Can. J. Math. Vol. **49** (6), 1997 pp. 1299–1322.

THE EXPLICIT SOLUTION OF THE $\bar\partial$-NEUMANN PROBLEM IN A NON-ISOTROPIC SIEGEL DOMAIN

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ABSTRACT. In this paper, we solve the $\bar\partial$-Neumann problem on $(0,q)$ forms, $0 \leq q \leq n$, in the strictly pseudoconvex non-isotropic Siegel domain:

$$U^{\mathbf{n}} = \left\{ (\mathbf{z}, z_{n+1}) : \begin{array}{c} \mathbf{z} = (z_1, \ldots, z_n) \in \mathrm{C}^n, \\ z_{n+1} \in \mathrm{C}; \end{array} \quad \operatorname{Im}(z_{n+1}) > \textstyle\sum_{j=1}^n a_j |z_j|^2 \right\},$$

where $a_j > 0$ for $j = 1, 2, \ldots, n$. The metric we use is invariant under the action of the Heisenberg group on the domain. The fundamental solution of the related differential equation is derived via the Laguerre calculus. We obtain an explicit formula for the kernel of the Neumann operator. We also construct the solution of the corresponding heat equation and the fundamental solution of the Laplacian operator on the Heisenberg group.

1**. Introduction.** The domain

$$U^{\mathbf{n}} = \left\{ (\mathbf{z},\ z_{n+1}) : \mathbf{z} \in \mathrm{C}^n, z_{n+1} \in \mathrm{C};\ \operatorname{Im} z_{n+1} > \sum_{j=1}^n a_j |z_j|^2 \right\},$$

where $a_j > 0$ for $j = 1, 2, \ldots, n$, is the non-isotropic Siegel domain. We give this name to the above domain because we obtain the classical Siegel domain if $a_j = 1$ for all $j = 1, 2, \ldots, n$, which is a well-known model for geometry and analysis on strongly pseudo-convex manifolds with boundary. We can identify the boundary $bU^{\mathbf{n}}$ with the Heisenberg group H_n, and H_n acts on $\bar{U}^n$ as a group of holomorphic isometries with respect to an appropriate Hermitian metric. In this paper, we construct the solution for the $\bar\partial$-Laplacian on $(0,q)$ forms on $\bar{U}^n$ satisfying $\bar\partial$-Neumann boundary conditions.

We begin with a few historical remarks. The $\bar\partial$-Neumann problem was first formulated by Garabedian and Spencer [7] for complex exterior differential forms on a compact complex-analytic manifold with strongly pseudo-convex boundary. In the case of strictly pseudo-convex domains, it was solved by Kohn [14] [15], who obtained C^∞ results, using L^2 methods; the interested reader can see, for instance, the exposition in Folland and Kohn [5] and Krantz [16].But it was Morrey [17] who first discovered and established the basic estimate of the problem for the special cases of $(0,0)$ and $(0,1)$ forms on certain tubular manifolds. Unfortunately there was error in [17] which was corrected in [18] by using the results of Kohn [14] [15]. The analysis relying on the Heisenberg group and leading to formulas for the solutions, together with sharp estimates, came later, see Folland and Stein [6], Rothschild and Stein [20] and Greiner and Stein [10].

Received by the editors 24 April, 1996.
AMS subject classification: Primary: 32F15, 32F20; secondary: 35N15.

An integral formula, in terms of the explicit kernels, was given for the solution of the $\bar{\partial}$-Neumann problem in the Siegel domain by N. K. Stanton [23]. Phong [19] announced an explicit construction of a parametrix on $(0,1)$ forms in the Siegel domain in $\mathbb{C}^{n+1}$, $n > 1$, and one can see the exposition in M. Beals, C. Fefferman, and R. Grossman [1] for Phong's construction. Harvey and Polking [11] constructed the Neumann kernel for the $\bar{\partial}$-Neumann problem on (p,q)-forms on the unit ball in $\mathbb{C}^{n+1}$, and Kimura [13] found the integral formula for the same problem but in $\mathbb{C}^2$ by a different method. N. K. Stanton [21] [22] has constructed the heat kernel associated with the $\bar{\partial}$-Neumann problem in the Siegel domain. The heat kernel of $\Box_b$ was constructed independently by B. Gaveau [8] and A. Hulanicki [12] for the Siegel domain and by R. Beals and P. C. Greiner [4] for $\boldsymbol{U^n}$.

In this paper, first we find the explicit formula of the fundamental solutions of the $\bar{\partial}$-Neumann problem and the corresponding heat equation for the non-isotropic Siegel domain by the method of Laguerre calculus. Then we construct the kernels of the $\bar{\partial}$-Neumann problem and the corresponding heat equation from the fundamental solutions. The method to construct the kernel of the correction term (5.17) is new. As a further application of the Laguerre calculus, we derive the fundamental solution of the Laplacian operator on the Heisenberg group. Our formulas are close to those of Stanton [21] [22] [23] for the Siegel domain, *i.e.*, $a_j = 1$ for $j = 1, 2, \ldots, n$, but she used a different method. Furthermore, we write the solutions of the $\bar{\partial}$-Neumann problem and the corresponding heat equation in terms of the complex distance and volume element on the Heisenberg group. Our formula can be extended to the domains whose boundaries have the structure of the Heisenberg manifolds. Hence we can get the explicit solution of the $\bar{\partial}$-Neumann problem for a large class of domains. We emphasize the explicit expressions of the formulas in this paper and postpone the regularity property of these formulas to a future publication.

In pursuing these objectives we shall proceed in the following order. First, we formulate the $\bar{\partial}$-Neumann problem, and identify the differential equations we have to solve in order to find the solution to the problem. To derive the fundamental solution, first we introduce the Laguerre calculus to $\mathbb{H}_n \times \mathbb{R}^d$, then apply it to derive the fundamental solutions of the related operators. We can now solve the $\bar{\partial}$-Neumann problem. Next, we solve the associated heat equation. Last, we derive the fundamental solution of the Laplacian operator of the Heisenberg group by applying the Laguerre calculus.

2. $\bar{\partial}$ Operators and $\bar{\partial}$ Laplacian on $\boldsymbol{U^n}$.

The non-isotropic Siegel domain $\boldsymbol{U^n}$ is

$$\boldsymbol{U^n} = \Big\{(\mathbf{z}, z_{n+1}) \in \mathbb{C}^{n+1} : \mathbf{z} \in \mathbb{C}^n, z_{n+1} \in \mathbb{C}; \operatorname{Im}(z_{n+1}) > \sum_{j=1}^{n} a_j |z_j|^2\Big\}$$

with $a_j > 0$ for $j = 1, 2, \ldots, n$. The boundary of $\boldsymbol{U^n}$ is the set

$$b\boldsymbol{U^n} = \Big\{(\mathbf{z}, z_{n+1}) \in \mathbb{C}^{n+1} :\ \operatorname{Im}(z_{n+1}) = \sum_{j=1}^{n} a_j |z_j|^2\Big\}.$$

Besides the ambient coordinates $(\mathbf{z}, z_{n+1})$, it is useful to deal with the Heisenberg coordinates $[\zeta, t, r]$ given by

$$\zeta = \mathbf{z}, \quad t = \operatorname{Re} z_{n+1}, \quad r = \operatorname{Im}(z_{n+1}) - \sum_{j=1}^{n} a_j |z_j|^2,$$

then the domain $U^{\mathbf{n}}$ and its boundary $bU^{\mathbf{n}}$ have the following simpler expressions:

$$U^{\mathbf{n}} = \{[\zeta, t, r] : r > 0\} \quad bU^{\mathbf{n}} = \{[\zeta, t, r] : r = 0\}.$$

We now come to the Heisenberg group, which gives the translation of the domain $U^{\mathbf{n}}$. Abstractly, the Heisenberg group consists of the underlying manifold

$$\mathrm{C}^n \times \mathrm{R} = \{[\zeta, t] : \zeta \in \mathrm{C}^n, t \in \mathrm{R}\}$$

with the multiplication law

$$[\zeta, t] \cdot [\eta, s] = \Big[\zeta + \eta, t + s + 2 \operatorname{Im} \sum_{j=1}^{n} a_j \zeta_j \bar{\eta}_j\Big]. \tag{2.1}$$

It is easy to check that the multiplication (2.1) does indeed make $\mathrm{C}^n \times \mathrm{R}$ into a group whose identity is the origin $[\mathbf{0}, 0]$ and where the inverse is given by $[\zeta, t]^{-1} = [-\zeta, -t]$. The space $\mathrm{C}^n \times \mathrm{R}$ with the multiplication structure (2.1) is the Heisenberg group and will be denoted by H_n.

To each element $[\zeta, t]$ of H_n, we associate the following holomorphic affine self-mapping of $U^{\mathbf{n}}$:

$$[\zeta, t]\colon (\mathbf{z}, z_{n+1}) \longmapsto \Big(\mathbf{z} + \zeta, z_{n+1} + t + i \sum_{j=1}^{n} a_j(|\zeta_j|^2 + 2z_j \bar{\zeta}_j)\Big). \tag{2.2}$$

In fact, since $|z_j + \zeta_j|^2 - |z_j|^2 = \operatorname{Im}\{i(2z_j \bar{\zeta}_j + |\zeta_j|^2)\}$, the mapping preserves the defining function

$$r(\mathbf{z}, z_{n+1}) = \operatorname{Im} z_{n+1} - \sum_{j=1}^{n} a_j |z_j|^2.$$

Hence the transformation (2.2) maps $U^{\mathbf{n}} = \{(\mathbf{z}, z_{n+1}) : r(\mathbf{z}) > 0\}$ to itself and preserves the boundary $bU^{\mathbf{n}} = \{(\mathbf{z}, z_{n+1}) : r(\mathbf{z}) = 0\}$. Observe next that the mapping (2.2) defines an action of the group H_n on the domain $U^{\mathbf{n}}$: if we compose the mapping (2.2) corresponding to elements $[\zeta, t]$ and $[\eta, s]$ of H_n, the resulting transformation corresponds to the element $[\zeta, t] \cdot [\eta, s]$. This follows easily from the identity

$$2\zeta_j \bar{\eta}_j + |\zeta_j|^2 + |\eta_j|^2 = |\zeta_j + \eta_j|^2 + 2i \operatorname{Im}(\zeta_j \bar{\eta}_j).$$

Thus (2.2) gives us a realization of H_n as a group of affine holomorphic bijections of $U^{\mathbf{n}}$.

The mappings (2.2) are simply transitive on the boundary $bU^{\mathbf{n}}$: for every two points in $bU^{\mathbf{n}}$, there is exactly one element of H_n mapping the first to the second. In particular, we have that

$$[\zeta, t]\colon (\mathbf{0}, 0) \longmapsto \Big(\zeta, t + i \sum_{j=1}^{n} a_j |\zeta_j|^2\Big),$$

so we can identify the Heisenberg group with $bU^{\mathbf{n}}$ via its action on the origin:

$$\mathrm{H}_n \ni [\zeta, t] \longmapsto \left(\zeta, t + i\sum_{j=1}^{n} a_j|\zeta_j|^2\right) \in bU^{\mathbf{n}}.$$

If we reconsider the Heisenberg coordinates $[\zeta, t, r]$ on $U^{\mathbf{n}}$, r represents the height of the point $(\mathbf{z}, z_{n+1}) \in U^{\mathbf{n}}$ and $[\zeta, t]$ represents its projection onto $bU^{\mathbf{n}}$, identified with H_n. Note, however, that the correspondence $(\mathbf{z}, z_{n+1}) \longmapsto [\zeta, t, r]$ is not holomorphic.

