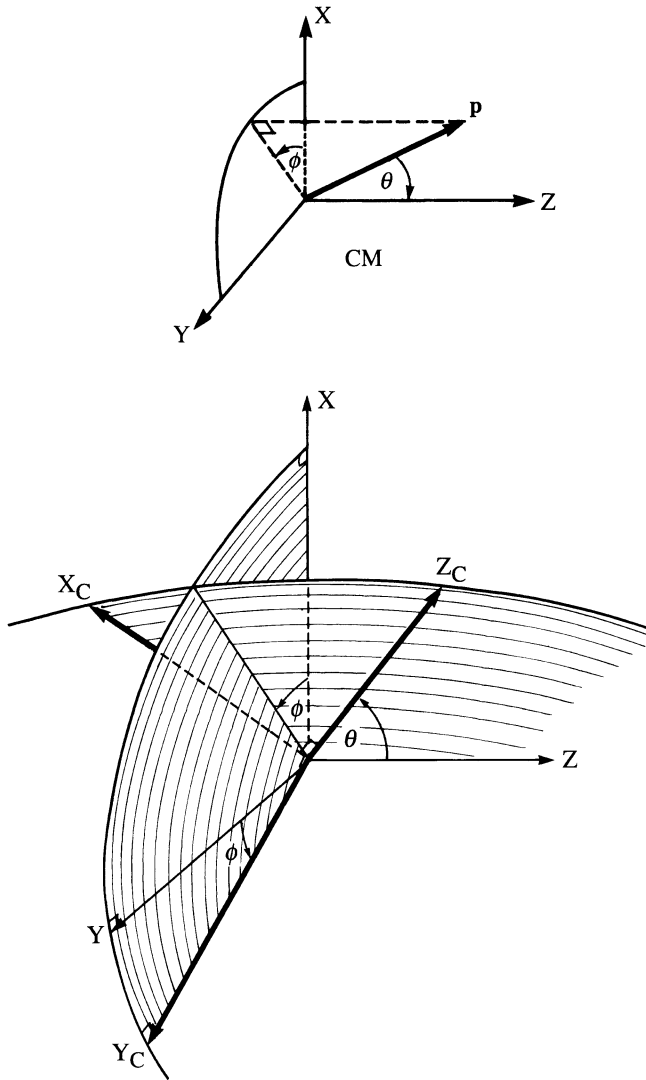


Fig. 3.3. Helicity rest frame for particle produced with angles θ, ϕ in CM.

(i) Resonance production

If one or several final state particles are unstable we will be interested in their decay distributions, which yield information about the production mechanism.

It is simplest to analyse the decay of some particle or resonance C in its own *helicity rest frame* (see subsection 1.2.2) reached from the CM frame of the production reaction $A + B \rightarrow C + D + E + \dots$, since in that

case the initial helicity density matrix of C before it decays is just equal to the helicity density matrix of C in the CM frame of the production reaction, i.e. it is given directly in terms of the CM helicity amplitudes for the production process.

Let C have momentum $\mathbf{p} = (p, \theta, \phi)$ in the CM frame. Then as explained in subsection 1.2.2 its helicity rest frame S_C has its Z -axis, Z_C , along \mathbf{p} and its Y -axis along $\mathbf{e}_{(z)} \times \mathbf{p}$, where $\mathbf{e}_{(z)}$ is a unit vector along the Z -axis of the CM frame. This is illustrated in Fig. 3.3. Note that for $\theta = 0$ or π we take Y_C along or opposite to the CM Y -axis.

The case of a reaction taking place in the XZ -plane is easier to visualize. The relative orientation of the helicity rest frames reached from the CM frame is shown in Fig. 3.4.

The relationship between the decay characteristics and the density matrix of C is discussed in Section 8.2. For an analysis done in S_C the relevant density matrix is then just the density matrix of C in the CM of the production reaction – no transformation is needed.

