Canad. Math. Bull. Vol. 62 (2), 2019 pp. 231–241
http://dx.doi.org/10.4153/S0008439518000024

Newton Complementary Duals of f-Ideals

Samuel Budd and Adam Van Tuyl

Abstract. A square-free monomial ideal I of $k[x_1, \dots, x_n]$ is said to be an f-ideal if the facet complex and non-face complex associated with I have the same f-vector. We show that I is an f-ideal if and only if its Newton complementary dual $\hat{I}$ is also an f-ideal. Because of this duality, previous results about some classes of f-ideals can be extended to a much larger class of f-ideals. An interesting by-product of our work is an alternative formulation of the Kruskal–Katona theorem for f-vectors of simplicial complexes.

1 Introduction

Let I be a square-free monomial ideal of $R = k[x_1, \dots, x_n]$ where k is a field. Associated with any such ideal are two simplicial complexes. The *non-face complex*, denoted $\delta_{\mathcal{N}}(I)$, (also called the *Stanley–Reisner complex*) is the simplicial complex whose faces are in one-to-one correspondence with the square-free monomials not in I. Faridi [7] introduced a second complex, the *facet complex* $\delta_{\mathcal{F}}(I)$, where the generators of I define the facets of the simplicial complex (see the next section for complete definitions). In general, the two simplicial complexes, $\delta_{\mathcal{N}}(I)$ and $\delta_{\mathcal{F}}(I)$, can be very different. For example, the two complexes can have different dimensions; as a consequence, the f-*vectors* of $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$, which enumerate all the faces of a given dimension, can be quite different.

If I is a square-free monomial ideal with the property that the f-vectors of $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ are the same, then I is called an f-*ideal*. The notion of an f-ideal was first introduced by Abbasi, Ahmad, Anwar, and Baig [1]. It is natural to ask if it is possible to classify all the square-free monomial ideals that are f-ideals. Abbasi *et al.* classified all the f-ideals generated in degree two. This result was later generalized by Anwar, Mahmood, Binyamin, and Zafar [3] who classified all the f-ideals I that are unmixed and generated in degree $d \geq 2$. An alternative proof for this result was found by Guo and Wu [10]. Gu, Wu, and Liu [9] later removed the unmixed restriction of [3]. Other work related to f-ideals includes the papers [14, 15].

The purpose of this note is to show that the property of being an f-ideal is preserved after taking the Newton complementary dual of I. The notion of a Newton complementary dual was first introduced in a more general context by Costa and Simis [5] in their study of Cremona maps; additional properties were developed by

Received by the editors April 2, 2018; revised August 8, 2018.
Published online on Cambridge Core April 11, 2019.
Parts of this paper appeared in the first author's MSc project [4]. The second author's research was supported in part by NSERC Discovery Grant 2014-03898.
AMS subject classification: **05E45,** 05E40, 13F55.
Keywords: f-ideals, facet ideal, Stanley–Reisner correspondence, f-vectors, Newton complementary dual, Kruskal–Katona.

Dória and Simis [6]. Ansaldi, Lin, and Shin [2] later investigated the Newton complementary duals of monomial ideals. Using the definition of [2], the *Newton complementary dual* of a square-free monomial ideal I is

$$\hat{I} = \left\langle \frac{x_1 \cdots x_n}{m} \;\middle|\; m \in \mathcal{G}(I) \right\rangle,$$

where $\mathcal{G}(I)$ denotes the minimal generators of I. With this notation, we prove the following theorem.

Theorem 1.1 (Theorem 4.1) *Let $I \subseteq R$ be a square-free monomial ideal. Then I is an f-ideal if and only if $\hat{I}$ is an f-ideal.*

Our proof involves relating the f-vectors of the four simplicial complexes $\delta_{\mathcal{F}}(I)$, $\delta_{\mathcal{N}}(I)$, $\delta_{\mathcal{F}}(\hat{I})$, and $\delta_{\mathcal{N}}(\hat{I})$. An interesting by-product of this discussion is to give a reformulation of the celebrated Kruskal–Katona theorem (see [12, 13]) which classifies what vectors can be the f-vector of a simplicial complex (see Theorem 3.7).

A consequence of Theorem 1.1 is that f-ideals come in "pairs". Note that when I is an f-ideal generated in degree d, $\hat{I}$ gives us an f-ideal generated in degree $n - d$. We can use the classification of [1] of f-ideals generated in degree two to also give us a classification of f-ideals generated in degree $n-2$. This corollary and others are given as applications of Theorem 1.1.

Our paper uses the following outline. In Section 2 we provide all the necessary background results. In Section 3, we introduce the Newton complementary dual of a square-free monomial ideal, and we study how the f-vector behaves under this duality. In Section 4 we prove Theorem 1.1 and devote the rest of the section to applications.

2 Background

In this section, we review the required background results.

Let $X = \{x_1, \dots, x_n\}$ be a set of vertices. A *simplicial complex* Δ on X is a subset of the power set of X that satisfies the following:

(i) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$;
(ii) $\{x_i\} \in \Delta$ for $i = 1, \dots, n$.

