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# Exact Filling of Figures with the Derivatives of Smooth Mappings Between Banach Spaces

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*Abstract.* We establish sufficient conditions on the shape of a set $A$ included in the space $\mathcal{L}_s^n(X, Y)$ of the $n$-linear symmetric mappings between Banach spaces $X$ and $Y$, to ensure the existence of a $C^n$-smooth mapping $f\colon X \to Y$, with bounded support, and such that $f^{(n)}(X) = A$, provided that $X$ admits a $C^n$-smooth bump with bounded $n$-th derivative and dens $X =$ dens $\mathcal{L}^n(X, Y)$. For instance, when $X$ is infinite-dimensional, every bounded connected and open set $U$ containing the origin is the range of the $n$-th derivative of such a mapping. The same holds true for the closure of $U$, provided that every point in the boundary of $U$ is the end point of a path within $U$. In the finite-dimensional case, more restrictive conditions are required. We also study the Fréchet smooth case for mappings from $\mathbb{R}^n$ to a separable infinite-dimensional Banach space and the Gâteaux smooth case for mappings defined on a separable infinite-dimensional Banach space and with values in a separable Banach space.

## 1 Introduction

Several properties related to the set of derivatives of smooth bumps have been studied recently. In particular, the questions as to how small, how large, and the shape of the set of derivatives of a smooth bump defined on a Banach space, have been considered.

Ekeland's variational principle [8] easily implies that if $b$ is a continuous Gâteaux smooth bump on a Banach space $X$, then the norm closure of $b'(X)$ contains the origin as an interior point. If, in addition, $X$ has the Radon–Nikodým property, it follows from Stegall's variational principle that the cone generated by the set of derivatives $C(b) := \{\lambda b'(x) : x \in X, \lambda \geq 0\}$ is a residual set in $X^*$. It was proved in [1] that if $X$ has a $C^1$-smooth and Lipschitzian bump, then there exists another $C^1$-smooth bump whose derivatives fill the whole dual space $X^*$. This result was generalized in [2] for higher orders of differentiability and for mappings, with bounded support, from $X$ to another Banach space $Y$, under certain conditions on $X$ and $Y$. Also, it was proved in [2] that, if $X$ is a separable Banach space, then there always exists a continuous Gâteaux smooth bump whose derivatives fill all of the dual space. On the other hand, in [4] there is an example of a $C^1$-smooth Lipschitzian bump on $\ell_2$ such that the cone generated by the set of its derivatives has empty interior. Also, as a consequence of a result of Hájek's [11], we know that if $f$ is a $C^1$-smooth function defined on $c_0$ with locally uniformly continuous derivative, then $f'(c_0)$ is contained in a countable union of compact sets, and thus the cone generated by the set of its

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derivatives is of the first Baire category. We refer to [3, 9] for more information on the size of the set of derivatives of a smooth bump.

The paper [6] studies the shape of the set of gradients of a $C^1$-smooth bump defined on $\mathbb{R}^n$. It was proved there that such a set cannot be locally contained in a hyperplane. It was also proved there that this set may fail to be simply connected, and there was constructed a $C^1$-smooth bump on $\mathbb{R}^n$ whose set of gradients fills in any pre-fixed "reasonably looking" compact set containing the origin as an interior point. For instance, this holds true for every compact convex set containing the origin in its interior. In [7], the same problem was considered for $C^1$-smooth and Lipschitz bumps defined on infinite-dimensional Banach spaces. They proved the following:

*Let $X$ be an infinite dimensional Banach space with a $C^1$-smooth and Lipschitz bump. Let $\Omega \subset X^*$ be an open connected set containing the origin and satisfying that there exists a summable sequence $a_0, a_1, a_2, \dots$ of positive numbers such that every $\eta \in \bar{\Omega}$ can be expressed as $\lim_{i\to\infty} \xi_i$ for some sequence $0 = \xi_0, \xi_1, \xi_2, \dots$ in $\Omega$ such that $\|\xi_{i+1} - \xi_i\| < a_i$, and that the linear segment $[\xi_i, \xi_{i+1}]$ lies in $\Omega$ for every $i = 0, 1, 2, \dots$. Then there exists a $C^1$-smooth and Lipschitz bump $b\colon X \to [0, 1]$ so that $b'(X) = \overline{\Omega}$.*

From this result it follows that every open connected set in $X$, containing the origin, is the range of the first derivative of a $C^1$-smooth bump defined on $X$.