(ii) Secondary scattering

Consider a stable particle K produced in the reaction CM with momentum (p_K, θ_K, ϕ_K) . We shall define the *natural analysing frame* for K , S_{LK} , to be a frame reached from the laboratory frame S_L by mean of a pure rotation such as to give particle K polar angles equal to zero. In other words

$$\mathbf{p}_K^{S_{LK}} = (p_K^L, 0, 0). \quad (3.3.10)$$

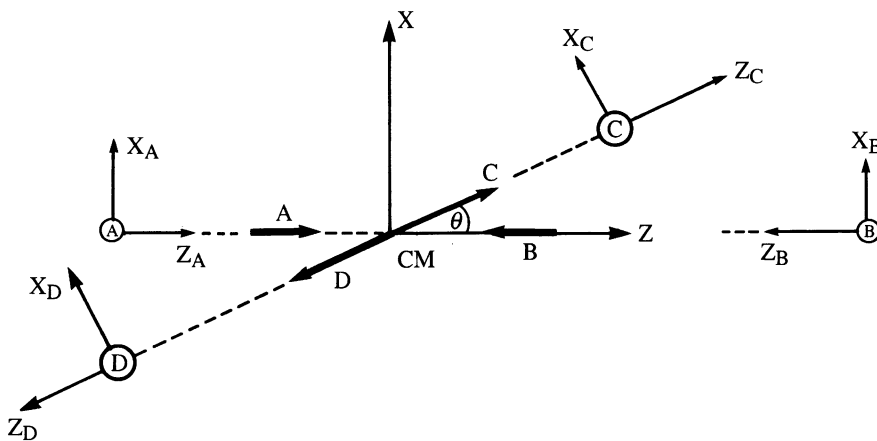


Fig. 3.4 Helicity rest frames reached from the CM of the reaction $A + B \rightarrow C + D$. Note that the Y -axes for B and D are opposite to those for A and C .

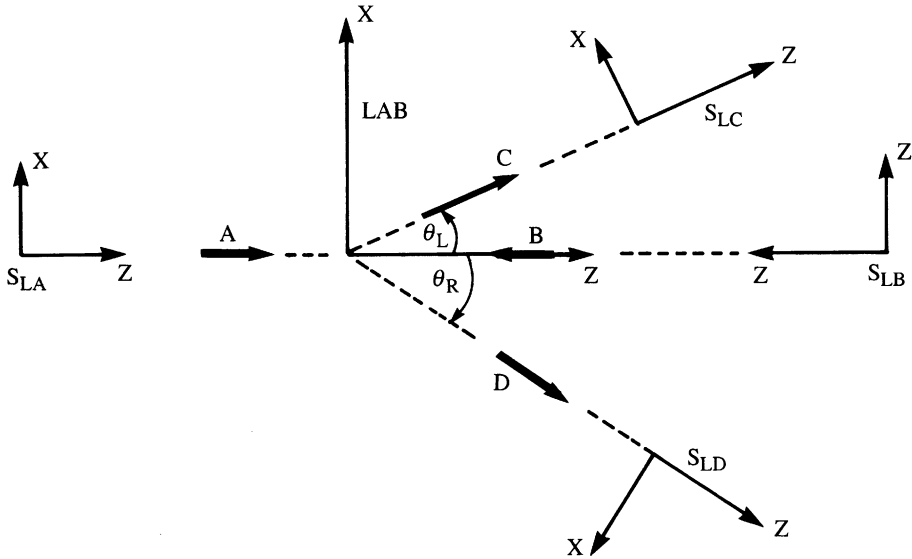


Fig. 3.5 The laboratory ‘analysing’ frames S_{LA} , S_{LB} , S_{LC} and S_{LD} for the reaction $A + B \rightarrow C + D$.

The natural analysing frames for the $2 \rightarrow 2$ reaction $AB \rightarrow CD$ are illustrated in Fig. 3.5. We include frames for the initial particles.

The Lab frame S_{LK} is the *simplest* and most natural frame in which to study the analysing reaction for K for the following reasons.

(α) Because S_{LK} is reached from S_L by the rotation

$$r_{y'}(\theta_K^L)r_z(\phi_K) = r(\phi_K, \theta_K^L, 0) \tag{3.3.11}$$

it is easy to see from (2.1.7) that the helicity density matrix for K is the same in S_{LK} as in S_L .

(β) Because the CM frame for the *analysing reaction* is reached from S_{LK} by boosting along the positive Z -axis of S_{LK} , the helicity density matrix for K in the CM of the analysing reaction is the same as it is in S_{LK} . By (α) it is then the same as in the main Lab frame.

Thus we have the result

$$[\rho(K)]_{\text{CM of analysing reaction}} = \rho^{S_{LK}}(K) = \rho^{S_L}(K) \tag{3.3.12}$$

and the *initial* helicity density matrix of K needed for the analysing reaction is simply given by $\rho^{S_L}(K)$.