When considering forms on $U^{\mathbf{n}}$ it is natural to choose a basis $\omega_1, \omega_2, \ldots, \omega_{n+1}$ of $(1, 0)$ forms so that

$$\omega_j = dz_j, \quad j = 1, 2, \ldots, n \quad \text{(the “tangential” forms);} \quad \text{and}$$

$$\omega_{n+1} = \sqrt{2}\partial r = -\sqrt{2}\sum_{j=1}^{n} a_j \bar{z}_j dz_j - \frac{i}{\sqrt{2}} dz_{n+1} \quad \text{(the “normal” form).}$$

The vector fields dual to these forms are then

$$\mathbf{Z}_j = \frac{\partial}{\partial z_j} + 2ia_j \bar{z}_j \frac{\partial}{\partial z_{n+1}}, \quad j = 1, 2, \ldots, n, \quad \mathbf{Z}_{n+1} = i\sqrt{2}\frac{\partial}{\partial z_{n+1}}.$$

Note that in the Heisenberg coordinates this gives:

$$\mathbf{Z}_j = \frac{\partial}{\partial \zeta_j} + ia_j\bar{\zeta}_j\frac{\partial}{\partial t}, j = 1, 2, \ldots, n; \mathbf{Z}_{n+1} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial r} + i\frac{\partial}{\partial t}\right);$$

also $\omega_j = d\zeta_j, j = 1, 2, \ldots, n$, $\omega_{n+1} = \sqrt{2}\partial r$. The proof is very simple, for one only needs to note that

$$\frac{\partial}{\partial z_j} = \frac{\partial}{\partial \zeta_j} - a_j\bar{\zeta}_j\frac{\partial}{\partial r} \quad \text{and} \quad \frac{\partial}{\partial z_{n+1}} = \frac{1}{2}\left(\frac{\partial}{\partial t} - i\frac{\partial}{\partial r}\right).$$

$\mathbf{Z}_j$, $j = 1, 2, \ldots, n$, are tangent to the level surface of r and hence are left-invariant vector fields on the group H_n. On the one hand, $\bar{\mathbf{Z}}_1, \bar{\mathbf{Z}}_2, \ldots, \bar{\mathbf{Z}}_n$ form a basis of the space of the tangential Cauchy-Riemann vector fields on $bU^{\mathbf{n}}$.

$$\mathbf{T} = \frac{i}{2a_j}[\mathbf{Z}_j, \bar{\mathbf{Z}}_j] = \frac{\partial}{\partial t}$$

is the vector field tangent to $bU^{\mathbf{n}}$ that generates the “missing” direction. On the other hand, $\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n, \mathbf{T}$ form a basis of the Lie algebra $\mathfrak{h}_n$ of the Heisenberg group H_n, where $\mathfrak{h}_n$ is the vector space of left-invariant vector fields on H_n equipped with the bracket $[\mathbf{V}_1, \mathbf{V}_2] = \mathbf{V}_1\mathbf{V}_2 - \mathbf{V}_2\mathbf{V}_1$.

We give C^{n+1} the invariant Hermitian metric for which $\{\omega_j = d\zeta_j,\ j = 1, 2, \ldots, n;\ \omega_{n+1} = \sqrt{2}\partial r\}$ forms an orthonormal basis for the cotangent space $T^*_{(1,0)}(U^{\mathbf{n}})$, and

$$\text{(2.3)} \qquad \left\{\bar{\omega}_J = \bar{\omega}_{j_1} \wedge \cdots \wedge \bar{\omega}_{j_q} : J = (j_1, \ldots, j_q), 1 \leq j_1 < j_2 < \cdots < j_q \leq n + 1\right\}$$

is an orthonormal basis for $\Lambda^{0,q}(U^{\mathbf{n}})$. This metric is not Kähler. The volume element with respect to this metric is 2^n times the standard Euclidean volume element.

https://doi.org/10.4153/CJM-1997-064-4 Published online by Cambridge University Press

In terms of the basis (2.3) and the metric, the $\bar\partial$ operator on $(0,q)$ form $f=\sum_J f_J\bar\omega_J$ is defined by

$$\bar\partial f=\sum_{j=1}^{n+1}\sum_J \bar{\mathbf{Z}}_j f_J\bar\omega_j\wedge\bar\omega_J,$$

and the $\bar\partial$-Laplacian $\Box=\bar\partial\bar\partial^*+\bar\partial^*\bar\partial$ and $\bar\partial$-Neumann boundary conditions act diagonally on $(0,q)$ forms. Interested readers can see [21] for details. Let $S^{0,q}(\bar U_n)$ denote the space of $(0,q)$ forms $f=\sum f_J\bar\omega_J$ on $U^{\mathbf{n}}$ such that each f_J is the restriction to $\bar U_n$ of a rapidly decreasing function on C^{n+1}. Then $f\in S^{0,q}(\bar U^n)\bigcap \mathrm{Dom}\,\Box$ if and only if

$$f_J|_{bU^{\mathbf{n}}}=0 \text{ if } n+1\in J;\ \bar{\mathbf{Z}}_{n+1}f_J|_{bU^{\mathbf{n}}}=\frac{1}{\sqrt2}\left(\frac{\partial}{\partial r}-i\frac{\partial}{\partial t}\right)f_J|_{bU^{\mathbf{n}}}=0 \text{ if } n+1\notin J. \tag{2.4}$$

For such f,

$$\Box f=\sum_{n+1\notin J}\Box^\tau f_J\bar\omega_J+\sum_{n+1\in J}\Box^\nu f_J\bar\omega_J \tag{2.5}$$

where

$$\Box^\tau=\mathfrak{L}_{n-2q}-\frac12\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^2}\right),\qquad \Box^\nu=\mathfrak{L}_{n-2(q-1)}-\frac12\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^2}\right) \tag{2.6}$$

with

$$\mathfrak{L}_\alpha=-\frac12\sum_{j=1}^n(\bar{\mathbf{Z}}_j\mathbf{Z}_j+\mathbf{Z}_j\bar{\mathbf{Z}}_j)+i\alpha\mathbf{T}.$$

For the detail derivation of (2.5), see [21]. If one takes into account the boundary conditions (2.4) for $\Box$, then the $\bar\partial$-Neumann problem can be split into a pair of problems

$$\Box^\tau U=f \quad \text{in } U^{\mathbf{n}} \text{ with } \bar{\mathbf{Z}}_{n+1}U|_{bU^{\mathbf{n}}}=0 \quad\text{and} \tag{2.7}$$
$$\Box^\nu U=f \quad \text{in } U^{\mathbf{n}} \text{ with } U|_{bU^{\mathbf{n}}}=0. \tag{2.8}$$

The problem (2.8), involving the normal component, is essentially the Dirichlet problem for the Laplacian, so it can be treated by the more standard methods used in elliptic boundary value problems. Our first goal is to derive the fundamental solution $k^\alpha(\mathbf{x},r)$ of the operator $\Box_\alpha$ by the Laguerre calculus on $\mathrm{H}_n\times\mathrm{R}$, where

$$\Box_\alpha=-\frac12\sum_{j=1}^n(\bar{\mathbf{Z}}_j\mathbf{Z}_j+\mathbf{Z}_j\bar{\mathbf{Z}}_j)+i\alpha\mathbf{T}-\frac12\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^2}\right).$$

Since $\Box^\tau=\Box_{n-2q}$ and $\Box^\nu=\Box_{n-2(q-1)}$, we can construct the solution for the $\bar\partial$-Neumann problem from the distribution k^α.

3. **The Laguerre Calculus on** $\mathrm{H}_n\times\mathrm{R}^d$**.** Laguerre calculus is the symbolic tensor calculus on the Heisenberg group H_n. It was first introduced on H_1 by Greiner [9] and extended to H_n and $\mathrm{H}_n\times\mathrm{R}^d$ by Beals, Gaveau, Greiner and Vauthier [3]. To make this paper self-contained, we include the basic definitions and results of the Laguerre calculus on $\mathrm{H}_n\times\mathrm{R}^d$ here. We will only present the basic definitions and refer [3] for details.

Our notation is

$$\mathrm{H}_n \times \mathrm{R}^d = \left\{[\zeta, t, \mathbf{x}''] : \zeta \in \mathrm{C}^n, t \in \mathrm{R}, \mathbf{x}'' \in \mathrm{R}^d\right\}.$$

The group law is defined by

$$[\zeta, t, \mathbf{x}''] \cdot [\eta, s, \mathbf{y}''] = \Big[\zeta + \eta, t + s + 2\,\mathrm{Im}\sum_{j=1}^{n} a_j\, \zeta_j \overline{\eta}_j, \mathbf{x}'' + \mathbf{y}''\Big].$$

We can also define the the left-invariant convolution on $\mathrm{H}_n \times \mathrm{R}^d$ by

$$f * g(\mathbf{x}) = \int_{\mathrm{H}_n \times \mathrm{R}^d} f(\mathbf{y}) g(\mathbf{y}^{-1} \cdot \mathbf{x})\, d\eta\, ds\, d\mathbf{y}''$$

where $\mathbf{x} = [\zeta, t, \mathbf{x}'']$, $\mathbf{y} = [\eta, s, \mathbf{y}'']$, $d\zeta\, ds$ is the Haar measure on H_n and $d\mathbf{y}''$ is the Euclidean measure on R^d. We set

$$\hat{f}(\zeta, \tau, \xi'') = \int_{\mathrm{R}^{d+1}} e^{-i\tau t - i\xi''\cdot \mathbf{x}''} f(\zeta, t, \mathbf{x}'')\, dt\, d\mathbf{x}''$$

to be the Euclidean Fourier transform with respect to t and $\mathbf{x}''$. Then a simple calculation yields

$$f * g(\mathbf{x}) = \frac{1}{(2\pi)^{d+1}} \int_{\mathrm{R}^{d+1}} e^{i\tau t + i\xi''\cdot \mathbf{x}''} \hat{f} *_\tau \hat{g}(\zeta, \tau, \xi'')\, d\tau\, d\xi''.$$

where $\hat{f} *_\tau \hat{g}$ is the twisted convolution and is given by:

$$\hat{f} *_\tau \hat{g}(\mathbf{z}, \xi'', \tau) = \int_{\mathrm{C}^n} \hat{f}(\mathbf{z} - \mathbf{w}, \tau, \xi'') \hat{g}(\mathbf{w}, \tau, \xi'') e^{-i\tau\langle \mathbf{z}, \mathbf{w}\rangle}\, d\mathbf{w}$$

with $\langle \mathbf{z}, \mathbf{w}\rangle = 2\,\mathrm{Im}\sum_{j=1}^{n} a_j z_j \overline{w}_j$. Thus we can treat ξ'' as a parameter and apply the results of the Laguerre calculus on H_n to the present situation. In particular, we have the Laguerre series expansion:

$$\hat{\mathbf{F}}(\zeta, \tau, \xi'') = \sum_{p_j, k_j = 1}^{\infty} F^{(p_1, \ldots, p_n)}_{k_1, \ldots, k_n}(\tau, \xi'') \prod_{j=1}^{n} a_j \tilde{\mathfrak{L}}^{(p_j - k_j)}_{p_j \wedge k_j - 1}(\sqrt{a_j}\, \zeta_j, \tau),$$

where $p_j \wedge k_j = \min p_j, k_j$ and $\tilde{\mathfrak{L}}^{(p)}_k(\zeta, \tau)$ for $\zeta = |\zeta| e^{i\theta}$; $k, \pm p = 0, 1, 2, \ldots$, are given by the Laguerre functions $\ell^{(p)}_k$:

$$\tilde{\mathfrak{L}}^{(p)}_k(\zeta, \tau) = \frac{2|\tau|}{\pi} \ell^{(p)}_k(2|\tau|\, |\zeta|^2) e^{ip\theta} \text{ and } \tilde{\mathfrak{L}}^{(-p)}_k(\zeta, \tau) = \frac{2|\tau|}{\pi} (-1)^p \ell^{(p)}_k(2|\tau|\, |\zeta|^2) e^{-ip\theta}.$$

The Laguerre functions $\ell^{(p)}_k$ are induced by the generalized Laguerre polynomials $L^{(p)}_k$:

$$\ell^{(p)}_k(x) = \left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)}\right]^{1/2} x^{p/2} L^{(p)}_k(x) e^{-x/2} \quad \text{where } x \geq 0 \text{ and } p, k = 0, 1, 2, \ldots.$$

Finally the generalized Laguerre polynomials $L^{(p)}_k(x)$ are defined by their usual generating function formula:

$$(3.1) \sum_{k=1}^{\infty} L^{(p)}_k(x) w^k = \frac{1}{(1-w)^{p+1}} \exp\left\{-\frac{xw}{1-w}\right\}, \qquad \text{for } p = 0, 1, 2, \ldots; x \geq 0,\ |w| < 1.$$

From the Laguerre series expansion, we define the Laguerre tensor $\mathfrak{L}(\mathbf{F})$:

DEFINITION 3.1. Let $\mathbf{F}$ induce a left-invariant convolution operator on $\mathrm{H}_n \times \mathrm{R}^d$. For all $\tau \in \mathrm{R} \setminus 0$ and $\xi'' \in \mathrm{R}^d$, we define the Laguerre tensor $\mathfrak{L}(\mathbf{F})$ by

$$\mathfrak{L}_{\tau,\xi''}(\hat{\mathbf{F}}) = \begin{cases} \left(F^{(p_1,\dots,p_n)}_{k_1,\dots,k_n}(\tau,\xi'')\right) & \text{for } \tau > 0, \\ \left(F^{(p_1,\dots,p_n)}_{k_1,\dots,k_n}(\tau,\xi'')\right)^t & \text{for } \tau < 0. \end{cases}$$

We recall the notion of the tensor contraction:

DEFINITION 3.2. Let $\mathbf{U} = (U^{(p_1,\dots,p_n)}_{k_1,\dots,k_n})$ and $\mathbf{V} = (V^{(q_1,\dots,q_n)}_{m_1,\dots,m_n})$ denote two infinite (n,n) tensors. Their product, $\mathbf{U} \cdot \mathbf{V}$, is defined to be

$$\mathbf{W} = \mathbf{U} \cdot \mathbf{V} = (W^{(p_1,\dots,p_n)}_{m_1,\dots,m_n}), \quad \text{where} \quad W^{(p_1,\dots,p_n)}_{m_1,\dots,m_n} = \sum_{k_1,\dots,k_n=1}^{\infty} U^{(p_1,\dots,p_n)}_{k_1,\dots,k_n} V^{(k_1,\dots,k_n)}_{m_1,\dots,m_n}.$$

The tensor $\mathbf{W}$ is the contraction of the tensors $\mathbf{U}$ and $\mathbf{V}$.