An element $F \in \Delta$ is called a *face*; maximal faces with respect to inclusion are called *facets*. If $F_1, \dots, F_r$ are the facets of Δ, then we write $\Delta = \langle F_1, \dots, F_r \rangle$.

For any face $F \in \Delta$, the *dimension* of F is given by $\dim(F) = |F| - 1$. Note that $\varnothing \in \Delta$ and $\dim(\varnothing) = -1$. The *dimension* of Δ is given by $\dim(\Delta) = \max\{\dim(F) \mid F \in \Delta\}$. If $d = \dim(\Delta)$, then the *f-vector* of Δ is the $d+2$ tuple

$$f(\Delta) = (f_{-1}, f_0, f_1, \dots, f_d),$$

where f_i is number of faces of dimension i in Δ. We write $f_i(\Delta)$ if we need to specify the simplicial complex.

Suppose that I is a square-free monomial ideal of $R = k[x_1, \dots, x_n]$ with k a field. We use $\mathcal{G}(I)$ to denote the unique set of minimal generators of I. If we identify the variables of R with the vertices X, we can associate with I two simplicial complexes.

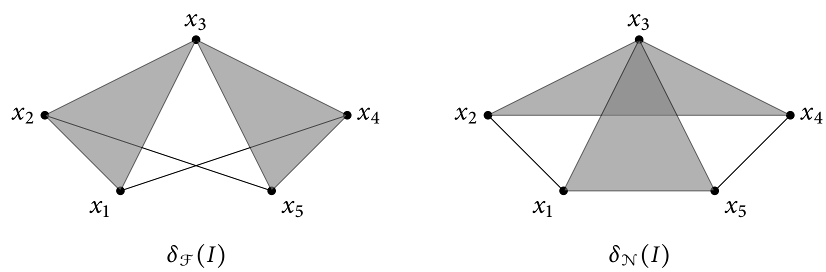

Figure 1: Facet and non-face complexes of $I = \langle x_1x_4, x_2x_5, x_1x_2x_3, x_3x_4x_5\rangle$.

The *non-face complex* (or *Stanley–Reisner complex*) is the simplicial complex

$$\delta_{\mathcal{N}}(I) = \left\{ \{x_{i_1}, \dots, x_{i_j}\} \subseteq X \mid x_{i_1} \cdots x_{i_j} \notin I \right\}.$$

In other words, the faces of $\delta_{\mathcal{N}}(I)$ are in one-to-one correspondence with the square-free monomials of R not in the ideal I. The *facet complex* is the simplicial complex

$$\delta_{\mathcal{F}}(I) = \left\langle \{x_{i_1}, \dots, x_{i_j}\} \subseteq X \mid x_{i_1} \cdots x_{i_j} \in \mathcal{G}(I) \right\rangle.$$

The facets of $\delta_{\mathcal{F}}(I)$ are in one-to-one correspondence with the minimal generator of I.

In general, the two simplicial complexes, $\delta_{\mathcal{N}}(I)$ and $\delta_{\mathcal{F}}(I)$, constructed from I are very different. In this note, we are interested in the following family of monomial ideals.

Definition 2.1 A square-free monomial ideal I is an *f-ideal* if $f(\delta_{\mathcal{N}}(I)) = f(\delta_{\mathcal{F}}(I))$.

Example 2.2 We illustrate the above ideas with the following example. Let $I = \langle x_1x_4, x_2x_5, x_1x_2x_3, x_3x_4x_5\rangle \subseteq R = k[x_1, x_2, x_3, x_4, x_5]$ be a square-free monomial ideal. Then Figure 1 shows both the facet and non-face complexes that are associated with I.

From Figure 1, one can see that $f(\delta_{\mathcal{F}}(I)) = f(\delta_{\mathcal{N}}(I)) = (1, 5, 8, 2)$, and therefore I is an f-ideal. We note that I in this example is generated by monomials of different degrees. In most of the other papers on this topic (*e.g.*, [1,3,9,10,14,15]) the focus has been on *equigenerated ideals*, *i.e.*, ideals where are all generators have the same degree.

Remark 2.3 It is important to note that $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ may be simplicial complexes on different sets of vertices, and, in particular, one must pay attention to the ambient ring.

For example, consider $I = \langle x_1, x_2x_3, x_2x_4, x_3x_4\rangle \subseteq k[x_1, \dots, x_5]$. For this ideal, $\delta_{\mathcal{F}}(I)$ is a simplicial complex on $\{x_1, x_2, x_3, x_4\}$ with facets $\{\{x_1\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}\}$. So $f(\delta_{\mathcal{F}}(I)) = (1, 4, 3)$. The vertices of $\delta_{\mathcal{N}}(I)$ are $\{x_1, x_2, x_3, x_4, x_5\} \backslash \{x_1\}$. Its facets are $\{\{x_2, x_5\}, \{x_3, x_5\}, \{x_4, x_5\}\}$. From this description, we see that I is in fact an f-ideal.