In this paper we provide, under weaker assumptions on $\Omega$, a $C^1$-smooth bump so that $b'(X) = \overline{\Omega}$. We study the analogous problem for higher order derivatives and establish some results that generalize the above mentioned theorems in both infinite-dimensional and finite-dimensional cases. If $X$ and $Y$ are Banach spaces and $n \in \{0, 1, 2, \dots\}$, then $\mathcal{L}_s^n(X, Y)$ stands for the (Banach) space of $n$-linear symmetric mappings from $X$ to $Y$. We define $\mathcal{L}_s^0(X, Y) = Y$. We prove, for $p \in \{0, 1, \dots, \infty\}$:

*If an infinite-dimensional Banach space $X$ has a $C^p$-smooth bump with bounded derivatives, and* $\operatorname{dens} X = \operatorname{dens} \mathcal{L}_s^n(X, Y)$ *for some $0 \le n \le p$, then there exists a $C^p$-smooth mapping $f\colon X \to Y$, with bounded support, such that $f^{(n)}(X) = U$, where $U \subset \mathcal{L}_s^n(X, Y)$ is a pre-set open bounded and connected set, containing the origin. If, in addition, every point of the closure $\overline{U}$ of $U$ is the end point of a path within $U$, then there exists a $C^n$-smooth mapping $g\colon X \to Y$, with bounded support, such that $g^{(n)}(X) = \overline{U}$.*

This result is close to being a characterization of the set of derivatives of a smooth bump since, if the set $\overline{U}$ is the range of the $n$-th derivative of a $C^n$-smooth mapping, then it is necessarily path-connected. However, when $X$ is finite-dimensional, the above result does not hold true and more restrictive conditions must be assumed, see an example below. We prove the following result:

*Let $n, m, p \in \mathbb{N}$ and consider an open, bounded, and connected subset $U \subset \mathcal{L}_s^p(\mathbb{R}^n, \mathbb{R}^m)$ containing the origin. If for every $\varepsilon > 0$ there is a finite family of open connected subsets of $U$, covering $U$, each one with diameter less than $\varepsilon$, then there exists a $C^p$-smooth mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$, with bounded support, so that $b^{(p)}(\mathbb{R}^n) = \overline{U}$.*

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In the last section, we first consider mappings from $\mathbb{R}^n$ to an infinite-dimensional separable Banach space $Y$. We prove:

*If a nonempty open subset $U \subset \mathcal{L}(\mathbb{R}^n, Y)$ satisfies the conditions required in the infinite-dimensional case (above), then there are Fréchet smooth mappings $b, g\colon \mathbb{R}^n \to Y$, with bounded support, so that $b'(\mathbb{R}^n) = U$ and $g'(\mathbb{R}^n) = \overline{U}$.*

Finally, we prove:

*When $X$ and $Y$ are separable Banach spaces and $X$ is infinite-dimensional, then there exists a uniformly Gâteaux smooth mapping with bounded support $f\colon X \to Y$ so that $f(X) = B_Y$ and $f'(X) = B_{\mathcal{L}_s^1(X,Y)}$.*

This sharpens results from [2] where smooth mappings $b$ and $g$ were constructed so that $B_{\mathcal{L}_s^1(\mathbb{R}^n,Y)} \subset b'(\mathbb{R}^n)$ in the first case, and $B_Y \subset g(X)$, $B_{\mathcal{L}_s^1(X,Y)} \subset g'(X)$ in the second case.

## 2 The Case of $C^n$-Smooth and Lipschitz Mappings with Bounded Support

We begin with a lemma which tells us that, for a polygonal curve $P$ in the space of symmetric $n$-linear mappings $\mathcal{L}_s^n(X, Y)$, one can always find a bump whose $n$-th derivative's range contains a suitable neighborhood of $P$ and is contained in another (larger but not much larger) neighborhood of $P$. This lemma is our main tool to construct bumps with a prescribed range of derivatives. By a *polygonal curve* in a Banach space $Z$ we understand any set $(z_0, z_1, \dots, z_k) := \bigcup_{i=0}^{k-1}[z_i, z_{i+1}]$, where $[z_i, z_{i+1}]$ is the linear segment joining the points $z_i$ and $z_{i+1}$ in $Z$, and $k$ is any positive integer. We say that this polygonal curve goes from $z_0$ to $z_k$.