We need the following two results of the Laguerre calculus:

THEOREM 3.1 (THE LAGUERRE CALCULUS ON $\mathrm{H}_n \times \mathrm{R}^d$). *Let* $\mathbf{F}$ *and* $\mathbf{G}$ *induce left-invariant convolution operators on* $\mathrm{H}_n \times \mathrm{R}^d$. *Then*

$$\mathfrak{L}_{\tau,\xi''}(\hat{\mathbf{F}} *_\tau \hat{\mathbf{G}}) = \mathfrak{L}_{\tau,\xi''}(\hat{\mathbf{F}}) \cdot \mathfrak{L}_{\tau,\xi''}(\hat{\mathbf{G}}) \quad \textit{for } \tau \in \mathrm{R} \setminus 0 \quad \textit{and} \quad \xi'' \in \mathrm{R}^d,$$

where the product on the right hand side denotes the tensor contraction.

THEOREM 3.2 (THE LAGUERRE TENSOR OF THE IDENTITY). *Let* $\mathbf{I}_\mathrm{H}$ *denote the identity operator on* $C_0^\infty(\mathrm{H}_n \times \mathrm{R}^d)$, *then* $\mathbf{I}_\mathrm{H}$ *is induced by the identity Laguerre tensor*

$$\mathfrak{L}_\pm(\hat{\mathbf{I}}_\mathrm{H}) = (\delta^{(p_1)}_{k_1} \cdots \delta^{(p_n)}_{k_n}).$$

We now apply the results of this section to find the fundamental solution $k^\alpha(\mathbf{x}, r)$ of $\Box_\alpha$ in the next section.

4. The Fundamental Solution of $\Box_\alpha$. On the space $\mathrm{H}_n \times \mathrm{R} = \{(\mathbf{x}, r) : \mathbf{x} = [\zeta, t] \in \mathrm{H}_n, r \in \mathrm{R}\}$. We let $k^\alpha(\mathbf{x}, r)$ be the distribution determined by

$$\Box_\alpha k^\alpha \overset{\text{def}}{=} \left[\mathfrak{L}_\alpha - \frac{1}{2}\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2}\right)\right] k^\alpha(\mathbf{x}, r) = \delta_0$$

with k^α vanishing at infinity; here δ_0 denotes the Dirac delta function at the origin. We shall first find $\mathfrak{L}(\hat\Box_\alpha^{-1})$, then obtain $k^\alpha(\mathbf{x}, r)$ via the inverse Fourier transform with respect to τ and ξ.

First we take the Fourier transform with respect to t and r for $\Box_\alpha$ to obtain

$$\hat\Box_\alpha = -\frac{1}{2}\sum_{j=1}^{n}(\tilde{\bar{\mathbf{Z}}}_j \tilde{\mathbf{Z}}_j + \tilde{\mathbf{Z}}_j \tilde{\bar{\mathbf{Z}}}_j) - \alpha\tau + \frac{1}{2}(\tau^2 + \xi^2),$$

where

$$\tilde{\mathbf{Z}}_j = \frac{\partial}{\partial z_j} - a_j \bar z_j \tau \quad \text{and} \quad \tilde{\bar{\mathbf{Z}}}_j = \frac{\partial}{\partial \bar z_j} + a_j z_j \tau.$$

We start with

$$\hat{\Box}_\alpha = \sum_{|\mathbf{k}|=0}^{\infty}\left[-\frac{1}{2}\sum_{j=1}^{n}(\tilde{\overline{\mathbf{Z}}}_j\tilde{\mathbf{Z}}_j+\tilde{\mathbf{Z}}_j\tilde{\overline{\mathbf{Z}}}_j)-\alpha\tau+\frac{1}{2}(\tau^2+\xi^2)\right]\prod_{j=1}^{n}a_j\tilde{\mathfrak{L}}_{k_j}^{(0)}(\sqrt{a_j}\,\zeta_j,\tau)*_\tau\,. \tag{4.1}$$

Now we apply

$$-\frac{1}{2}(\tilde{\overline{\mathbf{Z}}}_j\tilde{\mathbf{Z}}_j+\tilde{\mathbf{Z}}_j\tilde{\overline{\mathbf{Z}}}_j)\tilde{\mathfrak{L}}_{k_j}^{(0)}(\sqrt{a_j}\,\zeta_j,\tau)=(2k_j+1)a_j|\tau|\tilde{\mathfrak{L}}_{k_j}^{(0)}(\sqrt{a_j}\,\zeta_j,\tau) \tag{4.2}$$

to Eq. (4.1) and obtain

$$\hat{\Box}_\alpha = \sum_{|\mathbf{k}|=0}^{\infty}\left[\sum_{j=1}^{n}(2k_j+1)a_j|\tau|-\alpha\tau+\frac{1}{2}(\tau^2+\xi^2)\right]\prod_{j=1}^{n}a_j\tilde{\mathfrak{L}}_{k_j}^{(0)}(\sqrt{a_j}\,\zeta_j,\tau). \tag{4.3}$$

We refer to [3] for the proof of Eq. (4.2). Consequently, the Laguerre tensor of the convolution operator induced by $\Box_\alpha$ is

$$\mathfrak{L}(\hat{\Box}_\alpha)=|\tau|\left(\left[\sum_{j=1}^{n}(2k_j-1)a_j-\alpha\,\mathrm{sgn}(\tau)+\frac{1}{2}(|\tau|+\frac{\xi^2}{|\tau|})\right]\delta_{k_1}^{(p_1)}\cdots\delta_{k_n}^{(p_n)}\right), \tag{4.4}$$

and it is invertible as long as

$$|\alpha|\neq\sum_{j=1}^{n}(2k_j+1)a_j+\frac{1}{2}\left(|\tau|+\frac{\xi^2}{|\tau|}\right)\quad \text{for } \mathbf{k}=(k_1,\ldots,k_n)\in\mathbb{N}^n.$$

Under this condition, the inverse Laguerre tensor of (4.4) is

$$\mathfrak{L}(\hat{\Box}_\alpha^{-1})=|\tau|^{-1}\left(\left[\sum_{j=1}^{n}(2k_j-1)a_j-\alpha\,\mathrm{sgn}(\tau)+\frac{1}{2}\left(|\tau|+\frac{\xi^2}{|\tau|}\right)\right]^{-1}\delta_{k_1}^{(p_1)}\cdots\delta_{k_n}^{(p_n)}\right). \tag{4.5}$$

We next write Eq. (4.5) in terms of the Laguerre expansion:

$$\hat{\Box}_\alpha^{-1}=\frac{1}{|\tau|}\sum_{|\mathbf{k}|=0}^{\infty}\left[\sum_{j=1}^{n}(2k_j+1)a_j-\alpha\,\mathrm{sgn}(\tau)+\frac{1}{2}\left(|\tau|+\frac{\xi^2}{|\tau|}\right)\right]^{-1}\prod_{j=1}^{n}a_j\tilde{\mathfrak{L}}_{k_j}^{(0)}(\sqrt{a_j}\,\zeta_j,\tau). \tag{4.6}$$

Now we sum up this series. We first assume that

$$A_{\mathbf{k}}(\tau,\xi)\overset{\text{def}}{=}\sum_{j=1}^{n}(2k_j+1)a_j-\alpha\,\mathrm{sgn}(\tau)+\frac{1}{2}\left(|\tau|+\frac{\xi^2}{|\tau|}\right)>0\quad\text{for } \mathbf{k}\in\mathbb{N}^n,$$

then we can write $A_{\mathbf{k}}^{-1}$ in the integral form:

$$\frac{1}{\sum_{j=1}^{n}(2k_j+1)a_j-\alpha\,\mathrm{sgn}(\tau)+\frac{1}{2}\left(|\tau|+\frac{\xi^2}{|\tau|}\right)}=\int_0^\infty e^{-\left[\sum_{j=1}^{n}(2k_j+1)a_j-\alpha\,\mathrm{sgn}(\tau)+\frac{1}{2}\left(|\tau|+\frac{\xi^2}{|\tau|}\right)\right]s}\,ds. \tag{4.7}$$

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Next we substitute (4.7) into (4.6), and this yields:

$$\begin{aligned}
\hat{\Box}_\alpha^{-1} &= \frac{1}{|\tau|}\sum_{|\mathbf{k}|=0}^{\infty}\int_0^\infty e^{-\left[\sum_{j=1}^n(2k_j+1)a_j-\alpha\,\mathrm{sgn}(\tau)+\frac12\left(|\tau|+\frac{\xi^2}{|\tau|}\right)\right]s}\,ds\prod_{j=1}^n a_j\tilde{\mathfrak{L}}_{k_j}^{(0)}(\sqrt{a_j}\,\zeta_j,\tau)\\
&= \frac{1}{|\tau|}\int_0^\infty e^{-\left[-\alpha\,\mathrm{sgn}(\tau)+\frac12\left(|\tau|+\frac{\xi^2}{|\tau|}\right)\right]s}\prod_{j=1}^n\sum_{k_j=0}^\infty e^{-(2k_j+1)a_js}a_j\tilde{\mathfrak{L}}_{k_j}^{(0)}(\sqrt{a_j}\,\zeta_j,\tau)\,ds\\
&= \frac{|\tau|^{n-1}}{\pi^n}\int_0^\infty e^{-\left[-\alpha\,\mathrm{sgn}(\tau)+\frac12\left(|\tau|+\frac{\xi^2}{|\tau|}\right)\right]s}\\
&\qquad\times\Big(\prod_{j=1}^n 2a_je^{-a_js-a_j|\tau|\,|\zeta_j|^2}\sum_{k_j=0}^\infty e^{-2k_ja_js}L_{k_j}^{(0)}(2a_j|\tau|\,|\zeta_j|^2)\,ds\Big).
\end{aligned}$$

We apply the generating function formula (3.1) for the Laguerre polynomials to the last equation and obtain that:

$$\hat{\Box}_\alpha^{-1}=\frac{|\tau|^{n-1}}{\pi^n}\int_0^\infty e^{s\alpha\,\mathrm{sgn}(\tau)-\frac{s}{2}\left(|\tau|+\frac{\xi^2}{|\tau|}\right)}\left[\prod_{j=1}^n\frac{a_j}{\sinh(a_js)}\right]e^{-|\tau|\sum_{j=1}^n a_j|\zeta_j|^2\coth(a_js)}\,ds.$$

The fundamental solution $k^\alpha(\mathbf{x},r)$ should be the inverse Fourier transform of $\hat{\Box}_\alpha^{-1}(\tau,\xi)$ with respect to τ and ξ. We shall take the inverse Fourier transform first with respect to ξ, since it can be reduced to the Gaussian integral:

$$\text{(4.8)}\qquad \frac{1}{2\pi}\int_{-\infty}^\infty e^{-\frac{s}{2|\tau|}\xi^2+ir\xi}\,d\xi=\sqrt{\frac{|\tau|}{2\pi s}}e^{-\frac{|\tau|}{2s}r^2}.$$

Hence, after taking the inverse Fourier transform with respect to ξ, we have

$$\text{(4.9)}\qquad \tilde{\Box}_\alpha^{-1}(\tau,r)=\frac{|\tau|^{n-\frac12}}{\pi^n}\int_0^\infty\frac{1}{\sqrt{2\pi s}}e^{s\alpha\,\mathrm{sgn}(\tau)-\frac{|\tau|}{2}s-\frac{r^2}{2s}|\tau|}\prod_{j=1}^n\frac{a_j}{\sinh(a_js)}e^{-|\tau|\gamma(s,\zeta)}\,ds$$

with $\gamma(s,\zeta)=\sum_{j=1}^n a_j|\zeta_j|^2\coth(a_js)$. Next we take the inverse Fourier transform with respect to τ and obtain the fundamental solution:

$$\begin{aligned}
k^\alpha(\mathbf{x},r)&=\frac{\Gamma\left(n+\frac12\right)}{2^{\frac32}\pi^{n+\frac32}}\int_0^\infty\prod_{j=1}^n\frac{a_js}{\sinh(a_js)}\\
&\qquad\times\left\{\frac{e^{\alpha s}}{\left[\frac{s^2}{2}+\frac{r^2}{2}+\gamma(s,\zeta)-its\right]^{n+\frac12}}+\frac{e^{-\alpha s}}{\left[\frac{s^2}{2}+\frac{r^2}{2}+s\gamma(s,\zeta)+its\right]^{n+\frac12}}\right\}ds.
\end{aligned}$$