Note, however, that if I is an f-ideal, and if every generator of I has degree at least two, then $\delta_{\mathcal{F}}(I)$ and $\delta_{\mathcal{N}}(I)$ must be simplicial complexes on the vertex set $\{x_1, \dots, x_n\}$. To see why, since every generator of I has degree ≥ 2, this implies that

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$\{x_i\} \in \delta_{\mathcal{N}}(I)$ for all $i = 1, \dots, n$. So, $n = f_0(\delta_{\mathcal{N}}(I)) = f_0(\delta_{\mathcal{F}}(I))$, that is, $\delta_{\mathcal{F}}(I)$ must also have n vertices.

The above observation implies that the ideal $I = \langle x_1 x_2 \rangle \subseteq k[x_1, x_2, x_3]$ cannot be an f-ideal, since it is generated by a monomial of degree two, but $\delta_{\mathcal{F}}(I)$ is a simplicial complex on $\{x_1, x_2\}$, but the vertices of $\delta_{\mathcal{N}}(I)$ are $\{x_1, x_2, x_3\}$.

Remark 2.4 Although the f-vector counts faces of a simplicial complex, we can reinterpret the f_j's as counting square-free monomials of a fixed degree. In particular,

$$f_j(\delta_{\mathcal{N}}(I)) = \#\left\{ m \in R_{j+1} \;\middle|\; \begin{array}{c} m \text{ is a square-free monomial of degree} \\ j+1 \text{ and } m \notin I_{j+1} \end{array} \right\}.$$

On the other hand, for the f-vector of $\delta_{\mathcal{F}}(I)$ we have

$$f_j(\delta_{\mathcal{F}}(I)) = \#\left\{ m \in R_{j+1} \;\middle|\; \begin{array}{c} m \text{ is a square-free monomial of degree } j+1 \\ \text{that divides some } p \in \mathcal{G}(I) \end{array} \right\}.$$

Here, R_t, respectively I_t, denotes the degree t homogeneous elements of R, respectively I.

We refine Remark 2.4 by introducing a partition of the set of square-free monomials of degree d. This partition will be useful in Section 4. For each integer $d \geq 0$, let $M_d \subseteq R_d$ denote the set of square-free monomial of degree d in R_d. Given a square-free monomial ideal I with generating set $\mathcal{G}(I)$, set

$$\begin{aligned} A_d(I) &= \{m \in M_d \mid m \notin I_d \text{ and } m \text{ does not divide any element of } \mathcal{G}(I)\}, \\ B_d(I) &= \{m \in M_d \mid m \notin I_d \text{ and } m \text{ divides some element of } \mathcal{G}(I)\}, \\ C_d(I) &= \{m \in M_d \mid m \in \mathcal{G}(I)\}, \\ D_d(I) &= \{m \in M_d \mid m \in I_d \backslash \mathcal{G}(I)\}. \end{aligned}$$

So, for any square-free monomial ideal I and integer $d \geq 0$, we have the partition

$$M_d = A_d(I) \sqcup B_d(I) \sqcup C_d(I) \sqcup D_d(I). \tag{2.1}$$

Using this notation, we have the following characterization of f-ideals.

Lemma 2.5 *Let $I \subseteq k[x_1, \dots, x_n]$ be a square-free monomial ideal. Then I is an f-ideal if and only if $|A_d(I)| = |C_d(I)|$ for all $0 \leq d \leq n$.*

Proof Note that Remark 2.4 implies that

$$\begin{aligned} f_j(\delta_{\mathcal{N}}(I)) &= |A_{j+1}(I)| + |B_{j+1}(I)| \quad \text{for all } j \geq -1, \\ f_j(\delta_{\mathcal{F}}(I)) &= |B_{j+1}(I)| + |C_{j+1}(I)| \quad \text{for all } j \geq -1. \end{aligned}$$

The conclusion now follows, since $f_j(\delta_{\mathcal{F}}(I)) = f_j(\delta_{\mathcal{N}}(I))$ for all $-1 \leq j \leq n-1$ if and only if $|A_d(I)| = |C_d(I)|$ for all $0 \leq d \leq n$. ∎

3 The Newton Complementary Dual and f-Vectors

We introduce the generalized Newton complementary dual of a monomial ideal as defined in [2] (based on [5]). We then show how the f-vector behaves under this operation.

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Definition 3.1 Let $I \subseteq R = k[x_1, \dots, x_n]$ be a monomial ideal with $\mathcal{G}(I) = \{m_1, \dots, m_p\}$. Suppose that $m_i = x_1^{\alpha_{i,1}} x_2^{\alpha_{i,2}} \cdots x_n^{\alpha_{i,n}}$ for all $i = 1, \dots, p$. Let $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ be a vector such that $\beta_i \geq \alpha_{k,l}$ for all $l = 1, \dots, n$ and $k = 1, \dots, p$. The *generalized Newton complementary dual of I* determined by β is the ideal

$$\hat{I}^{[\beta]} = \left\langle \frac{x^\beta}{m} \;\middle|\; m \in \mathcal{G}(I) \right\rangle = \left\langle \frac{x^\beta}{m_1}, \frac{x^\beta}{m_2}, \dots, \frac{x^\beta}{m_p} \right\rangle \quad \text{where } x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}.$$

Remark 3.2 If $I \subseteq R = k[x_1, \dots, x_n]$ is a square-free monomial ideal, then one can take $\beta = (1, \dots, 1) = \mathbf{1}$, *i.e.*, $x^\beta = x_1 \cdots x_n$. For simplicity, we denote $\hat{I}^{[1]}$ by $\hat{I}$ and call it the *complementary dual* of I. Note that we have $\hat{\hat{I}} = I$.