***Lemma 2.1*** *Let $p \in \{0, 1, \dots, \infty\}$ and let $n$ be an integer with $0 \le n \le p$. Let $X$ be an infinite-dimensional Banach space admitting a $C^p$-smooth bump $b$ with bounded derivatives, and let $Y$ be another Banach space such that* $\operatorname{dens} \mathcal{L}_s^n(X, Y) = \operatorname{dens} X$. *Consider a polygonal curve $P$ in $\mathcal{L}_s^n(X, Y)$ from $0$ to a point $Q$. Then there is a constant $M > 1$ (which only depends on the spaces and not on the polygonal curve) so that for any $\varepsilon > 0$ there exists a $C^p$-smooth mapping $g\colon X \to Y$, with support in $B_X$ and with bounded derivatives, such that*

$$\|g^{(k)}\|_\infty := \sup\{\|g^{(k)}(x)\| : x \in X\} \le 4\varepsilon, \quad \textit{for } k = 0, 1, \dots, n-1,$$
$$P + \frac{\varepsilon}{M} B_{\mathcal{L}_s^n(X,Y)} \subset g^{(n)}(X) \subset P + 2\varepsilon\, B_{\mathcal{L}_s^n(X,Y)},$$
$$\textit{and } g^{(n)}\,|_{\delta B_X} \equiv Q \quad \textit{for some} \quad \delta > 0.$$

*Moreover, if $n < i \le p$, the $i$-th derivative $g^{(i)}$ is bounded by a constant which only depends on $i$, $\varepsilon$, $M$ and the length of the polygonal curve.*

**Proof** *First step:* If $X$ has a $C^p$-smooth bump with bounded $n$-th derivative, by composing it with a suitable $C^\infty$-smooth bump on $\mathbb{R}$, we obtain a $C^p$-smooth bump $b$

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with bounded image, bounded derivatives and with $b(\delta B_X) = 1$ for some $\delta > 0$. We may assume that the support of $b$ is included in $\frac{1}{2}B_X$. By the results obtained in [2], there is a $C^p$-smooth mapping $r\colon X \to Y$, with bounded derivatives, so that $r^{(n)}(X)$ contains $B_{\mathcal{L}^n_s(X,Y)}$. We may assume, up to suitable dilations (replacing $r$ by $x \mapsto \lambda r(\frac{x}{\lambda})$, for $\lambda > 0$ small enough) and a translation, that the support of $r$ is included in $B_X$ and is disjoint from the support of $b$.

*Second step:* Notice that the derivatives $b, b', \dots, b^{(n)}, r, r', \dots, r^{(n)}$ are all bounded. Denote by $B_{\mathcal{P}^n(X,Y)}$ the unit ball of the Banach space $\mathcal{P}^n(X,Y)$ of $n$-linear homogeneous and continuous polynomials from $X$ to $Y$. Then for any element $S \in B_{\mathcal{L}^n_s(X,Y)}$, take $R \in B_{\mathcal{P}^n(X,Y)}$, so that $S$ is the $n$-th derivative of $R$ and define then the mapping $h\colon X \to Y$, by

$$h(x) = b(x)R(x) + r(x), \quad x \in X.$$

Clearly, the mapping $h$ has support in $B_X$, is $C^p$-smooth with bounded derivatives and $h\ |_{rB_X} \equiv R\ |_{rB_X}$ for some $r > 0$; then $h^{(n)}\ |_{rB_X} \equiv S$. Let us fix $M > 1$ so that $M \geq \max\{\|h^{(i)}\|_\infty : i = 0, 1, \dots, n-1\}$. Equally, $h^{(i)}$ is bounded by a constant $M_i$, if $n < i \leq p$. The constants $M$ and $M_i$ do not depend on the considered $S \in B_{\mathcal{L}^n_s(X,Y)}$, since the derivatives of the corresponding $k$-homogeneous polynomial $R$ are bounded by 1.