Replacing s by $-s$ in the second part of the above integral we obtain:

$$\text{(4.10)}\qquad k^\alpha(\mathbf{x},r)=\frac{\Gamma\left(n+\frac12\right)}{2^{\frac32}\pi^{n+\frac32}}\int_{-\infty}^\infty\prod_{j=1}^n\frac{a_js}{\sinh(a_js)}\frac{e^{\alpha s}}{\left[\frac{s^2}{2}+\frac{r^2}{2}+s\gamma(s,\zeta)-its\right]^{n+\frac12}}\,ds.$$

We can write Eq. (4.10) in terms of the volume element and complex distance introduced by Beals, Gaveau and Greiner [2]. First we substitute s by $2s$ in (4.10):

$$k^\alpha(\mathbf{x},r)=\frac{\Gamma\left(n+\frac{1}{2}\right)}{2^{\frac{1}{2}}\pi^{n+\frac{3}{2}}}\int_{-\infty}^{\infty}\left[\prod_{j=1}^{n}\frac{2a_js}{\sinh(2a_js)}\right]\frac{e^{2\alpha s}\,ds}{\left[2s^2+\frac{r^2}{2}+2s\gamma(s,\zeta)-2ist\right]^{n+\frac{1}{2}}}. \tag{4.11}$$

We let the complex distance and volume element to be:

$$g(s;\mathbf{x})=\sum_{j=1}^{n}a_j|\zeta_j|^2\coth(2a_js)-it,\ \ \nu(s)=\prod_{j=1}^{n}\frac{2a_j}{\sinh(2a_js)}. \tag{4.12}$$

Then we write $k^\alpha(\mathbf{x},r)$ in closed form:

$$k^\alpha(\mathbf{x},r)=\frac{\Gamma\left(n+\frac{1}{2}\right)}{2^{\frac{1}{2}}\pi^{n+\frac{3}{2}}}\int_{-\infty}^{\infty}\frac{e^{2\alpha s}s^n\nu(s)\,ds}{\left[2s^2+\frac{r^2}{2}+2sg(s;\mathbf{x})\right]^{n+\frac{1}{2}}}. \tag{4.13}$$

Now we compare our formula with the results by N. Stanton [23] for the special case of $a_j=1$ for all $j=1,2,\ldots,n$. First the condition

$$|\alpha|<\sum_{j=1}^{n}(2k_j+1)a_j+\frac{1}{2}|\tau|+\frac{\xi^2}{2|\tau|} \tag{4.14}$$

holds for all $\mathbf{k}=(k_1,\ldots,k_n)\in\mathbb{N}^n$, $\alpha=n-2q$ and $\alpha=n-2(q-1)$ with $0\le q\le n$ if $a_j=1$ for all $j=1,2,\ldots,n$, since the minimum value of the right hand side of (4.14) is $n+|\xi|$ in this case. And setting $a_j=1$ in (4.10) yields

$$k^\alpha(\mathbf{x},r)=\frac{\Gamma(n+\frac{1}{2})}{2^{\frac{3}{2}}\pi^{n+\frac{3}{2}}}\int_{-\infty}^{\infty}\frac{s^n[\sinh(s)]^{-n}e^{\alpha s}}{\left[\frac{s^2}{2}+\frac{r^2}{2}+|\zeta|^2s\coth(s)-its\right]^{n+\frac{1}{2}}}\,ds.$$

This formula coincides with Stanton's results, see Lemma 2.4 of [23]. She obtained this formula by integration of the heat kernels, which she derived in [21] [22], with respect to time. We will also derive the heat kernel later.

We summarize the calculation of the fundamental solution k^α in the following theorem:

THEOREM 4.1. *The operator $K^\alpha(f)=f*k^\alpha$ given by the convolution in* $\mathrm{H}_n\times\mathrm{R}$ *with the kernel:*

$$k^\alpha(\mathbf{x},r)=\frac{\Gamma\left(n+\frac{1}{2}\right)}{2^{\frac{1}{2}}\pi^{n+\frac{3}{2}}}\int_{-\infty}^{\infty}\frac{e^{2\alpha s}s^n\nu(s)\,ds}{\left[2s^2+\frac{r^2}{2}+2sg(s;\mathbf{x})\right]^{n+1/2}}$$

is the fundamental solution of the operator:

$$\square_\alpha=-\frac{1}{2}\sum_{j=1}^{n}(\mathbf{Z}_j\bar{\mathbf{Z}}_j+\bar{\mathbf{Z}}_j\mathbf{Z}_j)+i\alpha\mathbf{T}-\frac{1}{2}\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^2}\right).$$

5. **$\bar\partial$-Neumann Problem.** We reduced the $\bar\partial$-Neumann problem to a pair of problems:

$$\square^{\tau} U = f \quad \text{in } U^{\mathbf{n}} \quad \text{with } \bar{\mathbf{Z}}_{n+1} U|_{bU^{\mathbf{n}}} = 0 \quad \text{and} \tag{5.1}$$

$$\square^{\nu} U = f \quad \text{in } U^{\mathbf{n}} \quad \text{with } U|_{bU^{\mathbf{n}}} = 0, \quad \text{where} \tag{5.2}$$

$$\square^{\tau} = \mathfrak{L}_{n-2q} - \frac{1}{2}\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2}\right) \quad \text{and} \quad \square^{\nu} = \mathfrak{L}_{n-2(q-1)} - \frac{1}{2}\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2}\right)$$
$$\text{with } \mathfrak{L}_{\alpha} = -\frac{1}{2}\sum_{j=1}^{n}(\bar{\mathbf{Z}}_j \mathbf{Z}_j + \mathbf{Z}_j \bar{\mathbf{Z}}_j) + i\alpha \mathbf{T}.$$

We have found the fundamental solution $k^{\alpha}(\mathbf{x}, r)$ of the operator $\square_{\alpha}$, where $\square^{\tau} = \square_{n-2q}$ and $\square^{\nu} = \square_{n-2q+2}$.

The problem (5.2), involving the normal component, is essentially the Dirichlet problem. From the fundamental solution of $\square_{\alpha}$, we can easily construct Green's operator.

Note that

$$k^{n-2q+2}(\mathbf{x}, r) = \frac{\Gamma\left(n+\frac{1}{2}\right)}{2^{\frac{1}{2}}\pi^{n+\frac{3}{2}}} \int_{-\infty}^{\infty} \frac{e^{2(n-2q+2)s} s^n \nu(s)\, ds}{\left[2s^2 + \frac{r^2}{2} + 2sg(s; \mathbf{x})\right]^{n+\frac{1}{2}}} \tag{5.3}$$

is the fundamental solution of $\square^{\nu}$ and is even in r. We set

$$G_{\nu}(\mathbf{x}, \mathbf{y}, r, \rho) = k^{n-2q+2}(\mathbf{y}^{-1}\cdot \mathbf{x}, r-\rho) - k^{n-2q+2}(\mathbf{y}^{-1}\cdot \mathbf{x}, r+\rho).$$

Then the operator G_{ν}, defined by

$$(G_{\nu} f)(\mathbf{x}, r) = \int_{\mathrm{H}_n\times \mathrm{R}^+} G_{\nu}(\mathbf{x}, \mathbf{y}, r, \rho) f(\mathbf{y}, \rho)\, d\mathbf{y}\, d\rho \tag{5.4}$$

is Green's operator for $\square^{\nu}$; that is

$$\square^{\nu} G_{\nu}(f) = f \text{ in } U^{\mathbf{n}}, \text{ and } G_{\nu}(f)|_{bU^{\mathbf{n}}} = G_{\nu}(f)|_{r=0} = 0. \tag{5.5}$$

The proof of (5.5) is quite simple. Indeed, we let

$$f_1(\mathbf{x}, r) = \begin{cases} f(\mathbf{x}, r) & \text{if } r \ge 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(\mathbf{x}, r) = \begin{cases} f(\mathbf{x}, -r) & \text{if } r \le 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then we can write (5.4) in the following form:

$$G_{\nu} f(\mathbf{x}, r) = f_1 * k^{n-2q+2}(\mathbf{x}, r) - f_2 * k^{n-2q+2}(\mathbf{x}, r).$$

$k^{n-2q+2}(\mathbf{x}, r) = k^{n-2q+2}(\mathbf{x}, -r)$ implies $G_{\nu}(f)|_{r=0} = 0$. And since $k^{n-2q+2}(\mathbf{x}, r)$ is the fundamental solution of $\square^{\nu}$, this yields

$$\square^{\nu} G_{\nu}(f) = f_1 - f_2 = f, \quad \text{for } r > 0.$$

Hence $U = G_{\nu}(f)(\mathbf{x}, r)$ solves the problem (5.2), and we summarize its solution in the following theorem:

THEOREM 5.1. *$k^{n-2q+2}(\mathbf{x}, r)$ is given by (5.3). Then*

$$U = \int_{\mathrm{H}_n \times \mathrm{R}^+} \left[k^{n-2q+2}(\mathbf{y}^{-1} \cdot \mathbf{x}, r - \rho) - k^{n-2q+2}(\mathbf{y}^{-1} \cdot \mathbf{x}, r + \rho)\right] f(\mathbf{y}, \rho)\, d\mathbf{y}\, d\rho$$

solves the problem $\Box^{\nu} U = f$ in $U^{\mathbf{n}}$ with $U|_{bU^{\mathbf{n}}} = 0$.

The problem (5.1), involving the tangential part, has the $\bar{\partial}$-Neumann boundary condition. First we will also construct the related Green's operator G_τ as above but with Neumann boundary condition. The fundamental solution for the operator $\Box^{\tau}$ is

$$k^{n-2q}(\mathbf{x}, r) = \frac{\Gamma\left(n + \frac{1}{2}\right)}{2^{\frac{1}{2}} \pi^{n+\frac{3}{2}}} \int_{-\infty}^{\infty} \frac{e^{2(n-2q)s} s^n \nu(s)\, ds}{\left[2s^2 + \frac{r^2}{2} + 2sg(s; \mathbf{x})\right]^{n+\frac{1}{2}}}. \tag{5.6}$$

Similarly, we set

$$G_\tau(\mathbf{x}, \mathbf{y}, r, \rho) = k^{n-2q}(\mathbf{y}^{-1} \cdot \mathbf{x}, r - \rho) + k^{n-2q}(\mathbf{y}^{-1} \cdot \mathbf{x}, r + \rho). \tag{5.7}$$

Then the operator G_τ, defined by

$$(G_\tau f)(\mathbf{x}, r) = \int_{\mathrm{H}_n \times \mathrm{R}^+} G_\tau(\mathbf{x}, \mathbf{y}, r, \rho) f(\mathbf{y}, \rho)\, d\mathbf{y}\, d\rho$$

is Green's operator for $\Box^{\tau}$ with the Neumann boundary condition; that is

$$\Box^{\tau} G_\tau(f) = f \text{ in } U^{\mathbf{n}}, \ \frac{\partial}{\partial r} G_\tau(f)|_{bU^{\mathbf{n}}} = \frac{\partial}{\partial r} G_\tau(f)|_{r=0} = 0 \tag{5.8}$$

The proof of (5.8) is similar to that of (5.5).

Next we write the solution of (5.1) as

$$U = G_\tau(f) + P(f), \tag{5.9}$$

where $P(f)$ is the correction term so that $\bar{\mathbf{Z}}_{n+1} U|_{r=0} = 0$ and $\Box^{(\tau)} U = f$. By looking at this problem this may, with a given f, P, more precisely its kernel, is to be determined.

If we apply the equation and the boundary condition in (5.1) to (5.9) we find that

$$\Box^{\tau} P(f) = 0 \quad \text{and} \quad \bar{\mathbf{Z}}_{n+1} P(f)|_{bU^{\mathbf{n}}} = -\bar{\mathbf{Z}}_{n+1} G_\tau(f)|_{bU^{\mathbf{n}}}. \tag{5.10}$$

We assume that P is the convolution operator with kernel $p(\mathbf{x}, r)$, *i.e.*,

$$P(f)(\mathbf{x}, r) = \int_{\mathrm{H}_n \times \mathrm{R}^+} p(\mathbf{y}^{-1} \cdot \mathbf{x}, r + \rho) f(\mathbf{y}, \rho)\, d\mathbf{y}\, d\rho,$$

where $p(\mathbf{x}, r)$ decays very fast as $r \longrightarrow \infty$.