Example 3.3 We return to the ideal I of Example 2.2. For this ideal we have

$$\hat{I} = \left\langle \frac{x_1 \cdots x_5}{x_1 x_4}, \frac{x_1 \cdots x_5}{x_2 x_5}, \frac{x_1 \cdots x_5}{x_1 x_2 x_3}, \frac{x_1 \cdots x_5}{x_3 x_4 x_5} \right\rangle = \langle x_2 x_3 x_5, x_1 x_3 x_4, x_4 x_5, x_1 x_2 \rangle.$$

The next lemma is key to understanding how the f-vector behaves under the duality.

Lemma 3.4 *Let $I \subseteq R = k[x_1, \dots, x_n]$ be a square-free monomial ideal. For all integers $j = -1, \dots, n-1$, there is a bijection*

$$\{m \in R_{j+1} \mid m \text{ a square-free monomial that divides some } p \in \mathcal{G}(I)\}$$
$$\longleftrightarrow \{m \in \hat{I}_{n-j-1} \mid m \text{ a square-free monomial}\}.$$

Proof Fix a $j \in \{-1, \dots, n-1\}$, let A denote the first set, and let B denote the second set. We claim that the map $\varphi : A \to B$ given by

$$\varphi(m) = \frac{x_1 x_2 \cdots x_n}{m}$$

gives the desired bijection. This map is defined, because if $m \in A$, there is a generator $p \in \mathcal{G}(I)$ such that $m | p$. But that then means that $\frac{x_1 \cdots x_n}{p}$ divides $\varphi(m) = \frac{x_1 \cdots x_n}{m}$, and consequently, $\varphi(m) \in \hat{I}$. Moreover, since $\deg(m) = j+1$, we have $\deg(\varphi(m)) = n - j - 1$. Finally, since m is a square-free monomial, so is $\varphi(m)$.

It is immediate that the map is injective. For surjectivity, let $m \in B$. It suffices to show that the square-free monomial $m' = \frac{x_1 \cdots x_n}{m} \in A$, since $\varphi(m') = m$. By our construction of m' it follows that $\deg(m') = j + 1$. Also, because $m \in B$, there is some $p \in \mathcal{G}(I)$ such that $\frac{x_1 \cdots x_n}{p}$ divides m. But this then means that m' divides p, *i.e.*, $m' \in A$. ∎

Remark 3.5 Using the notation introduced before Lemma 2.5, Lemma 3.4 gives a bijection between $B_{j+1}(I) \sqcup C_{j+1}(I)$ and $C_{n-j-1}(\hat{I}) \sqcup D_{n-j-1}(\hat{I})$ for all $j = -1, \dots, n-1$.

Lemma 3.4 can be used to relate the f-vectors of $\delta_{\mathcal{N}}(I)$, $\delta_{\mathcal{F}}(I)$, $\delta_{\mathcal{N}}(\hat{I})$, and $\delta_{\mathcal{F}}(\hat{I})$.

Corollary 3.6 *Let $I \subseteq R = k[x_1, \dots, x_n]$ be a square-free monomial ideal.*

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(i) *If* $f(\delta_{\mathcal{F}}(I)) = (f_{-1}, f_0, \ldots, f_d)$, *then*

$$f(\delta_{\mathcal{N}}(\hat{I})) = \left(\binom{n}{0} - f_{n-1}, \ldots, \binom{n}{i} - f_{n-i-1}, \ldots, \binom{n}{n-1} - f_0, \binom{n}{n} - f_{-1}\right).$$

(ii) *If* $f(\delta_{\mathcal{N}}(I)) = (f_{-1}, f_0, \ldots, f_d)$, *then*

$$f(\delta_{\mathcal{F}}(\hat{I})) = \left(\binom{n}{0} - f_{n-1}, \ldots, \binom{n}{i} - f_{n-i-1}, \ldots, \binom{n}{n-1} - f_0, \binom{n}{n} - f_{-1}\right).$$

In both cases, $f_i = 0$ *if* $i > d$.

Proof (i) Fix some $j \in \{-1, 0, \ldots, n-1\}$. By Remark 2.4 and Lemma 3.4, we have

$$\begin{aligned} f_j(\delta_{\mathcal{F}}(I)) &= \#\{m \in R_{j+1} \mid m \text{ a square-free monomial that divides some } p \in \mathcal{G}(I)\} \\ &= \#\{m \in \hat{I}_{n-j-1} \mid m \text{ a square-free monomial}\} \\ &= \#\{m \in R_{n-j-1} \mid m \text{ a square-free monomial}\} \\ &\quad - \#\{m \notin \hat{I}_{n-j-1} \mid m \text{ a square-free monomial}\} \\ &= \binom{n}{n-j-1} - f_{n-j-2}(\delta_{\mathcal{N}}(\hat{I})). \end{aligned}$$

Rearranging, and letting $l = n - j - 2$ gives

$$f_l(\delta_{\mathcal{N}}(\hat{I})) = \binom{n}{l+1} - f_{n-l-2}(\delta_{\mathcal{F}}(I)) \quad \text{for } l = -1, \ldots, n-1,$$

as desired.