*Third step:* Let $P = (R_0 = 0, R_1, \dots, R_s = Q)$ be a given polygonal. Then there is a family of points $\{Q_0 = 0, Q_1, \dots, Q_k = Q\}$ satisfying that $\|Q_j - Q_{j-1}\| \leq \frac{2\varepsilon}{M}$, the polygonal curve $P$ is included in $\bigcup_j (Q_j + \frac{2\varepsilon}{M} B_{\mathcal{L}^n_s(X,Y)})$ and $k\frac{2\varepsilon}{M} \leq l + 1$, where $l$ denotes the length of the polygonal curve $P$. By the second step, there are $C^p$-smooth mappings $h_j\colon X \to Y$, $j = 1, 2, \dots, k$, with support in $B_X$, with bounded derivatives, and with $\|h_j^{(i)}\|_\infty \leq 2\varepsilon$ for $i = 0, 1, \dots, n$, $\|h_j^{(i)}\|_\infty \leq \frac{2\varepsilon M_i}{M}$ for $i = n+1, \dots, p$, and with

$$\frac{2\varepsilon}{M} B_{\mathcal{L}^n_s(X,Y)} \subset h_j^{(n)}(X) \subset 2\varepsilon B_{\mathcal{L}^n_s(X,Y)},$$
$$h_j^{(n)}\ |_{\delta B_X} \equiv Q_j - Q_{j-1}, \text{ for } j = 1, \dots, n \text{ and some } \delta > 0.$$

Then we define the mapping $g\colon X \to Y$ by

$$g(x) = h_1(x) + \left(\frac{\delta}{2}\right)^n h_2\left(\frac{2}{\delta}x\right) + \dots + \left(\frac{\delta}{2}\right)^{n(k-1)} h_k\left(\left(\frac{2}{\delta}\right)^{k-1} x\right), \quad x \in X.$$

Notice that the support of $g$ is included in $B_X$ and if we take $\gamma = \frac{\delta^k}{2^k}$ then $g^{(n)}\ |_{\gamma B_X} \equiv Q$. Clearly, $g$ is $C^p$-smooth,

$$\|g^{(i)}\|_\infty \leq \|h_1^{(i)}\|_\infty + \frac{\delta}{2}\|h_2^{(i)}\|_\infty + \dots + \left(\frac{\delta}{2}\right)^{k-1} \|h_k^{(i)}\|_\infty \leq 4\varepsilon, \quad i = 0, 1, \dots, n-1.$$

and

$$\bigcup_{j=1}^{k} (Q_j + \frac{2\varepsilon}{M} B_{\mathcal{L}^n_s(X,Y)}) \subset g^{(n)}(X) \subset \bigcup_{j=1}^{k} (Q_j + 2\varepsilon B_{\mathcal{L}^n_s(X,Y)}).$$

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This implies that $P + \frac{\varepsilon}{M} B_{\mathcal{L}_s^n(X,Y)} \subset g^{(n)}(X) \subset P + 2\varepsilon\, B_{\mathcal{L}_s^n(X,Y)}$.

Finally, for $n < i \leq p$ we have

$$\|g^{(i)}\|_\infty \leq \sum_{j=1}^{k} \frac{2\varepsilon M_i}{M}\Big(\frac{2}{\delta}\Big)^{(j-1)(i-n)} = \frac{2\varepsilon\, M_i}{M}\, \frac{(\frac{2}{\delta})^{(i-n)k} - 1}{(\frac{2}{\delta})^{(i-n)} - 1}$$

$$\leq \frac{4\varepsilon\, M_i}{M}\Big(\frac{2}{\delta}\Big)^{(i-n)((l+1)M\varepsilon^{-1})} \qquad \blacksquare$$

***Remark 2.2*** When checking the proof of the above Lemma we observe that:

(i) When $\dim X < \infty$ and $\dim Y = \infty$, Lemma 2.1 holds if we replace $C^p$-smoothness with Fréchet smoothness and the inequalities and inclusions there hold for $n = 1$.

(ii) When $X$ and $Y$ are separable Banach spaces and $\dim X = \infty$, Lemma 2.1 holds if we replace $C^p$ smoothness with uniformly Gâteaux smoothness and the inequalities and inclusions there hold for $n = 0, 1$.

Now, when $X$ is infinite-dimensional and under the above hypotheses on $X$ and $Y$, we can easily deduce that every open connected subset of $\mathcal{L}_s^n(X, Y)$ that contains the origin can be regarded as the range of a higher order derivative of some mapping with bounded support. Thus we see that there are no restrictions on the shape of an open connected set in order to be the range of a higher derivative of a smooth mapping with bounded support.