The boundary conditions in (5.8) and (5.10) yield that

$$\left(\frac{\partial}{\partial r} - i \frac{\partial}{\partial t}\right) P(f)|_{r=0} = i \frac{\partial}{\partial t} G_\tau(f)|_{r=0}. \tag{5.11}$$

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It, in turn, implies that

$$\int_{\mathrm{H}_n\times\mathrm{R}^+}\left(\frac{\partial}{\partial r}-i\frac{\partial}{\partial t}\right)p(\mathbf{y}^{-1}\cdot\mathbf{x},r+\rho)f(\mathbf{y},\rho)\,d\mathbf{y}\,d\rho\Big|_{r=0}$$
$$=\int_{\mathrm{H}_n\times\mathrm{R}^+}i\frac{\partial}{\partial t}\Big[k^{n-2q}(\mathbf{y}^{-1}\cdot\mathbf{x},r-\rho)+k^{n-2q}(\mathbf{y}^{-1}\cdot\mathbf{x},r+\rho)\Big]f(\mathbf{y},\rho)\,d\mathbf{y}\,d\rho\Big|_{r=0}.$$

Since we want to find $p(\mathbf{x},r)$ such that (5.9) solves the equation (5.1) for all f taking from some class of functions (*e.g.*, $C_0^\infty(\mathrm{H}_n\times R^+)$), we obtain, from the above equation, that

$$\left(\frac{\partial}{\partial r}-i\frac{\partial}{\partial t}\right)p(\mathbf{x},r)=2i\frac{\partial}{\partial t}k^{n-2q}(\mathbf{x},r),\quad\text{for all }\mathbf{x}\in\mathrm{H}_n,\ r>0.\tag{5.12}$$

We will find $p(\mathbf{x},r)$ from (5.12) under the condition $\lim_{r\to\infty}\tilde{p}(\zeta,\tau,r)e^{\tau r}=0$ for any $\tau\in\mathrm{R}$. Taking the Fourier transform with respect to t in (5.12), we obtain

$$\left(\frac{\partial}{\partial r}+\tau\right)\tilde{p}(\zeta,\tau,r)=-2\tau\tilde{k}^{n-2q}(\zeta,\tau,r)\quad\Longrightarrow\quad\frac{\partial}{\partial r}[e^{\tau r}\tilde{p}]=-2\tau e^{\tau r}\tilde{k}^{n-2q}.$$

Next we integrate both sides from r to ∞, and the condition $\lim\limits_{r\to\infty}\tilde{p}(\zeta,\tau,r)e^{\tau r}=0$ for any $\tau\in\mathrm{R}$ yields:

$$\tilde{p}(\zeta,\tau,r)=2\tau\int_0^\infty e^{\tau v}\tilde{k}^{n-2q}(\zeta,\tau,r+v)\,dv.\tag{5.13}$$

We now take the inverse Fourier transform with respect to τ and find that

$$p(\mathbf{x},r)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty\tau e^{\tau(v+it)}\tilde{k}^{n-2q}(\zeta,\tau,r+v)\,d\tau\,dv.\tag{5.14}$$

We substitute $\tilde{k}^{n-2q}=\tilde{\Box}_{n-2q}^{-1}$, where (see Eq. (4.9) of section 4):

$$\tilde{\Box}_{n-2q}^{-1}(\zeta,\tau,r)=\frac{|\tau|^{n-\frac12}}{\pi^n}\int_0^\infty\frac{1}{\sqrt{2\pi s}}\prod_{j=1}^n\frac{a_j}{\sinh(a_js)}e^{s(n-2q)\operatorname{sgn}(\tau)-\frac{|\tau|}{2}s-\frac{r^2}{2s}|\tau|-\gamma(s,\zeta)|\tau|}\,ds,\tag{5.15}$$

with $\gamma(s,\zeta)=\sum_{j=1}^n a_j|\zeta_j|^2\coth(a_js)$, into (5.14) and obtain

$$\begin{aligned}p(\mathbf{x},r)&\\=&\frac{1}{\sqrt2\pi^{n+\frac32}}\int_0^\infty\int_0^\infty s^{-\frac12}\prod_{j=1}^n\frac{a_j}{\sinh(a_js)}\\&\times\int_{-\infty}^\infty\tau|\tau|^{n-\frac12}\exp\left\{\tau(v+it)+s(n-2q)\operatorname{sgn}(\tau)-\left[\frac{s}{2}+\frac{(r+v)^2}{2s}|+\gamma(s,\zeta)\right]|\tau|\right\}d\tau\,ds\,dv.\end{aligned}$$

We calculate the integral with respect to τ first and obtain:

$$\int_{-\infty}^\infty\tau|\tau|^{n-\frac12}\exp\left\{\tau(v+it)+s(n-2q)\operatorname{sgn}\tau-\left[\frac{s}{2}+\frac{(r+v)^2}{2s}+\gamma(s,\zeta)\right]|\tau|\right\}d\tau$$
$$=\frac{\Gamma\left(n+\frac32\right)e^{(n-2q)s}}{\left[\gamma(s,\zeta)-it-v+\frac{s}{2}+\frac{(r+v)^2}{2s}\right]^{n+\frac32}}-\frac{\Gamma\left(n+\frac32\right)e^{-(n-2q)s}}{\left[\gamma(s,\zeta)+it+v+\frac{s}{2}+\frac{(r+v)^2}{2s}\right]^{n+\frac32}}.$$

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This yields that

$$p(\mathbf{x},r) = \frac{\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{2}\pi^{n+\frac{3}{2}}}\int_0^\infty\int_0^\infty s^{-\frac{1}{2}}\prod_{j=1}^n\frac{a_j}{\sinh(a_j s)}$$
$$\times\left\{\frac{e^{(n-2q)s}}{\left[\gamma(s,\zeta)-it-v+\frac{s}{2}+\frac{(r+v)^2}{2s}\right]^{n+\frac{3}{2}}}-\frac{e^{-(n-2q)s}}{\left[\gamma(s,\zeta)+it+v+\frac{s}{2}+\frac{(r+v)^2}{2s}\right]^{n+\frac{3}{2}}}\right\}ds\,dv.$$

Simplify the above equation by replacing s by $-s$ for the second part of s-integration, we obtain

$$p(\mathbf{x},r) = \frac{\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{2}\pi^{n+\frac{3}{2}}}\int_0^\infty\int_{-\infty}^\infty\prod_{j=1}^n\frac{a_j}{\sinh(a_j s)}\frac{e^{(n-2q)s}s^{n+1}\,ds\,dv}{\left[\left(\gamma(s,\zeta)-it-v\right)s+\frac{s^2}{2}+\frac{(r+v)^2}{2}\right]^{n+\frac{3}{2}}}. \tag{5.16}$$

We can also write (5.16) in terms of the complex distance $g(s,\mathbf{x})$ and volume element $\nu(s)$ which have been defined in (4.12):

$$p(\mathbf{x},r) = \frac{\Gamma\left(n+\frac{3}{2}\right)}{2^n\pi^{n+\frac{3}{2}}}\int_0^\infty\int_{-\infty}^\infty\frac{e^{2(n-2q)s}s^{n+1}\nu(s)}{\left[\left(g(s,\mathbf{x})-v\right)s+s^2+\frac{(r+v)^2}{4}\right]^{n+\frac{3}{2}}}\,ds\,dv. \tag{5.17}$$

Some simple calculation will yield that $\Box^\tau p(\mathbf{x},r)=0$ for all $\mathbf{x}\in \mathrm{H}_n$, $r>0$.

The following theorem summarizes our solution to the problem (5.1).

THEOREM 5.2.

$$U=\int_{\mathrm{H}_n\times\mathrm{R}^+}\left[k^{n-2q}(\mathbf{y}^{-1}\cdot\mathbf{x},r+\rho)+k^{n-2q}(\mathbf{y}^{-1}\cdot\mathbf{x},r-\rho)+p(\mathbf{y}^{-1}\cdot\mathbf{x},r+\rho)\right]f(\mathbf{y},\rho)\,d\mathbf{y}\,d\rho$$

will solve the problem $\Box^\tau U=f$ *in* $U^{\mathbf{n}}$ *with* $\bar{\mathbf{Z}}_{n+1}U|_{bU^{\mathbf{n}}}=0$, *where* $k^{n-2q}(\mathbf{x},r)$ *is given by (5.6) and* $p(\mathbf{x},r)$ *is given by (5.17).*

6**. The Associated Heat Equation.** In this section, we consider the heat equation for the $\bar\partial$-Neumann problem. This problem is analogue of some classical problems in differential geometry.

We fix q, $0\le q\le n$, and work on the $(0,q)$ forms on $\bar{U^{\mathbf{n}}}$. Let $\Lambda_0^{(0,q)}(\bar{U^{\mathbf{n}}})$ denote the C^∞ $(0,q)$ forms on $U^{\mathbf{n}}$ which can be extended to compactly supported $(0,q)$ forms on C^{n+1}, $\Lambda_0^{(0,q)}$ the C^∞ $(0,q)$ forms with compact support in $U^{\mathbf{n}}$, and $L_2^{(0,q)}(\bar{U^{\mathbf{n}}})$ the square integrable $(0,q)$ forms on $U^{\mathbf{n}}$.

Let $F(\cdot,s)\in L_2^{(0,q)}(\bar{U^{\mathbf{n}}})$ for all $s\in\mathrm{R}^+$ and suppose that the coefficients of F are differentiable in s. $F(\mathbf{z},t,r,s)$ solves the heat equation for the $\bar\partial$-Neumann problem if for fixed s,

$$\left(\frac{\partial}{\partial s}+\Box\right)F=0 \quad\text{and}\quad F(\mathbf{z},t,r,s)\in\operatorname{Dom}\Box. \tag{6.1}$$

The initial value problem for the heat equation (6.1) is to find a solution $F(\cdot,s)$ of (6.1) with specified initial value $f\in\Lambda_0^{(0,q)}(\bar{U^{\mathbf{n}}})$, *i.e.*,

$$\lim_{s\to 0^+}F(\cdot,s)=f \quad \text{in } L_2^{(0,q)}(\bar{U^{\mathbf{n}}}). \tag{6.2}$$

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A fundamental solution of the heat equation is a one-parameter family of bounded operator H_s, $s \in \mathrm{R}^+$, on $L_2^{(0,q)}(U^{\mathbf{n}})$ such that for $f \in \Lambda_0^{(0,q)}(U^{\mathbf{n}})$:

$$\left(\frac{\partial}{\partial s} + \Box\right) H_s(f) = 0, H_s(f) \in \mathrm{Dom}\,\Box \quad \text{with } \lim_{s\to 0^+} H_s(f) = f \text{ in } L_2^{(0,q)}(\bar{U}^{\mathbf{n}}).$$

Stanton [21] proved the existence and uniqueness of the fundamental solution for the Siegel domain. Her proof can be extended to our domain. Furthermore, she proved that H_s is the semi-group generated by $-\Box$ and can be written as the convolution operator on $\mathrm{H}_n \times \mathrm{R}$. For each $s \in \mathrm{R}^+$, H_s is self-adjoint and positive.

Similarly to the corresponding $\bar\partial$-Neumann problem, since the $\bar\partial$-Neumann boundary conditions and $\Box$ act diagonally on $(0, q)$ forms, the heat equation can be split into a pair of problems:

$$\left(\frac{\partial}{\partial s} + \Box^{\nu}\right) F^{\nu} = 0 \quad \text{with } F^{\nu}|_{r=0} = 0 \quad \text{and} \quad \lim_{s\to 0^+} F^{\nu}(\cdot, s) = f \text{ in } L_2^{(0,q)}(\bar{U}^{\mathbf{n}}) \tag{6.3}$$

$$\left(\frac{\partial}{\partial s} + \Box^{\tau}\right) F^{\tau} = 0 \quad \text{with } \bar{\mathbf{Z}}_{n+1}F^{\tau}|_{r=0} = 0 \quad \text{and} \quad \lim_{s\to 0^+} F^{\tau}(\cdot, s) = f \text{ in } L_2^{(0,q)}(\bar{U}^{\mathbf{n}}). \tag{6.4}$$

6.1. *The interior problem.* Following the classical method of reduction to the boundary, we break the problem into two parts. First, we will use the Laguerre calculus to find the kernel $k_s^{\alpha}(\mathbf{x}, r) = \exp\{-s\Box_{\alpha}\}$ such that

$$(K_s^{\alpha} f)(\mathbf{x}, r) = \int_{\mathrm{H}_n\times\mathrm{R}} k_s^{\alpha}(\mathbf{y}^{-1}\cdot\mathbf{x}, r-\rho) f(\mathbf{y}, \rho)\, d\mathbf{y}\, d\rho$$

solves the initial value problem:

$$\left(\frac{\partial}{\partial s} + \Box_{\alpha}\right) (K_s^{\alpha} f)(\mathbf{x}, r) = 0, \quad \lim_{s\to 0^+} K_s^{\alpha} f = f \text{ for } f \in C_0^{\infty}(\bar{U}^{\mathbf{n}}). \tag{6.5}$$

We call (6.5) the interior problem. From $k_s^{\alpha}(\mathbf{x}, r)$, we can construct the solutions of (6.3) and (6.4).