Finally, then, in terms of the helicity density matrix in the CM of the production reaction we have

$$[\rho(K)]_{\text{CM of analysing reaction}} = d(\alpha_K)\rho(K)d^\dagger(\alpha_K) \tag{3.3.13}$$

with α_K given by (2.2.13), or, equivalently,

$$[t_m^l(K)]_{\text{CM of analysing reaction}} = d_{mm'}^l(\alpha_K)t_{m'}^l(K). \tag{3.3.14}$$

We remind the reader that α_K takes on special values when the production reaction is an elastic reaction; see (2.2.15) and (2.2.16).

3.4 Covariant spin vectors

In Section 1.1 the spin-polarization vector for a non relativistic state $|\chi\rangle$ was defined as

$$\mathcal{P}_\chi \equiv \frac{1}{s}\langle\chi|\hat{\mathbf{s}}|\chi\rangle \equiv \frac{\mathbf{s}_\chi}{s}, \tag{3.4.1}$$

where \mathbf{s}_χ is the mean spin vector, and this was generalized in subsection 3.1.5 to an ensemble or mixture of pure states. As stressed in Section 3.2 these non-relativistic quantities and, more generally, the multipole parameters can continue to be used in the relativistic case for a massive particle, provided that any physical statements about the spin are understood to hold in the helicity rest frame of the particle.

It is nonetheless advantageous sometimes to deal with relativistic, covariant generalizations of these quantities. We showed in Section 1.2 that the natural covariant generalization of the non-relativistic spin operators \hat{s}_j is given by the space components of the Pauli–Lubanski operators \hat{W}^μ ; namely, when acting on the state of a massive particle at rest,

$$\hat{W}^j|\overset{\circ}{p}; s_z = \lambda\rangle = m\hat{s}^j|\overset{\circ}{p}; s_z = \lambda\rangle. \tag{3.4.2}$$

Note, in addition, that

$$\hat{W}^0|\overset{\circ}{p}; s_z = \lambda\rangle = 0. \tag{3.4.3}$$

Now for any rest state $|\overset{\circ}{p}; \chi\rangle$ that is a linear superposition of states of spin s with different values of s_z , we can define

$$\overset{\circ}{\mathcal{P}}_\chi^\mu \equiv \frac{1}{s}\langle\overset{\circ}{p}; \chi|\hat{W}^\mu|\overset{\circ}{p}; \chi\rangle \tag{3.4.4}$$

and we see from (3.4.2), (3.4.3) and (3.4.1) that

$$\overset{\circ}{\mathcal{P}}_\chi^\mu = \frac{m}{s}(0, \mathbf{s}_\chi) = m(0, \mathcal{P}_\chi). \tag{3.4.5}$$

(The reason for our convention of including the factor m in this definition will become clear later.)

Moreover, because $\overset{\circ}{p} = (m, 0, 0, 0)$, we have

$$\begin{aligned} \mathcal{S}\chi \cdot \overset{\circ}{p} &= 0 \\ \mathcal{S}\chi \cdot \mathcal{S}\chi &= -\frac{m^2}{s^2} \mathbf{s}_\chi^2. \end{aligned} \tag{3.4.6}$$

Now consider the expectation value of \hat{W}^μ for a relativistic helicity state $|\mathbf{p}; \lambda\rangle$. We have from (1.2.25)

$$\begin{aligned} \langle \mathbf{p}; \lambda | \hat{W}^\mu | \mathbf{p}; \lambda \rangle &= \langle \overset{\circ}{p}; \lambda | U^{-1}[h(\mathbf{p})] \hat{W}^\mu U[h(\mathbf{p})] | \overset{\circ}{p}; \lambda \rangle \\ &= \Lambda^\mu_\nu \langle \overset{\circ}{p}; \lambda | \hat{W}^\nu | \overset{\circ}{p}; \lambda \rangle \end{aligned} \tag{3.4.7}$$

since the \hat{W}^μ transform covariantly as a 4-vector. Hence we can define the *covariant helicity mean spin vector*

$$\mathcal{S}^\mu(\mathbf{p}, \lambda) \equiv \frac{1}{s} \langle \mathbf{p}; \lambda | \hat{W}^\mu | \mathbf{p}; \lambda \rangle \tag{3.4.8}$$

and we then have, in full detail,

$$\mathcal{S}^\mu(\mathbf{p}, \lambda) = (\Lambda[h(\mathbf{p})])^\mu_\nu \overset{\circ}{\mathcal{S}}^\nu_\lambda \tag{3.4.9}$$

where

$$p^\mu = (E, \mathbf{p}) = (\Lambda[h(\mathbf{p})])^\mu_\nu \overset{\circ}{p}^\nu.$$