***Theorem 2.3*** *Let $p \in \{0, 1, \dots, \infty\}$ and let $X, Y$ be Banach spaces with $\dim X = \infty$. Assume that $X$ admits a $C^p$-smooth bump with bounded derivatives and*

$$\operatorname{dens} \mathcal{L}_s^n(X, Y) = \operatorname{dens} X$$

*for some $0 \leq n \leq p$. If $U \subset \mathcal{L}_s^n(X, Y)$ is a pre-fixed open, bounded and connected set with $0 \in U$, then there is a $C^p$-smooth mapping $h\colon X \to Y$ with bounded support such that $h^{(n)}(X) = U$.*

**Proof** Let $U$ be as above. Let $D$ be a dense subset in $U$ whose cardinality is equal to the density of $X$. Add $0$ to $D$. Let $\mathcal{P}$ denote the set of all polygonal curves lying in $U$, beginning at $0$ and with vertices in $D$. Clearly, the cardinality of $\mathcal{P}$ is equal to $\operatorname{dens} X$. For every rational number $0 < \varepsilon < 1$ and for every $P \in \mathcal{P}$ such that $P + 2\varepsilon\, B_{\mathcal{L}^n(X,Y)} \subset U$ we find a mapping $g_{P,\varepsilon}\colon X \to Y$ satisfying the properties from Lemma 2.1. Let us relabel the family of these bumps as $\{g_\alpha\}_{\alpha\in\Gamma}$, where $\operatorname{card}\Gamma = \operatorname{dens} X$.

Consider a bounded family of 3-separated points $\{x_\alpha\}_{\alpha\in\Gamma}$. Define

$$h(x) = \sum_{\alpha\in\Gamma} g_\alpha(x - x_\alpha), \quad x \in X.$$

Let us check that the bump $h$ fulfills the required conditions. Notice that for every $x \in X$ there is at most one non-zero summand in the above definition, which remains

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the same in a neighborhood of $x$. Thus, $h$ is a $C^p$-smooth mapping with bounded support and $h^{(n)}(X) \subset U$. Let us check that $U \subset h^{(n)}(X)$. Since $U$ is connected and open, for any $Q \in U$, there is a polygonal curve $P = (Q_0 = 0, Q_1, \ldots, Q_m = Q)$ and $0 < \varepsilon < 1/2$ so that $P + 4\varepsilon B_{\mathcal{L}_s^n(X,Y)} \subset U$. We may assume, by the density of $D$ in $U$, that $Q_0, Q_1, \ldots, Q_{m-1} \in D$. Also, take $Q'_m \in D$ so that $\|Q'_m - Q\| < \frac{\varepsilon}{M}$. The polygonal curve $P' = \{Q_0 = 0, Q_1, \ldots, Q_{m-1}, Q'_m\} \in \mathcal{P}$ and, since $P' + 2\varepsilon B_{\mathcal{L}_s^n(X,Y)} \subset P + 4\varepsilon B_{\mathcal{L}_s^n(X,Y)} \subset U$, the associated mapping $g_{P',\varepsilon}$ belongs to the family $\{g_\alpha\}$. From Lemma 2.1 we obtain that $Q \in P' + \frac{\varepsilon}{M} B_{\mathcal{L}_s^n(X,Y)} \subset g^{(n)}_{P',\varepsilon}(X) \subset h^{(n)}(X)$ and the proof is finished. ■

***Remark 2.4*** Notice that the above result is a generalization for the case $0 \le n \le p$ of the results given in [7] for $n = p = 1$. We cannot deduce that $h^{(n+1)}$ is bounded. Actually, if the $n+1$-th derivative of $h$ were bounded, then $U$ should satisfy the following property: *there exists $M > 0$ so that for every point $x \in U$, the points $0$ and $x$ can be connected by a path within $U$ of length bounded by $M$.* However, for every Banach space of dimension bigger than one, there are open sets which do not have this property. Our next theorem further extends the family of sets which can be written as $h^{(n)}(X)$.

***Theorem 2.5*** *Let $X, Y$ be a Banach space with $\dim X = \infty$. Assume that $X$ has a $C^n$-smooth bump with bounded $n$-th derivative and assume that*

$$\operatorname{dens} X = \operatorname{dens} \mathcal{L}_s^n(X,Y).$$

*Let $U \subset \mathcal{L}_s^n(X,Y)$ be an open, bounded, connected set with $0 \in U$, so that for every $Q$ in the boundary $\partial U$ of $U$ there exists a (continuous) path from $0$ to $Q$ through points of $U$. Then there exists a $C^n$-smooth mapping $h\colon X \to Y$ such that $h^{(n)}(X) = \overline{U}$.*

**Proof** Since we have already constructed a mapping with the required smoothness conditions so that the image of the $n$-th derivative is $U$, we just need to construct a mapping (with the same kind of smoothness) whose $n$-th derivative is included in the closure of $U$ and covers $\partial U$. If the reader prefers a direct proof (which does not rely on Theorem 2.3), consider the argument below for every point of $\overline{U}$.