We shall first compute $\hat{k}_s^{\alpha}(\zeta, \tau, \xi) = \exp\{-s\hat{\Box}_{\alpha}\}$. Eq.(4.3)

$$\hat{\Box}_{\alpha}(\zeta, \tau, \xi) = \sum_{|\mathbf{k}|=0}^{\infty} \left[\sum_{j=1}^{n}(2k_j+1)a_j|\tau| - \alpha\tau + \frac{1}{2}(\tau^2+\xi^2)\right] \prod_{j=1}^{n} a_j \tilde{\mathfrak{L}}_{k_j}^{(0)}(\sqrt{a_j}\,\zeta_j, \tau)$$

and Theorem 3.2 imply that

$$\begin{aligned}
\hat{k}_s^{\alpha}(\zeta, \tau, \xi) &= \sum_{|\mathbf{k}|=0}^{\infty} e^{-s\left[\sum_{j=1}^{n}(2k_j+1)a_j|\tau| - \alpha\tau + \frac{1}{2}(\tau^2+\xi^2)\right]} \prod_{j=1}^{n} a_j \tilde{\mathfrak{L}}_{k_j}^{(0)}(\sqrt{a_j}\,\zeta_j, \tau) \\
&= e^{\alpha\tau s - \frac{s}{2}(\tau^2+\xi^2)} \sum_{|\mathbf{k}|=0}^{\infty} \prod_{j=1}^{n} \frac{2|\tau|a_j}{\pi} e^{-|\tau|s(2k_j+1)a_j - a_j|\tau|\,|\zeta_j|^2} L_{k_j}^{(0)}(2a_j|\tau|\,|\zeta_j|^2) \\
&= \frac{|\tau|^n}{\pi^n} e^{\alpha\tau s - \frac{s}{2}(\tau^2+\xi^2)} \prod_{j=1}^{n} 2a_j e^{-a_j|\tau|s - a_j|\tau|\,|\zeta_j|^2} \sum_{k_j=0}^{\infty} e^{-2|\tau|sk_ja_j} L_{k_j}^{(0)}(2a_j|\tau|\,|\zeta_j|^2)
\end{aligned}$$

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We now sum up this series via the generating function for the Laguerre polynomials and obtain:

$$\begin{aligned}\hat{k}_s^\alpha(\mathbf{z},\tau,\xi) &= \frac{|\tau|^n}{\pi^n} e^{\alpha\tau s-\frac{s}{2}(\tau^2+\xi^2)}\left[\prod_{j=1}^n \frac{a_j}{\sinh(a_j|\tau|s)}\right]\exp\Big\{-|\tau|\sum_{j=1}^n a_j|\zeta_j|^2\coth(a_j|\tau|s)\Big\}\\ &= \frac{|\tau|^n}{\pi^n}\left[\prod_{j=1}^n \frac{a_j}{\sinh(a_j|\tau|s)}\right]\exp\left\{\alpha\tau s-\frac{s}{2}(\tau^2+\xi^2)-|\tau|\gamma(|\tau|s,\zeta)\right\}.\end{aligned}$$

We then take the inverse Fourier transform with respect to ξ by applying the Gaussian integral

$$\frac{1}{2\pi}\int_{-\infty}^\infty e^{-\frac{s}{2}\xi^2+ir\xi}\,d\xi = \frac{1}{\sqrt{2\pi s}}e^{-\frac{r^2}{2s}},$$

and this yields

$$\tilde{k}_s^\alpha(\zeta,\tau,r) = \frac{|\tau|^n e^{-\frac{r^2}{2s}}}{\pi^n\sqrt{2\pi s}}\prod_{j=1}^n \frac{a_j}{\sinh(a_j|\tau|s)}\exp\left\{\alpha\tau s-\frac{1}{2}s\tau^2-|\tau|\gamma(|\tau|s,\zeta)\right\}. \tag{6.6}$$

Next we take the inverse Fourier transform with respect to τ and find the heat operator:

$$k_s^\alpha(\mathbf{x},r) = \frac{e^{-\frac{r^2}{2s}}}{2^{\frac{3}{2}}\pi^{n+\frac{3}{2}}s^{\frac{1}{2}}}\int_{-\infty}^\infty\prod_{j=1}^n \frac{a_j|\tau|}{\sinh(a_j|\tau|s)}\exp\left\{it\tau+\alpha\tau s-\frac{1}{2}s\tau^2-|\tau|\gamma(|\tau|s,\zeta)\right\}\,d\tau.$$

Substitute τ by τ/s in the above equation, we have

$$k_s^\alpha(\mathbf{x},r) = \frac{e^{-\frac{r^2}{2s}}}{2^{\frac{3}{2}}(\pi s)^{n+\frac{3}{2}}}\int_{-\infty}^\infty\prod_{j=1}^n \frac{a_j|\tau|}{\sinh(a_j|\tau|)}\exp\left\{\frac{it}{s}\tau+\alpha\tau-\frac{\tau^2}{2s}-\frac{|\tau|}{s}\gamma(|\tau|,\zeta)\right\}\,d\tau.$$

Since

$$\prod_{j=1}^n \frac{a_j|\tau|}{\sinh(a_j|\tau|)} = \prod_{j=1}^n \frac{a_j\tau}{\sinh(a_j\tau)} \quad\text{and}\quad |\tau|\coth(a_j|\tau|)=\tau\coth(a_j\tau),$$

we can simplify the above equation and obtain the heat kernel:

$$k_s^\alpha(\mathbf{x},r) = \frac{e^{-\frac{r^2}{2s}}}{2^{\frac{3}{2}}(\pi s)^{n+\frac{3}{2}}}\int_{-\infty}^\infty\prod_{j=1}^n \frac{a_j\tau}{\sinh(a_j\tau)}\exp\left\{\frac{it}{s}\tau+\alpha\tau-\frac{\tau^2}{2s}-\frac{\tau}{s}\gamma(\tau,\zeta)\right\}\,d\tau. \tag{6.7}$$

We can also write (6.7) in terms of the volume $\nu(\tau)$ and complex distance $g(\tau,\mathbf{x})$ (see Eq. (4.12)) by replacing τ by 2τ in Eq. (6.7):

$$k_s^\alpha(\mathbf{x},r) = \frac{e^{-\frac{r^2}{2s}}}{\sqrt{2}(\pi s)^{n+\frac{3}{2}}}\int_{-\infty}^\infty \tau^n\nu(\tau)\exp\left\{2\alpha\tau-\frac{2\tau^2}{s}-\frac{2\tau}{s}g(\tau,\mathbf{x})\right\}\,d\tau. \tag{6.8}$$

Now we compare our formula with Stanton's results. If we set $a_j=1$ for all $j=1,2,\ldots,n$ and substitute τ by $-\tau$ in (6.7), then we obtain

$$k_s^\alpha(\mathbf{x},r) = \frac{e^{-\frac{r^2}{2s}}}{2^{\frac{3}{2}}(\pi s)^{n+\frac{3}{2}}}\int_{-\infty}^\infty \frac{\tau^n}{\sinh^n(\tau)}\exp\left\{-i\frac{s}{t}\tau-\alpha\tau-\frac{\tau^2}{2s}-\frac{\tau}{s}|\zeta|^2\coth(\tau)\right\}\,d\tau.$$

This is N. K. Stanton's formula (1.8) of [23].

We summarize the solution to the interior problem (6.5) in the following theorem:

THEOREM 6.1. *The convolution operator* $K_s^\alpha f = f * k_s^\alpha$ *on* $\mathrm{H}_n \times \mathrm{R}$ *with* k_s^α *given by (6.8) solves the initial problem:*

$$\left(\frac{\partial}{\partial s} + \Box_\alpha\right)(K_s^\alpha f)(\mathbf{x}, r) = 0 \quad \textit{with} \quad \lim_{s\to 0^+} K_s^\alpha f = f \quad \textit{for } f \in C_0^\infty(\bar{U}^{\mathbf{n}}).$$

As an application of the heat kernel, we consider the operator

$$H_s^\alpha(\mathbf{x}, r) = \Box_\alpha^{-\frac{1}{2}} \exp\{-s\Box_\alpha^{\frac{1}{2}}\}\delta_0(\mathbf{x}, r) \quad s > 0. \tag{6.9}$$

This operator will give the solution of the corresponding wave equation by analytical continuation $s > 0$ to is with $s \in \mathrm{R}$. Here we just consider $H_s^\alpha(\mathbf{x}, r)$ for $s > 0$ and will consider its analytical continuation in a future publication. Using the subordination identity:

$$A^{-1}e^{-sA} = \pi^{-\frac{1}{2}} \int_0^\infty e^{-s^2/4\mu} \mu^{-\frac{1}{2}} e^{-\mu A^2}\, d\mu \tag{6.10}$$

with $A = \Box_\alpha^{\frac{1}{2}}$, we get

$$H_s^\alpha(\mathbf{x}, r) = \pi^{-\frac{1}{2}} \int_0^\infty e^{-s^2/4\mu} \mu^{-\frac{1}{2}} \exp\{-\mu\Box_\alpha\}\, d\mu. \tag{6.11}$$

We substitute Eq. (6.8) into Eq. (6.11),

$$\begin{aligned} &H_s^\alpha(\mathbf{x}, r) \\ &\quad = \frac{1}{\sqrt{2}\pi^{n+2}} \int_0^\infty \frac{e^{-\frac{s^2}{4\mu}-\frac{r^2}{2\mu}}}{\mu^{n+2}} \int_{-\infty}^\infty \tau^n \nu(\tau) \exp\left\{2\alpha\tau - \frac{2\tau^2}{\mu} - \frac{2\tau}{\mu} g(\tau, \mathbf{x})\right\} d\tau\, d\mu. \end{aligned} \tag{6.12}$$

We change the order of the integrations:

$$H_s^\alpha(\mathbf{x}, r) = \frac{1}{\sqrt{2}\pi^{n+2}} \int_{-\infty}^\infty e^{2\alpha\tau} \tau^n \nu(\tau) \int_0^\infty e^{-\left(\frac{s^2}{4}+\frac{r^2}{2}+2\tau^2+\frac{2\tau}{g}(\tau,\mathbf{x})\right)\frac{1}{\mu}} \frac{d\mu}{\mu^{n+2}}\, d\tau. \tag{6.13}$$

Next substituting μ by $1/\mu$ in (6.13) yields:

$$\begin{aligned} H_s^\alpha(\mathbf{x}, r) &= \frac{1}{\sqrt{2}\pi^{n+2}} \int_{-\infty}^\infty e^{2\alpha\tau} \tau^n \nu(\tau) \int_0^\infty \mu^n e^{-\left[\frac{s^2}{4}+\frac{r^2}{2}+2\tau^2+2\tau g(\tau,\mathbf{x})\right]\mu}\, d\mu\, d\tau \\ &= \frac{n!}{\sqrt{2}\pi^{n+2}} \int_{-\infty}^\infty \frac{e^{2\alpha\tau}\tau^n\nu(\tau)}{\left[\frac{s^2}{4}+\frac{r^2}{2}+2\tau^2+2\tau g(\tau,\mathbf{x})\right]^{n+1}}\, d\tau. \end{aligned}$$

In conclusion, we find

$$H_s^\alpha(\mathbf{x}, r) = \frac{n!}{\sqrt{2}\pi^{n+2}} \int_{-\infty}^\infty \frac{e^{2\alpha\tau}\tau^n\nu(\tau)}{\left[\frac{s^2}{4}+\frac{r^2}{2}+2\tau^2+2\tau g(\tau,\mathbf{x})\right]^{n+1}}\, d\tau. \tag{6.14}$$

We now integrate $H_s^\alpha(\mathbf{x}, r)$ with respect to time s, formally, this should give us the fundamental solution $k^\alpha(\mathbf{x}, r)$ of the operator $\Box_\alpha$ (see Eq. (4.13)). We will see that this is indeed true. We carry out the calculations formally:

$$k^\alpha(\mathbf{x}, r) = \int_0^\infty H_s^\alpha(\mathbf{x}, r)\, ds = \frac{n!}{\sqrt{2}\pi^{n+2}} \int_0^\infty \int_{-\infty}^\infty \frac{e^{2\alpha\tau}\tau^n\nu(\tau)\, d\tau\, ds}{\left[\frac{s^2}{4}+\frac{r^2}{2}+2\tau^2+2\tau g(\tau,\mathbf{x})\right]^{n+1}}. \tag{6.15}$$

https://doi.org/10.4153/CJM-1997-064-4 Published online by Cambridge University Press

Exchange the order of the integrations in Eq. (6.15):

$$k^\alpha(\mathbf{x},r)=\frac{n!}{\sqrt{2}\pi^{n+2}}\int_{-\infty}^{\infty}e^{2\alpha\tau}\tau^n\nu(\tau)\int_0^\infty\frac{ds}{\left[\frac{s^2}{4}+\frac{r^2}{2}+2\tau^2+2\tau g(\tau,\mathbf{x})\right]^{n+1}}\,d\tau. \tag{6.16}$$