For a state χ that is a linear combination of states with different values of s_z we clearly have

$$\mathcal{S}^\mu(\mathbf{p}, \chi) = (\Lambda[h(\mathbf{p})])^\mu_\nu \overset{\circ}{\mathcal{S}}^\nu_\chi. \tag{3.4.10}$$

If one uses the canonical spin states (see subsection 1.2.1) then one defines

$$\mathcal{S}^\mu_{\text{can}}(\mathbf{p}, s_z) \equiv \frac{1}{s} \text{can} \langle \mathbf{p}; s_z | \hat{W}^\mu | \mathbf{p}; s_z \rangle_{\text{can}} \tag{3.4.11}$$

and so, in contrast to (3.4.10),

$$\mathcal{S}^\mu_{\text{can}}(\mathbf{p}, \chi) = (\Lambda[l(\mathbf{v})])^\mu_\nu \overset{\circ}{\mathcal{S}}^\nu_\chi \tag{3.4.12}$$

where $l(\mathbf{v})$ is the pure boost that takes $\overset{\circ}{p}$ to p^μ .

Note that (3.4.6) generalizes to

$$\mathcal{S}(\mathbf{p}, \chi) \cdot p = 0 \tag{3.4.13}$$

and one has

$$\mathcal{S}^2(\mathbf{p}, \chi) = -\frac{m^2}{s^2} \mathbf{s}_\chi^2. \tag{3.4.14}$$

We shall see in subsection 6.3.1 that \mathcal{S}^μ provides a convenient approach to the relativistic motion of the mean spin vector.

As an example let us compute the covariant helicity mean spin vector for a spin-1/2 particle in a definite helicity state $\lambda = \pm 1/2$. In this case from (3.4.1)

$$\mathbf{s}_\chi \equiv \mathbf{s}(\pm 1/2) = (0, 0, \pm 1/2)$$

and using eqn (1.2.23) we arrive at

$$\mathcal{S}^\mu(\mathbf{p}, \lambda) = 2\lambda(p, E\hat{\mathbf{p}}) \tag{3.4.15}$$

where $\hat{\mathbf{p}}$ is a unit vector along \mathbf{p} . Note the important result that, for spin 1/2,

$$\lim_{m \rightarrow 0} \mathcal{S}^\mu(\mathbf{p}, \lambda) = 2\lambda p^\mu. \tag{3.4.16}$$

Had we not included the factor m in the definition of $\overset{\circ}{\mathcal{S}}_\mu$ the limit $m \rightarrow 0$ of (3.4.15) could not have been taken. Thus, with our convention, \mathcal{S}^μ applies equally well for massless particles.

In the general case of massless particles we replace (3.4.4), (3.4.5) by

$$\mathcal{S}^\mu(\mathbf{p}_{\text{st}}, \lambda) \equiv \frac{1}{s} \langle \mathbf{p}_{\text{st}}; \lambda | \hat{W}^\mu | \mathbf{p}_{\text{st}}; \lambda \rangle \tag{3.4.17}$$

using the ‘standard’ states, defined in subsection 1.2.3, where $p_{\text{st}} = (\bar{p}, 0, 0, \bar{p})$. It follows from (1.2.39) and (1.2.41) that, for a massless helicity state with arbitrary 4-momentum p^μ ,

$$\mathcal{S}^\mu(\mathbf{p}, \lambda) = \frac{\lambda}{s} p^\mu, \tag{3.4.18}$$

which is perfectly consistent with (3.4.15) when $s = 1/2$.

The tensor operators \hat{T}_{ij} , introduced in subsection 3.1.12 for *massive* spin-1 particles, are all expressed as products of the basic spin operators \hat{s}_j . A relativistic generalization of (3.1.62) is then

$$\hat{T}^{\mu\nu} = \frac{1}{2} \sqrt{\frac{3}{2}} \left[\frac{1}{m^2} (\hat{W}^\mu \hat{W}^\nu + \hat{W}^\nu \hat{W}^\mu) + \frac{4}{3} \left(g^{\mu\nu} - \frac{\hat{P}^\mu \hat{P}^\nu}{m^2} \right) \right] \tag{3.4.19}$$