(ii) The proof is similar to (i). Indeed, if we replace I with $\hat{I}$ we show that

$$f_j(\delta_{\mathcal{F}}(\hat{I})) = \binom{n}{n-j-1} - f_{n-j-2}(\delta_{\mathcal{N}}(I)) = \binom{n}{j+1} - f_{n-j-2}(\delta_{\mathcal{N}}(I))$$

for all $j \in \{-1, 0, \ldots, n-1\}$. ∎

We end this section with some consequences related to the Kruskal–Katona theorem; although we do not use this result in the sequel, we feel it is of independent interest.

We follow the notation of Herzog–Hibi [11, Section 6.4]. The *Macaulay expansion* of a with respect to j is the expansion

$$a = \binom{a_j}{j} + \binom{a_{j-1}}{j-1} + \cdots + \binom{a_k}{k},$$

where $a_j > a_{j-1} > \cdots > a_k \geq k \geq 1$. This expansion is unique (see [11, Lemma 6.3.4]). For a fixed a and j, we use the Macaulay expansion of a with respect to j to define

$$a^{(j)} = \binom{a_j}{j+1} + \binom{a_{j-1}}{j} + \cdots + \binom{a_k}{k+1}.$$

Kruskal–Katona's theorem [12,13] then classifies what vectors can be the f-vector of a simplicial complex using the Macaulay expansion operation. This equivalence, as well as two new equivalent statements that use the complementary dual, are given below.

Theorem 3.7 *Let $(f_{-1}, f_0, \ldots, f_d) \in \mathbb{N}_+^{d+2}$ with $f_{-1} = 1$. Then the following are equivalent:*

(i) $(f_{-1}, f_0, f_1, \ldots, f_d)$ *is the f-vector of a simplicial complex on $n = f_0$ vertices.*
(ii) $f_t \le f_{t-1}^{(t)}$ *for all* $1 \le t \le d$.
(iii)

$$\left(\binom{n}{0} - f_{n-1}, \ldots, \binom{n}{i} - f_{n-i-1}, \ldots, \binom{n}{n-1} - f_0, \binom{n}{n} - f_{-1}\right)$$

is the f-vector of a simplicial complex on $\binom{n}{1} - f_{n-2}$ *vertices (where $f_i = 0$ if $i > d$).*
(iv)

$$\binom{n}{t+1} - \left[\binom{n}{t+2} - f_{t+1}\right]^{(n-t-2)} \le f_t \quad \textit{for all } 0 \le t \le d-1.$$

Proof (i) $\Leftrightarrow$ (ii). This equivalence is the Kruskal–Katona theorem (see [12,13]).

(i) $\Leftrightarrow$ (iii). This equivalence follows from Corollary 3.6 and the Kruskal–Katona equivalence of (i) $\Leftrightarrow$ (ii). In particular, one lets I be the square-free monomial ideal with $f(\delta_{\mathcal{N}}(I)) = (f_{-1}, f_0, \ldots, f_d)$, and then one uses Corollary 3.6 to show that (iii) is a valid f-vector. The duality of I and $\hat{I}$ is used to show the reverse direction.

(iii) $\Leftrightarrow$ (iv). Corollary 3.6 and the equivalence of (i) $\Leftrightarrow$ (ii) implies that the vector of (iii) is an f-vector of a simplicial complex if and only if, for each $0 \le i \le n-2$,

$$\binom{n}{i+2} - f_{n-i-3} \le \left[\binom{n}{i+1} - f_{n-i-2}\right]^{(i+1)}.$$

The result now follows if we take $i = n - 3 - t$ and rearrange the above equation. ∎

Discussion 3.8 Although this material is not required for our paper, it is prudent to make some observations about the Alexander dual. Recall that for any simplicial complex Δ on a vertex set X, the *Alexander dual* of Δ is the simplicial complex on X given by

$$\Delta^\vee = \{F \subseteq X \mid X \backslash F \notin \Delta\}.$$

It is well known (for example, see [11, Corollary 1.5.5]) that if $\Delta = \langle F_1, \ldots, F_s \rangle$, then $\mathcal{N}(\Delta^\vee)$, the non-face ideal of $\Delta^\vee$ (*i.e.*, the ideal generated by the square-free monomials $x_{i_1} \cdots x_{i_j}$ where $\{x_{i_1}, \ldots, x_{i,j}\} \notin \Delta$) is given by

$$\mathcal{N}(\Delta^\vee) = \langle m_{F_1^c}, \ldots, m_{F_s^c} \rangle,$$

where $m_{F_i^c} = \prod_{x \in F_i^c} x$ with $F_i^c = X \backslash F_i$. But we can also write $m_{F_i^c} = (\prod_{x \in X} x)/m_{F_i} = \frac{x_1 \cdots x_n}{m_{F_i}}$. Now tracing through the definitions, if I is square-free monomial ideal, then

$$\hat{I} = \mathcal{N}\big((\delta_{\mathcal{F}}(I))^\vee \big);$$

i.e., the complementary dual of I is the non-face ideal of the Alexander dual of the facet complex of I. This, in turn, implies that $\delta_{\mathcal{N}}(\hat{I}) = \delta_{\mathcal{N}}(\mathcal{N}((\delta_{\mathcal{F}}(I))^\vee) = (\delta_{\mathcal{F}}(I))^\vee$.