Let $\Lambda(\tau,\mathbf{x},r)=2\tau^2+r^2/2+2\tau g(\tau,\mathbf{x})$, and then apply the integral formula

$$\int_0^\infty\frac{ds}{(s^2+a^2)^{n+1}}=\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2a^{2n+1}}$$

to Eq. (6.16), we get

$$\begin{aligned}
k^\alpha(\mathbf{x},r)&=\frac{n!}{\sqrt{2}\pi^{n+2}}\int_{-\infty}^{\infty}e^{2\alpha\tau}\tau^n\nu(\tau)\int_0^\infty\frac{ds}{\left[\frac{s^2}{4}+\Lambda\right]^{n+1}}\,d\tau\\
&=\frac{4^{n+1}n!}{\sqrt{2}\pi^{n+2}}\int_{-\infty}^{\infty}e^{2\alpha\tau}\tau^n\nu(\tau)\int_0^\infty\frac{ds}{\left[s^2+4\Lambda\right]^{n+1}}\,d\tau\\
&=\frac{4^{n+1}n!}{\sqrt{2}\pi^{n+2}}\int_{-\infty}^{\infty}e^{2\alpha\tau}\tau^n\nu(\tau)\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2(2\Lambda^{\frac12})^{2n+1}}\,d\tau\\
&=\frac{4^{n+1}n!}{\sqrt{2}\pi^{n+2}}\cdot\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2^{2n+2}}\int_{-\infty}^{\infty}\frac{e^{2\alpha\tau}\tau^n\nu(\tau)}{\Lambda^{n+\frac12}}\,d\tau\\
&=\frac{\Gamma\left(n+\frac12\right)}{\sqrt{2}\pi^{n+1}\Gamma\left(\frac12\right)}\int_{-\infty}^{\infty}\frac{e^{2\alpha\tau}\tau^n\nu(\tau)}{\Lambda^{n+\frac12}}\,d\tau.
\end{aligned}$$

Then the definitions of Λ and $\Gamma\left(\frac12\right)=\sqrt{\pi}$ lead to

$$k^\alpha(\mathbf{x},r)=\frac{\Gamma\left(n+\frac12\right)}{\sqrt{2}\pi^{n+\frac32}}\int_{-\infty}^{\infty}\frac{e^{2\alpha\tau}\tau^n\nu(\tau)}{\left[2\tau^2+\frac{r^2}{2}+2\tau g(\tau,\mathbf{x})\right]^{n+\frac12}}\,d\tau. \tag{6.17}$$

This coincides with (4.13). Of course, the above calculations are purely formal. To carry out the integration rigorously, we need estimates on the $H_s^\alpha(\mathbf{x},r)$ for large time s. N. Stanton proved the estimates for the case of $a_j=1$ for all $j=1,2,\ldots,n$. I will carry out the estimates for our problem in future publication.

6.2. *The solution of the heat equation.* We will solve the heat equation for the $\bar\partial$-Neumann problem. In the beginning of this section, we reduced this problem to a pair of problems:

$$\left(\frac{\partial}{\partial s}+\square^\nu\right)F^\nu=0 \text{ in } U^{\mathbf{n}} \quad \text{with } F^\nu|_{bU^{\mathbf{n}}}=0 \quad \text{and} \quad \lim_{s\to0^+}F^\nu(\cdot,s)=f \tag{6.18}$$

$$\left(\frac{\partial}{\partial s}+\square^\tau\right)F^\tau=0 \text{ in } U^{\mathbf{n}} \quad \text{with } \bar{\mathbf{Z}}_{n+1}F^\tau|_{bU^{\mathbf{n}}}=0 \quad \text{and} \quad \lim_{s\to0^+}F^\tau(\cdot,s)=g, \tag{6.19}$$

where $\square^\tau=\mathfrak{L}_{n-2q}-\frac12\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^2}\right)$ and $\square^\nu=\mathfrak{L}_{n-2(q-1)}-\frac12\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^2}\right)$ with $\mathfrak{L}_\alpha=-\frac12\sum_{j=1}^n(\bar{\mathbf{Z}}_j\mathbf{Z}_j+\mathbf{Z}_j\bar{\mathbf{Z}}_j)+i\alpha\mathbf{T}$.

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Recall that $\Box^\tau = \Box_{n-2q}$ and $\Box^\nu = \Box_{n-2q+2}$.

The problem (6.18), involving the normal component, is essentially the Dirichlet problem. From the solution of the interior problem (6.5), we can easily construct Green's operator. For (6.18), we set $\alpha = n-2q+2$ in the interior problem.

$$k_s^{n-2q+2}(\mathbf{x}, r) = \frac{e^{-\frac{r^2}{2s}}}{\sqrt{2}(\pi s)^{n+\frac{3}{2}}} \int_{-\infty}^{\infty} \tau^n \nu(\tau) e^{2(n-2q+2)\tau - \frac{2\tau^2}{s} - \frac{2\tau}{s} g(\tau,\mathbf{x})}\, d\tau \tag{6.20}$$

is even in r. We set

$$G_s^\nu(\mathbf{x}, \mathbf{y}, r, \rho) = k_s^{n-2q+2}(\mathbf{y}^{-1}\cdot \mathbf{x}, r-\rho) - k_s^{n-2q+2}(\mathbf{y}^{-1}\cdot \mathbf{x}, r+\rho). \tag{6.21}$$

Then the operator G_s^ν, defined by

$$(G_s^\nu f)(\mathbf{x}, r) = \int_{\mathrm{H}_n\times \mathrm{R}^+} G_\nu(\mathbf{x}, \mathbf{y}, r, \rho) f(\mathbf{y}, \rho)\, d\mathbf{y}\, d\rho \tag{6.22}$$

is Green's operator for $(\frac{\partial}{\partial s} + \Box^\nu)$; that is

(6.23)
$$\left(\frac{\partial}{\partial s} + \Box^\nu\right) G_s^\nu(f) = 0 \text{ in } U^{\mathbf{n}}, G_s^\nu(f)|_{bU^{\mathbf{n}}} = G_s^\nu(f)|_{r=0} = 0 \quad \text{and} \quad \lim_{s\to 0^+} G_s^\nu(f) = f.$$

The proof of (6.23) is quite simple. Indeed, we let

$$f_1(\mathbf{x}, r) = \begin{cases} f(\mathbf{x}, r) & \text{if } r \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(\mathbf{x}, r) = \begin{cases} f(\mathbf{x}, -r) & \text{if } r \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then we can write (6.22) in the following form:

$$G_s^\nu f(\mathbf{x}, r) = f_1 * k_s^{n-2q-2}(\mathbf{x}, r) - f_2 * k_s^{n-2q+2}(\mathbf{x}, r).$$

$k_s^{n-2q+2}(\mathbf{x}, r) = k_s^{n-2q+2}(\mathbf{x}, -r)$ implies $G_s^\nu(f)|_{r=0} = 0$. And since $k_s^{n-2q+2}(\mathbf{x}, r)$ is the corresponding kernel of the solution of the interior problem, it yields

$$\left(\frac{\partial}{\partial s} + \Box^\nu\right) G_s^\nu(f) = 0 \quad \text{for } r > 0 \quad \text{and} \quad \lim_{s\to 0^+} G_s^\nu(f) = f_1 - f_2 = f.$$

Hence $F^\nu(\mathbf{x}, r, s) = G_s^\nu(f)(\mathbf{x}, r)$ solves the problem (6.18).

The problem (6.19), involving the tangential part, has the $\bar{\partial}$-Neumann boundary condition. First we will also construct the related Green's operator G_s^τ as above but with Neumann boundary condition. We set $\alpha = n-2q$ in the interior problem. The kernel of the solution for the interior problem is

$$k_s^{n-2q}(\mathbf{x}, r) = \frac{e^{-\frac{r^2}{2s}}}{\sqrt{2}(\pi s)^{n+\frac{3}{2}}} \int_{-\infty}^{\infty} \tau^n \nu(\tau) \exp\left\{2(n-2q)\tau - \frac{2\tau^2}{s} - \frac{2\tau}{s} g(\tau, \mathbf{x})\right\} d\tau. \tag{6.24}$$

Similarly, we set

$$G_s^\tau(\mathbf{x}, \mathbf{y}, r, \rho) = k_s^{n-2q}(\mathbf{y}^{-1}\cdot \mathbf{x}, r-\rho) + k_s^{n-2q}(\mathbf{y}^{-1}\cdot \mathbf{x}, r+\rho). \tag{6.25}$$

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Then the operator G_s^τ, defined by

$$(G_s^\tau f)(\mathbf{x}, r) = \int_{\mathrm{H}_n \times \mathrm{R}^+} G_s^\tau(\mathbf{x}, \mathbf{y}, r, \rho) f(\mathbf{y}, \rho)\, d\mathbf{y}\, d\rho \tag{6.26}$$

is the Green's operator for $(\frac{\partial}{\partial s} + \Box^\tau)$ with the Neumann boundary condition; that is

$$\left(\frac{\partial}{\partial s} + \Box^\tau\right) G_s^\tau(f) = 0 \text{ in } U^{\mathbf{n}},\ \frac{\partial}{\partial r} G_s^\tau(f)|_{r=0} = 0 \quad \text{and} \quad \lim_{s \longrightarrow 0^+} G_s^\tau(f) = f. \tag{6.27}$$

The proof of (6.27) is similar to that of (6.23).

Next we write the solution of (6.19) as

$$F^\tau(\mathbf{x}, r, s) = G_s^\tau(f) + P_s(f), \tag{6.28}$$

where $P_s(f)$ is the correction term so that

$$\left(\frac{\partial}{\partial s} + \Box^\tau\right) F^\tau = 0 \quad \text{with } \bar{\mathbf{Z}}_{n+1} F^\tau(\mathbf{x}, r, s)|_{r=0} = 0 \quad \text{and} \quad \lim_{s \longrightarrow 0^+} F^\tau = f.$$

In this way of looking at the problem, f is given and P_s, more precisely its kernel $p_s(\mathbf{x}, r)$, is to be determined.

If we apply the equation and the boundary condition in (6.19) to (6.28) we find that

$$\left(\frac{\partial}{\partial s} + \Box^{(\tau)}\right) P_s(f) = 0;\ \bar{\mathbf{Z}}_{n+1} P_s(f)|_{bU^{\mathbf{n}}} = -\bar{\mathbf{Z}}_{n+1} G_s^\tau(f)|_{bU^{\mathbf{n}}} \quad \text{and} \quad \lim_{s \longrightarrow 0^+} P_s(f) = 0. \tag{6.29}$$

We assume that P_s is the convolution operator with kernel $p_s(\mathbf{x}, r)$, *i.e.*,

$$P_s(f)(\mathbf{x}, r) = \int_{\mathrm{H}_n \times \mathrm{R}^+} p_s(\mathbf{y}^{-1} \cdot \mathbf{x}, r + \rho) f(\mathbf{y}, \rho)\, d\mathbf{y}\, d\rho$$

and $p_s(\mathbf{x}, r)$ decays very fast as $r \longrightarrow \infty$.

The boundary conditions in (6.27) and (6.29) yield that

$$\left(\frac{\partial}{\partial r} - i\frac{\partial}{\partial t}\right) P_s(f)|_{r=0} = i\frac{\partial}{\partial t} G_s^\tau(f)|_{r=0}. \tag{6.30}$$

This, in turn, implies that

$$\begin{aligned} &\int_{\mathrm{H}_n \times \mathrm{R}^+} \left(\frac{\partial}{\partial r} - i\frac{\partial}{\partial t}\right) p_s(\mathbf{y}^{-1} \cdot \mathbf{x}, r + \rho) f(\mathbf{y}, \rho)\, d\mathbf{y}\, d\rho \Big|_{r=0} \\ &\quad = \int_{\mathrm{H}_n \times \mathrm{R}^+} i\frac{\partial}{\partial t}\left[k_s^{n-2q}(\mathbf{y}^{-1} \cdot \mathbf{x}, r - \rho) + k_s^{n-2q}(\mathbf{y}^{-1} \cdot \mathbf{x}, r + \rho)\right] f(\mathbf{y}, \rho)\, d\mathbf{y}\, d\rho \Big|_{r=0}. \end{aligned}$$

Since we want to find $p_s(\mathbf{x}, r)$ such that (6.28) solves the problem (6.19) for all f taking from some class of functions (*e.g.*, $C_0^\infty(\mathrm{H}_n \times R^+)$), we obtain, from the above equation, that

$$\left(\frac{\partial}{\partial r} - i\frac{\partial}{\partial t}\right) p_s(\mathbf{x}, r) = 2i\frac{\partial}{\partial t} k_s^{n-2q}(\mathbf{x}, r), \quad \text{for all } \mathbf{x} \in \mathrm{H}_n,\ r > 0,\ s > 0. \tag{6.31}$$