with expectation values

$$\mathcal{T}^{\mu\nu}(\mathbf{p}, \chi) = \langle \mathbf{p}; \chi | \hat{T}^{\mu\nu} | \mathbf{p}; \chi \rangle \tag{3.4.20}$$

such that

$$\mathcal{T}^{\mu\nu}(\mathbf{p}, \chi) = \Lambda^\mu_\alpha \Lambda^\nu_\beta \overset{\circ}{\mathcal{T}}^{\alpha\beta} \tag{3.4.21}$$

where $\Lambda = \Lambda[h(\mathbf{p})]$. The relation to the non-relativistic expectation values T_{ij} is then

$$\overset{\circ}{\mathcal{T}}_{ij} = T_{ij} \quad \overset{\circ}{\mathcal{T}}_{0j} = \overset{\circ}{\mathcal{T}}_{j0} = \overset{\circ}{\mathcal{T}}_{00} = 0. \quad (3.4.22)$$

We note that

$$\sum_{\mu} \mathcal{T}^{\mu}_{\mu} = 0 \quad (3.4.23)$$

$$p_{\mu} \mathcal{T}^{\mu\nu}(\mathbf{p}, \chi) = \mathcal{T}^{\mu\nu}(\mathbf{p}, \chi) p_{\nu} = 0.$$

To specify the most general state for a massive spin-1 particle at rest we introduced in eqn (3.1.76) a polarization vector ϵ (in general complex). Copying the above procedure, we can define

$$\overset{\circ}{\epsilon}^{\mu} \equiv (0, \epsilon) \quad (3.4.24)$$

and take

$$\epsilon^{\mu}(\mathbf{p}) = (\Lambda[h(\mathbf{p})])^{\mu}_{\nu} \overset{\circ}{\epsilon}^{\nu}. \quad (3.4.25)$$

We have

$$p_{\mu} \epsilon^{\mu}(\mathbf{p}) = 0 \quad (3.4.26)$$

and from (3.1.77)

$$\epsilon^{*}(\mathbf{p}) \cdot \epsilon(\mathbf{p}) = -1. \quad (3.4.27)$$

The relations between $\mathcal{S}^{\mu}(\mathbf{p}, \epsilon)$, $\mathcal{T}^{\mu\nu}(\mathbf{p}, \epsilon)$ and $\epsilon^{\mu}(\mathbf{p})$, which generalize (3.1.78) and (3.1.79), are

$$\mathcal{S}_{\mu}(\mathbf{p}, \epsilon) = -\epsilon_{\mu\alpha\beta\gamma} p^{\alpha} \operatorname{Im} \left(\epsilon^{*\beta} \epsilon^{\gamma} \right) \quad (3.4.28)$$

and

$$\mathcal{T}_{\mu\nu}(\mathbf{p}, \epsilon) = -\sqrt{\frac{3}{2}} \left[\operatorname{Re}(\epsilon_{\mu}^* \epsilon_{\nu}) + \frac{1}{3} \left(g_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{m^2} \right) \right]. \quad (3.4.29)$$

For states of definite helicity, ϵ is given by eqn (3.1.80). Equation (3.4.28) illustrates very clearly that the spin-polarization vector for a spin-1 particle is quite different from the polarization vector ϵ^{μ} . Indeed, the complex polarization vector contains all the information needed to specify the state of the particle whereas this is not true for the spin-polarization vector.

It is instructive to link the above discussion of the polarization vector ϵ^{μ} with the more familiar use of polarization vectors in field theory when describing a field of spin-1 quanta by means of a 4-vector field $A_{\mu}(x)$. In that case the analogue of eqn (2.4.5) is usually written

$$A_{\mu}(x) = \sum_{\lambda} \int \frac{d^3 \mathbf{p}}{(2\pi)^{3/2} 2p^0} \left[\epsilon_{\mu}(\mathbf{p}, \lambda) a(\mathbf{p}, \lambda) e^{-ip \cdot x} + \epsilon_{\mu}^*(\mathbf{p}, \lambda) b^{\dagger}(\mathbf{p}, \lambda) e^{ip \cdot x} \right]. \quad (3.4.30)$$

To ensure that there are no spin-0 quanta present one imposes the invariant condition

$$\partial^\mu A_\mu(x) = 0 \tag{3.4.31}$$

from which we have the requirement

$$p^\mu \epsilon_\mu(\mathbf{p}, \lambda) = 0 \tag{3.4.32}$$

as in (3.4.26).