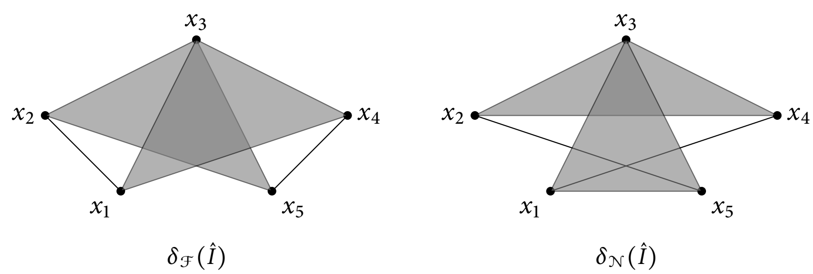

Figure 2: Facet and non-face complexes of $\hat{I} = \langle x_1x_2, x_4x_5, x_1x_3x_4, x_2x_3x_5\rangle$.

4 f-Ideals and Applications

We use the tools of the previous sections to prove our main theorem about f-ideals and to deduce some new consequences about this class of ideals. Our main theorem is an immediate application of Corollary 3.6.

Theorem 4.1 *Let I be a square-free monomial ideal of $R = k[x_1, \dots, x_n]$. Then I is an f-ideal if and only if $\hat{I}$ is an f-ideal.*

Proof Suppose $f(\delta_{\mathcal{N}}(I)) = f(\delta_{\mathcal{F}}(I)) = (f_{-1}, f_0, \dots, f_d)$. Then by Corollary 3.6, both $\delta_{\mathcal{N}}(\hat{I})$ and $\delta_{\mathcal{F}}(\hat{I})$ will have the same f-vector. For the reverse direction, simply replace I with $\hat{I}$ and use the same corollary. ∎

Remark 4.2 Theorem 4.1 was first proved by the first author (see [4]) using the characterization of f-ideals of [9]. The proof presented here avoids the machinery of [9].

Example 4.3 In Example 3.3 we computed the ideal $\hat{I}$ of the ideal I in Example 2.2. By Theorem 4.1, the ideal $\hat{I}$ is an f-ideal. Indeed, the simplicial complexes $\delta_{\mathcal{N}}(\hat{I})$ and $\delta_{\mathcal{N}}(\hat{I})$ are given in Figure 2, and both simplicial complexes have f-vector $(1, 5, 8, 2)$. Note that the f-vector of $\delta_{\mathcal{N}}(I)$ and $\delta_{\mathcal{F}}(I)$ was $(1, 5, 8, 2)$, so by Corollary 3.6,

$$\begin{aligned} f(\delta_{\mathcal{F}}(\hat{I})) = f(\delta_{\mathcal{N}}(\hat{I})) &= \left(\binom{5}{0} - 0, \binom{5}{1} - 0, \binom{5}{2} - 2, \binom{5}{3} - 8, \binom{5}{4} - 5, \binom{5}{5} - 1\right) \\ &= (1, 5, 8, 2). \end{aligned}$$

Theorem 4.1 implies that f-ideals come in "pairs". This observation allows us to extend many known results about f-ideals to their complementary duals. For example, we can now classify the f-ideals that are equigenerated in degree $n-2$.

Theorem 4.4 *Let I be a square-free monomial of $k[x_1, \dots, x_n]$ equigenerated in degree $n-2$. Then the following are equivalent:*

(i) *I is an f-ideal.*
(ii) *$\hat{I}$ is an f-ideal.*
(iii) *$\hat{I}$ is an unmixed ideal of height $n-2$ (i.e., all of the associated primes of I have height $n-2$) with $p = \frac{1}{2}\binom{n}{2}$.*

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Proof The equivalence of (i) and (ii) is Theorem 4.1. Because the ideal $\hat{I}$ is equigenerated in degree two, the equivalence of (ii) and (iii) is [1, Theorem 3.5]. ∎

Following Guo, Wu, and Liu [9], let $V(n,d)$ denote the set of f-ideals of $k[x_1,\dots,x_n]$ that are equigenerated in degree d. We can now extend Guo, *et al.*'s results.

Theorem 4.5 *Using the notation above, we have the following.*

(i) *For all* $1 \le d \le n-1$, $|V(n,d)| = |V(n,n-d)|$.
(ii) *If* $n \ne 2$, *then* $V(n,1) = V(n,n-1) = \varnothing$. *If* $n = 2$, *then* $|V(2,1)| = 2$.
(iii) $V(n,n-2) \ne \varnothing$ *if and only if* $n \equiv 0$ *or* $1 \pmod 4$.