Let $D$ be a dense subset of $U$, with cardinality equal to the density of $X$. Take any $Q \in \partial U$. By hypotheses, we select a path from $0$ to $Q$ through points of $U$. Therefore we may choose a sequence $Q_0, = 0, Q_1, Q_2, \ldots,$ of elements of $D$, with limit $Q$ (in norm), and polygonal curves $P_i$, included in $U$, from $Q_{i-1}$ to $Q_i$, so that, for every $i \ge 2$, the polygonal curves $P_i, P_{i+1}, P_{i+2}, \ldots$ are included in $Q + \frac{1}{2^{i+2}} B_{\mathcal{L}_s^n(X,Y)}$. Let us denote by $\mathcal{P}$ the family of these obtained "infinite polygonal curves". Then, every $P \in \mathcal{P}$ can be identified with the infinite sequence of the points $Q_i$, $P = \{Q_0 = 0, Q_1, Q_2, \ldots\}$. Finally, for every $k \in \mathbb{N}$, we define

$$\mathcal{P}_k = \Big\{ R = \{Q_0 = 0, Q_1, \ldots, Q_k\} : \text{there is } P \in \mathcal{P} \text{ whose first } k+1 \text{ points are } Q_0 = 0, Q_1, \ldots, Q_k \Big\}.$$

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For every $R \in \bigcup_{k=1}^{\infty} \mathcal{P}_k$ we construct a $C^n$-smooth mapping $g_R\colon X \to Y$, with bounded $n$-th derivative, and $\varepsilon_R > 0$ in the following way:

*Step 1.* For every $R = \{0, Q_1\} \in \mathcal{P}_1$ there is a polygonal curve $P_R$ and $0 < \varepsilon_R < 1/2^3$ so that $P_R + 2\varepsilon_R B_{\mathcal{L}_s^n(X,Y)} \subset U$. By Lemma 2.1, there is a $C^n$-smooth mapping $g_R\colon X \to Y$, with support in $B_X$, such that $\|g_R^{(k)}\|_\infty \le 4\varepsilon_R$ for $k = 0, 1, \dots, n-1$, and there is $0 < \delta_R < 1$ such that

$$g_R^{(n)}\,|_{\delta_R B_X} \equiv Q_1 - Q_0 = Q_1, \quad \text{and} \quad g_R^{(n)}(X) \subset P_R + 2\varepsilon_R\, B_{\mathcal{L}_s^n(X,Y)}.$$

*Step 2.* For every pair $(Q_1, Q_2)$, where $R = \{0, Q_1, Q_2\} \in \mathcal{P}_2$ we select a polygonal curve $P_{Q_1,Q_2}$ from $Q_1$ to $Q_2$ so that $P_{Q_1,Q_2}$ is included in a ball of radius $\frac{1}{2^4}$. Take $0 < \varepsilon_R < \frac{1}{2^4}$ so that $P_{Q_1,Q_2} + 2\varepsilon_R B_{\mathcal{L}_s^n(X,Y)} \subset U$. By Lemma 2.1, there is a $C^n$-smooth mapping $f_R\colon X \to Y$, with support in $B_X$, such that $\|f_R^{(k)}\|_\infty \le 4\varepsilon_R$, for $k = 0, 1, \dots, n-1$, and there is $0 < \gamma_R < 1$ such that

$$f_R^{(n)}\,|_{\gamma_R B_X} \equiv Q_2 - Q_1, \quad \text{and} \quad Q_1 + f_R^{(n)}(X) \subset P_{Q_1,Q_2} + 2\varepsilon_R B_{\mathcal{L}_s^n(X,Y)}.$$

Then, if $R' = \{0, Q_1\}$, we define

$$g_R(x) = \Big(\frac{\delta_{R'}}{8}\Big)^n\, f_R\Big(\frac{x}{\delta_{R'}/8}\Big), \quad x \in X.$$