Moreover, using (3.4.25) and (3.1.80), if one takes the simple case $\mathbf{p} = (0, 0, p)$ then, via eqn (1.2.23),

$$\epsilon^\mu(p_z, \lambda = \pm 1) = \frac{1}{\sqrt{2}}(0, \mp 1, -i, 0) \tag{3.4.33}$$

$$\epsilon^\mu(p_z, \lambda = 0) = \frac{1}{m}(p, 0, 0, E) \tag{3.4.34}$$

and we check that

$$\sum_{\lambda=0,\pm 1} \epsilon^{*\mu}(\mathbf{p}, \lambda) \epsilon^\nu(\mathbf{p}, \lambda) = \frac{p^\mu p^\nu}{m^2} - g^{\mu\nu}. \tag{3.4.35}$$

But (3.4.33), (3.4.34) and (3.4.35) are just the usual properties of the polarization vectors in (3.4.30). (See, for example, Gasiorowicz, 1967.) Thus the polarization vectors introduced in (3.4.25) coincide exactly with those used in conventional field theory.

Finally, let us note from (2.4.10) that $\epsilon_\mu(\mathbf{p}, \lambda)$ plays the rôle of the wave function for the single-particle state $|\mathbf{p}; \lambda\rangle$ annihilated by the field $A_\mu(x)$.

For photons, our standard state $|\mathbf{p}_{st}; \lambda\rangle$ consists of the photon moving along OZ with momentum $p_{st}^\mu = (\bar{p}, 0, 0, \bar{p})$ and helicity ± 1 and we may take $\epsilon^\mu(\mathbf{p}_{st}; \lambda = \pm 1)$ as given by (3.4.33).

For a photon in the state $|\mathbf{p}; \lambda\rangle = U[h(\mathbf{p}, \mathbf{p}_{st})]|\mathbf{p}_{st}; \lambda\rangle$ the polarization vector is then

$$\epsilon^\mu(\mathbf{p}, \lambda) = (\Lambda[h(\mathbf{p}, \mathbf{p}_{st})])^\mu{}_\nu \epsilon^\nu(\mathbf{p}_{st}; \lambda). \tag{3.4.36}$$

Explicitly, one finds, when $\mathbf{p} = (p, \theta, \varphi)$, that

$$\epsilon^\mu(\mathbf{p}, \pm 1) = \frac{1}{\sqrt{2}}(0, \mp \cos \theta \cos \varphi + i \sin \varphi, \mp \cos \theta \sin \varphi - i \cos \varphi, \pm \sin \theta). \tag{3.4.37}$$

Using this, one can check that the connection between $\mathcal{S}_\mu(\mathbf{p}, \boldsymbol{\varepsilon})$ and ϵ^μ given for massive spin-1 particles in (3.4.28) continues to hold for photons, and correctly gives (3.4.18).

It is simple to check that, as expected, for the spatial part of the vectors

$$\boldsymbol{\varepsilon}(\mathbf{p}) \cdot \mathbf{p} = 0. \tag{3.4.38}$$

To define a covariant spin tensor $\mathcal{T}^{\mu\nu}$ for photons is a little clumsy. The rôle of $\hat{p}^\mu/m = (1, 0, 0, 0)$ must here be taken by a unit time-like vector n_{st}^μ defined to have components

$$n_{\text{st}}^\mu = (1, 0, 0, 0) \quad (3.4.39)$$

in the standard frame. Then

$$\mathcal{T}^{\mu\nu}(\mathbf{p}, \boldsymbol{\varepsilon}) = \sqrt{\frac{3}{2}} \left[\frac{1}{3} (n^\mu n^\nu - g^{\mu\nu}) - \text{Re}(\epsilon^{\mu*} \epsilon^\nu) \right] \quad (3.4.40)$$

where ϵ^μ is given by (3.4.37) and

$$\begin{aligned} n^\mu &\equiv n^\mu(\mathbf{p}) = (\Lambda[h(\mathbf{p}, \mathbf{p}_{\text{st}})])^\mu{}_\nu n_{\text{st}}^\nu \\ &= \frac{1}{2p\bar{p}} \left[p^2 + \bar{p}^2, (p^2 - \bar{p}^2)\hat{\mathbf{p}} \right]. \end{aligned} \quad (3.4.41)$$

This $\mathcal{T}^{\mu\nu}$ satisfies eqns (3.4.21) and (3.4.22) with $\hat{\mathcal{T}}$ replaced by \mathcal{T}_{st} .