Proof (i) By Theorem 4.1, the complementary dual gives a bijection between the sets $V(n,d)$ and $V(n,n-d)$.

(ii) Suppose that $I \in V(n,1)$, *i.e.*, I is an f-ideal generated by a subset of the variables. So the facets of $\delta_{\mathcal{F}}(I)$ are vertices, while $\delta_{\mathcal{N}}(I)$ is a simplex. Then $f_0(\delta_{\mathcal{F}}(I))$, the number of variables that generate I, must be the same as $f_0(\delta_{\mathcal{N}}(I))$, the number of variables not in I. This implies that n cannot be odd. Furthermore, if $n \ge 4$ is even, then $\dim \delta_{\mathcal{F}}(I) = 0$, but $\dim \delta_{\mathcal{N}}(I) = \frac{n}{2} - 1 \ge 1$, contradicting the fact that I is an f-ideal.

When $n = 2$, $I_1 = \langle x_1 \rangle$ and $I_2 = \langle x_2 \rangle$ are f-ideals of $k[x_1,x_2]$.

(iii) By (i), $|V(n,n-2)| \ne 0$ if and only if $|V(n,2)| \ne 0$. Now [9, Proposition 3.4] shows that $V(n,2) \ne \varnothing$ if and only if $n \equiv 0, 1 \pmod 4$. ∎

Remark 4.6 [9, Proposition 4.10] gives an explicit formula for $|V(n,2)|$, which we will not present here. So by Theorem 4.5(i), there is an explicit formula for $|V(n,n-2)|$.

We now explore some necessary conditions on the f-vector of $\delta_{\mathcal{N}}(I)$ (equivalently, $\delta_{\mathcal{F}}(I)$) when I is an f-ideal. We also give a necessary condition on the generators of an f-ideal. As we shall see, Theorem 4.1 plays a role in some of our proofs.

We first recall some notation. If $I \subseteq R$ is a square-free monomial ideal, then we let

$$\alpha(I) = \min\{\deg(m) \mid m \in \mathcal{G}(I)\} \quad \text{and} \quad \omega(I) = \max\{\deg(m \mid m \in \mathcal{G}(I)\}.$$

We present some conditions on the f-vector; some of these results were known.

Theorem 4.7 *Suppose that I is an f-ideal in $R = k[x_1,\dots,x_n]$ with associated f-vector $f = f(\delta_{\mathcal{F}}(I)) = f(\delta_{\mathcal{N}}(I))$. Let $\alpha = \alpha(I)$ and $\omega = \omega(I)$. Then*

(i) $f_i = \binom{n}{i+1}$ *for* $i = 0,\dots,\alpha-2$;
(ii) $f_{\alpha-1} \ge \frac{1}{2}\binom{n}{\alpha}$;
(iii) $f_{\omega-1} \le \frac{1}{2}\binom{n}{\omega}$;
(iv) *if* $\alpha = \omega$ (i.e., *I is equigenerated*), *then* $f_{\alpha-1} = \frac{1}{2}\binom{n}{\alpha}$;
(v) $\dim \delta_{\mathcal{F}}(I) = \dim \delta_{\mathcal{N}}(I) = \omega - 1 \le n-2$.

Proof (i) See [3, Lemma 3.7].

(ii) If I is generated by monomials of degree α or larger, then (2.1) becomes

$$M_\alpha = A_\alpha(I) \sqcup B_\alpha(I) \sqcup C_\alpha(I)$$

since $D_\alpha(I) = \varnothing$. Suppose $f_{\alpha-1} < \frac{1}{2}\binom{n}{\alpha}$. Because $f_{\alpha-1}(\delta_{\mathcal{N}}(I)) = |A_\alpha(I)| + |B_\alpha(I)|$, we have $|C_\alpha(I)| > \frac{1}{2}\binom{n}{\alpha}$. But since I is an f-ideal, by Lemma 2.5 we have

$$\frac{1}{2}\binom{n}{\alpha} > f_{\alpha-1}(\delta_{\mathcal{N}}(I)) \geq |A_\alpha(I)| = |C_\alpha(I)| > \frac{1}{2}\binom{n}{\alpha}.$$

We now have the desired contradiction.

(iii) Suppose that $f_{\omega-1}(\delta_{\mathcal{F}}(I)) > \frac{1}{2}\binom{n}{\omega} = \frac{1}{2}\binom{n}{n-\omega}$. Since $\omega = \omega(I)$, we must have that $\alpha(\hat{I}) = n - \omega$. Since $\hat{I}$ is also an f-ideal by Theorem 4.1, (ii) implies that $f_{n-\omega-1}(\delta_{\mathcal{F}}(\hat{I})) \geq \frac{1}{2}\binom{n}{n-\omega}$. But by Corollary 3.6, and since $\hat{I}$ is an f-ideal,

$$f_{n-\omega-1}\big(\delta_{\mathcal{F}}(\hat{I})\big) = f_{n-\omega-1}\big(\delta_{\mathcal{N}}(\hat{I})\big) = \binom{n}{n-\omega} - f_{\omega-1}(\delta_{\mathcal{F}}(I)) < \frac{1}{2}\binom{n}{n-\omega}.$$

This gives the desired contradiction.