We will find $p_s(\mathbf{x}, r)$ from (6.31) under the condition $\lim_{r\to\infty} p_s(\mathbf{x}, r)e^{\tau r} = 0$ for any $\tau \in \mathbb{R}$. Taking the Fourier transform with respect to t in Eq. (6.31), we obtain

$$\left(\frac{\partial}{\partial r} + \tau\right) \tilde{k}_s^{n-2q}(\zeta, \tau, r) = -2\tau \tilde{k}_s^{n-2q}(\zeta, \tau, r) \Longrightarrow \frac{\partial}{\partial r}[e^{\tau r}\tilde{p}_s] = -2\tau e^{\tau r}\tilde{k}_s^{n-2q}.$$

Next we integrate both sides from r to ∞, this implies that

$$\tilde{p}(\zeta, \tau, r) = 2\tau \int_0^\infty e^{\tau v}\tilde{k}_s^{n-2q}(\zeta, \tau, r + v)\, dv. \tag{6.32}$$

We now take the inverse Fourier transform with respect to τ and find that

$$p_s(\mathbf{x}, r) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \tau e^{\tau(v+it)}\tilde{k}_s^{n-2q}(\zeta, \tau, r + v)\, d\tau\, dv. \tag{6.33}$$

We substitute $\tilde{k}_s^{n-2q}(\zeta, \tau, r)$ (see equation (6.6)):

$$\tilde{k}_s^{n-2q}(\zeta, \tau, r) = \frac{|\tau|^n e^{-\frac{r^2}{2s}}}{\pi^n\sqrt{2\pi s}} \prod_{j=1}^n \frac{a_j}{\sinh(a_j|\tau|s)} \exp\left\{(n - 2q)\tau s - \frac{1}{2}s\tau^2 - |\tau|\gamma(|\tau|s, \zeta)\right\}.$$

into (6.33) to obtain:

$$\begin{aligned} p_s(\mathbf{x}, r) \;=\; & \frac{1}{\sqrt{2s}\pi^{n+\frac{3}{2}}} \int_0^\infty \int_{-\infty}^\infty \tau^{n+1} \prod_{j=1}^n \frac{a_j}{\sinh(a_j\tau s)} \\ & \times \exp\left\{\tau(v + it) + (n - 2q)\tau s - \frac{s}{2}\tau^2 - \frac{(r + v)^2}{2s} - \gamma(\tau s, \zeta)\tau\right\}\, d\tau\, dv. \end{aligned} \tag{6.34}$$

where we set $\gamma(s, \zeta) = \sum_{j=1}^n a_j|\zeta_j|^2 \coth(a_j s)$. Next we substitute τ by τ/s in (6.34):

$$\begin{aligned} p_s(\mathbf{x}, r) \;=\; & \frac{1}{\sqrt{2s}\pi^{n+\frac{3}{2}}s^{n+2}} \int_0^\infty \int_{-\infty}^\infty \tau^{n+1} \prod_{j=1}^n \frac{a_j}{\sinh(a_j\tau)} \\ & \times \exp\left\{\frac{\tau(v + it)}{s} + (n - 2q)\tau - \frac{\tau^2}{2s} - \frac{(r + v)^2}{2s} - \frac{\tau}{s}\gamma(\tau, \zeta)\right\}\, d\tau\, dv. \end{aligned} \tag{6.35}$$

Calculating the integral with respect to v first we obtain:

$$\int_0^\infty \exp\left\{\frac{\tau v}{s} - \frac{\tau^2}{2s} - \frac{(r + v)^2}{2s}\right\}\, dv = \sqrt{2s}e^{-\frac{r}{s}\tau} \int_{\frac{r-\tau}{\sqrt{2s}}}^\infty e^{-\mu^2}\, d\mu. \tag{6.36}$$

(6.34) and (6.36) yield that

$$\begin{aligned} p_s(\mathbf{x}, r) \;=\; & \frac{1}{\pi^{n+\frac{3}{2}}s^{n+2}} \int_{-\infty}^\infty \tau^{n+1} \prod_{j=1}^n \frac{a_j}{\sinh(a_j\tau)} \\ & \times \exp\left\{\frac{i\tau t}{s} + (n - 2q)\tau - \frac{\tau}{s}\gamma(\tau, \zeta) - \frac{r}{s}\tau\right\} \left(\int_{\frac{r-\tau}{\sqrt{2s}}}^\infty e^{-\mu^2}\, d\mu\right)\, d\tau. \end{aligned} \tag{6.37}$$

Replacing τ by 2τ in (6.37), we can also write (6.37) in terms of the complex distance $g(\tau,\mathbf{x})$ and volume element $\nu(\tau)$:

$$(6.38)\qquad p_s(\mathbf{x},r)=\frac{4}{\pi^{n+\frac{3}{2}}s^{n+2}}\int_{-\infty}^{\infty}\tau^{n+1}\nu(\tau)e^{\left[(n-2q)\tau-\frac{2\tau}{s}g(\tau,\mathbf{x})-\frac{r}{s}\tau\right]}\left(\int_{\frac{r-2\tau}{\sqrt{2s}}}^{\infty}e^{-\mu^2}\,d\mu\right)d\tau,$$

where $g(\tau,\mathbf{x})$ and $\nu(\tau)$ are given in (4.12).

In conclusion, we summarize the solutions to (6.18) and (6.19) in the following theorem:

THEOREM 6.2. *$k_s^{\alpha}(\mathbf{x},r)$ is given by (6.8) and $p_s(\mathbf{x},r)$ is given by (6.38). Then*

$$F^{\nu}=\int_{\mathrm{H}_n\times\mathrm{R}^+}\left[k_s^{n-2q}(\mathbf{y}^{-1}\cdot\mathbf{x},r+\rho)-k_s^{n-2q}(\mathbf{y}^{-1}\cdot\mathbf{x},r-\rho)\right]f(\mathbf{y},\rho)\,d\mathbf{y}\,d\rho$$

solves the problem

$$\left(\frac{\partial}{\partial s}+\Box^{\nu}\right)F^{\tau}=0\quad\text{in } U^{\mathbf{n}}\quad\text{with } F^{\nu}|_{bU^{\mathbf{n}}}=0\quad\text{and}\quad\lim_{s\longrightarrow0^+}F^{\nu}=f.$$

$F^{\tau}=\int_{\mathrm{H}_n\times\mathrm{R}^+}[k_s^{n-2q}(\mathbf{y}^{-1}\cdot\mathbf{x},r+\rho)+k_s^{n-2q}(\mathbf{y}^{-1}\cdot\mathbf{x},r-\rho)+p_s(\mathbf{y}^{-1}\cdot\mathbf{x},r+\rho)]f(\mathbf{y},\rho)\,d\mathbf{y}\,d\rho$ *solves the problem*

$$\left(\frac{\partial}{\partial s}+\Box^{\tau}\right)F^{\tau}=0\quad\text{in } U^{\mathbf{n}}\quad\text{with } \bar{\mathbf{Z}}_{n+1}F^{\tau}|_{bU^{\mathbf{n}}}=0\quad\text{and}\quad\lim_{s\longrightarrow0^+}F^{\tau}=f.$$

7. **The Laplacian Operator in the Heisenberg Group.** Another operator which is closely related to the $\mathfrak{L}_\alpha$ is the Laplacian operator

$$P_\alpha\stackrel{\text{def}}{=}\mathfrak{L}_\alpha+\lambda\mathbf{T}^2=-\frac{1}{2}\sum_{j=1}^{n}(\bar{\mathbf{Z}}_j\mathbf{Z}_j+\mathbf{Z}_j\bar{\mathbf{Z}}_j)+i\alpha\mathbf{T}+\lambda\mathbf{T}^2\quad\text{with } \lambda>0,$$

which is closely related to the operators $\Box_+$ and $\Box_-$ introduced in [10]. We can find its fundamental solution easily by the Laguerre calculus. Since the calculation is similar to those in Section 4, we omit details here and only list the main steps.

As in Section 4, we start with the Fourier transform with respect to t and obtain

$$\tilde{P}_\alpha=-\frac{1}{2}\sum_{j=1}^{n}(\tilde{\bar{\mathbf{Z}}}_j\tilde{\mathbf{Z}}_j+\tilde{\mathbf{Z}}_j\tilde{\bar{\mathbf{Z}}}_j)-\alpha\tau-\lambda\tau^2.$$

Then, the Laguerre tensor of the convolution operator induced by P_α is

$$(7.1)\qquad \mathfrak{L}(\tilde{P}_\alpha)=|\tau|\left(\left[\sum_{j=1}^{n}(2k_j-1)a_j-\alpha\,\mathrm{sgn}(\tau)+\lambda|\tau|\right]\delta_{k_1}^{(p_1)}\cdots\delta_{k_n}^{(p_n)}\right),$$

and it is invertible as long as

$$|\alpha|\neq\sum_{j=1}^{n}(2k_j+1)a_j+\lambda|\tau|\quad\text{for } \mathbf{k}=(k_1,\ldots,k_n)\in\mathbb{N}^n.$$

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Under this condition, the inverse Laguerre tensor of (7.1) is

$$\mathfrak{L}(\tilde{P}_\alpha^{-1}) = |\tau|^{-1}\left(\left[\sum_{j=1}^{n}(2k_j-1)a_j - \alpha\,\mathrm{sgn}(\tau) + \lambda|\tau|\right]^{-1}\delta_{k_1}^{(p_1)}\cdots\delta_{k_n}^{(p_n)}\right). \tag{7.2}$$

We write (7.2) in terms of the Laguerre expansion as follows:

$$\tilde{P}_\alpha^{-1} = \frac{1}{|\tau|}\sum_{|\mathbf{k}|=0}^{\infty}\left[\sum_{j=1}^{n}(2k_j+1)a_j - \alpha\,\mathrm{sgn}(\tau) + \lambda|\tau|\right]^{-1}\prod_{j=1}^{n}a_j\tilde{\mathfrak{L}}_{k_j}^{(0)}(\sqrt{a_j}\zeta_j,\tau). \tag{7.3}$$

Similar to the previous calculation, one applies the generating function formula for the Laguerre polynomials and obtains

$$\tilde{P}_\alpha^{-1} = \frac{|\tau|^{n-1}}{\pi^n}\int_0^\infty e^{-[-\alpha\,\mathrm{sgn}(\tau)+\lambda|\tau|]s}\left[\prod_{j=1}^{n}\frac{a_j}{\sinh(a_js)}\right]e^{-|\tau|\gamma(s,\zeta)}\,ds.$$

The fundamental solution of P_α is the inverse Fourier transform of $\tilde{P}_\alpha^{-1}(\tau,\zeta)$ with respect to τ. We can also write the fundamental solution in terms of the volume element and complex distance introduced by Beals, Gaveau and Greiner [2].

$$F(\mathbf{x}) = \frac{\Gamma(n)}{2^n\pi^{n+1}}\int_{-\infty}^{\infty}\frac{e^{2\alpha s}\nu(s)\,ds}{\left[2\lambda s + g(s;\mathbf{x})\right]^n} \tag{7.4}$$

with

$$g(s;\mathbf{x}) = \sum_{j=1}^{n}a_j|\zeta_j|^2\coth(2a_js) - it \quad\text{and}\quad \nu(s) = \prod_{j=1}^{n}\frac{2a_j}{\sinh(2a_js)}.$$

The fundamental solution (7.4) also coincides with Stanton's results if we set $a_j = 1$ for all $j = 1, 2, \ldots, n$ and $\lambda = \frac{1}{2}$. See [23], Theorem 2.12 for the details.

We make some additional remarks about the assumption $a_j > 0$ for $j = 1, 2, \ldots, n$. We have expressed all the formulas in terms of the complex distance $g(s,\mathbf{x})$ and the volume element $\nu(s)$ (see Eq. (4.12)). But $g(s,\mathbf{x})$ and $\nu(s)$ depend on $|a_j|$ only. This implies that our formulas still hold no matter whether a_j, $j = 1, 2, \ldots, n$, are positive or negative. As to the assumption about α, we derived the fundamental solution $k^\alpha(\mathbf{x}, r)$ of $\Box_\alpha$ under the condition:

$$|\alpha| < \sum_{j=1}^{n}(2k_j+1)a_j + \frac{1}{2}\left(|\tau| + \frac{\xi^2}{|\tau|}\right) \quad \text{for } \mathbf{k}\in\mathbb{N}^n, \tau\neq 0.$$

We can extend k^α by analytic continuation to the case:

$$|\alpha| \neq \sum_{j=1}^{n}(2k_j+1)a_j + \frac{1}{2}\left(|\tau| + \frac{\xi^2}{|\tau|}\right) \quad \text{for } \mathbf{k}\in\mathbb{N}^n, \tau\neq 0.$$

This is the necessary and sufficient condition for the Laguerre tensor $\mathfrak{L}(\hat{\Box}_\alpha)$ to be invertible.

ACKNOWLEDGMENTS. This paper was part of my Ph.D thesis at the University of Toronto. I thank my thesis advisor, Peter C. Greiner, for his guidance.

https://doi.org/10.4153/CJM-1997-064-4 Published online by Cambridge University Press

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