(iv) We simply combine the inequalities of (ii) and (iii).

(v) Since $\omega = \omega(I)$, there is a generator m of I of degree ω, and furthermore, every other generator has smaller degree. So the facet of $\delta_{\mathcal{F}}(I)$ of largest dimension has dimension $\omega - 1$. Since I is an f-ideal, this also forces $\delta_{\mathcal{N}}(I)$ to have a facet of dimension of $\omega - 1$. Note that $\omega(I) \leq n - 1$, since no f-ideal has $x_1 \cdots x_n$ as a generator. ∎

Our final result shows that if I is not an equigenerated f-ideal, then in some cases we can deduce the existence of generators of other degrees.

Theorem 4.8 *Suppose that I is an f-ideal of $k[x_1,\ldots,x_n]$ with $\alpha = \alpha(I) < \omega(I) = \omega$, and let $f = f(\delta_{\mathcal{F}}(I)) = f(\delta_{\mathcal{N}}(I))$.*

(i) *If $f_{\alpha-1} > \binom{n}{\alpha} - n + \alpha$, then I also has a generator of degree $\alpha + 1$.*

(ii) *If $f_{\omega-1} < \omega$, then I also has a generator of degree $\omega - 1$.*

Proof To prove (i), it is enough to prove (ii) and apply Theorem 4.1. Indeed, suppose that $f_{\alpha-1} > \binom{n}{\alpha} - n + \alpha$. Then the ideal $\hat{I}$ is an f-ideal with $\omega(\hat{I}) = n - \alpha$ and

$$f_{\omega(\hat{I})-1}\big(\delta_{\mathcal{F}}(\hat{I})\big) = \binom{n}{n-\alpha} - f_{\alpha-1} < n - \alpha = \omega(\hat{I}).$$

So by (ii) the ideal $\hat{I}$ will have a generator of degree $\omega(\hat{I}) - 1$, which implies that I has a generator of degree $\alpha + 1$.

(ii) Note that if $\alpha = \omega - 1$, then the conclusion immediately follows. So suppose that $\alpha < \omega - 1$. We use the partition (2.1). Since I is generated in degrees $\leq \omega$, we have $B_\omega(I) = \varnothing$. It then follows by Lemma 2.5 and Remark 2.4 that

$$f_{\omega-1} = |A_\omega(I)| = |C_\omega(I)| < \omega.$$

Now suppose that I has no generators of degree $\omega - 1$. So $|C_{\omega-1}(I)| = 0$, and consequently, $|A_{\omega-1}(I)| = 0$, because I is an f-ideal. Because $\alpha < \omega - 1$, we have $D_{\omega-1}(I) \neq \varnothing$. Then, again by Lemma 2.5 and Remark 2.4, we must have $f_{\omega-2} = |B_{\omega-1}(I)|$. Let $m \in A_\omega(I)$. After relabeling, we can assume that $m = x_1 x_2 \cdots x_\omega$. Note that $m/x_i \notin I$ for $i = 1,\ldots,\omega$. Indeed, if $m/x_i \in I$, this implies that $m \in I$, contradicting the fact that all elements of $A_\omega(I)$ are not in I. So $m/x_i \in B_{\omega-1}(I)$ for all i. By definition,

https://doi.org/10.4153/S0008439518000024 Published online by Cambridge University Press

every element of $B_{\omega-1}(I)$ must divide an element of $C_\omega(I)$ (since $B_\omega(I) = \varnothing$). Because $|C_\omega(I)| < \omega$, there is one monomial $z \in C_\omega(I)$ such that m/x_i and m/x_j both divide z. But since $\deg z = \omega$, this forces $m = z$. We now arrive at a contradiction, since $m \in A_\omega(I) \cap C_\omega(I)$, but these two sets are disjoint. ∎

Remark 4.9 The ideal I of Example 2.2 is an f-ideal with $\alpha = \alpha(I) = 2$, and $f_{2-1} = 8 > \binom{5}{2} - 5 + 2 = 7$. So by Theorem 4.8, the ideal I should have a generator of degree $\alpha + 1 = 3$, which it does. Alternatively, we could have deduced that I has a generator of degree 2 from the fact that $\omega(I) = 3$ and $f_{3-1} = 2$.

In our computer experiments, we only found f-ideals that had either $\alpha(I) = \omega(I)$, *i.e.*, the f-ideals were equigenerated, or $\alpha(I) + 1 = \omega(I)$. It would be interesting to determine the existence of f-ideals with the property that $\alpha(I) + d = \omega(I)$ for any $d \in \mathbb{N}$. Theorem 4.8 would imply a necessary condition on the generators of these ideals.

Acknowledgments We thank Hasan Mahmood for his feedback and comments. `Macaulay2` [8] was used in the initial stages of this project. We also thank the referee for his/her helpful suggestions and improvements.

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