The mapping $g_R\colon X \to Y$ is $C^n$-smooth, the support is included in $\frac{\delta_{R'}}{8} B_X$ and $\|g_R^{(k)}\|_\infty \le \frac{\delta_{R'}}{8} 4\varepsilon_R \le 4\varepsilon_R$, for $k = 0, 1, \dots, n-1$. Also, there is $0 < \delta_R < \frac{\delta_{R'}}{8}$ with

$$g_R^{(n)}\,|_{\delta_R B_X} \equiv Q_2 - Q_1, \quad \text{and} \quad Q_1 + g_R^{(n)}(X) \subset P_{Q_1,Q_2} + 2\varepsilon_R B_{\mathcal{L}_s^n(X,Y)}.$$

*Step 3.* In general, for $k \ge 2$ and for every $R = \{0, Q_1, \dots, Q_{k-1}, Q_k\} \in \mathcal{P}_k$, we select a polygonal curve $P_{Q_{k-1},Q_k}$ from $Q_{k-1}$ to $Q_k$ so that $P_{Q_{k-1},Q_k}$ is included in a ball of radius $\frac{1}{2^{k+2}}$. Take $0 < \varepsilon_R < \frac{1}{2^{k+2}}$ so that $P_{Q_{k-1},Q_k} + 2\varepsilon_R B_{\mathcal{L}_s^n(X,Y)} \subset U$. By induction on $k \in \mathbb{N}$ we define, for every $R = \{0, Q_1, \dots, Q_{k-1}, Q_k\} \in \mathcal{P}_k$, a $C^n$-smooth mapping $g_R\colon X \to Y$, whose support is included in $\frac{\delta_{R'}}{8} B_X$, where $R' = \{0, Q_1, \dots, Q_{k-1}\}$ and

$$\|g_R^{(k)}\|_\infty \le 4\,\varepsilon_R, \quad \text{for } k = 0, 1, \dots, n-1;$$

$$g_R^{(n)}\,|_{\delta_R B_X} \equiv Q_k - Q_{k-1}, \quad \text{for some} \quad 0 < \delta_R < \frac{\delta_{R'}}{8};$$

$$Q_{k-1} + g_R^{(n)}(X) \subset P_{Q_{k-1},Q_k} + 2\varepsilon_R B_{\mathcal{L}_s^n(X,Y)}.$$

If $j \ge i$ and $R = \{0, Q_1, \dots, Q_j\} \in \mathcal{P}_j$, $R' = \{0, Q_1', \dots, Q_i'\} \in \mathcal{P}_i$, we shall write $R' \le R$ whenever $Q_1 = Q_1', \dots, Q_i = Q_i'$. Let us select within $4\,B_X$ a family of

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points $\{x_R : R \in \mathcal{P}_k, k \in \mathbb{N}\}$ with the properties:

$$\|x_R - x_S\| > 2, \quad \text{if } R, S \in \mathcal{P}_1 \text{ and } R \neq S,$$

$$\|x_R - x_S\| \geq \frac{\delta_{R'}}{3}, \quad \text{if } R' \leq R, S,\ R' \in \mathcal{P}_k \text{ and } R, S \in \mathcal{P}_{k+1}$$

$$\|x_R - x_{R'}\| \leq \frac{\delta_{R'}}{2}, \quad \text{if } R' \leq R,$$

$$x_R + \delta_R B_X \subset 4B_X.$$

From the above properties we deduce that $(x_R + B_X) \cap (x_S + B_X) = \varnothing$, for $R, S \in \mathcal{P}_1$ and $R \neq S$; also, if $R, S \in \mathcal{P}_{k+1}, R' \in P_k, R' \leq R, S$ and $R \neq S$, then

$$(x_R + \frac{\delta_{R'}}{8} B_X) \cap (x_S + \frac{\delta_{R'}}{8} B_X) = \varnothing, \quad (x_R + \frac{\delta_{R'}}{8} B_X) \subset (x_{R'} + \delta_{R'} B_X).$$

Now, we define the mapping $h\colon X \to Y$ as the sum $\sum_{k=1}^{\infty} h_k$, where

$$h_k(x) = \sum_{R \in \mathcal{P}_k} g_R(x - x_R), \quad x \in X,\ k \in \mathbb{N}.$$

Let us check that $h$ fulfills the required conditions. For every $x \in X$ there is a neighborhood $U_x$ of $x$ and $R_0 \in \mathcal{P}_k$, where $h_k(y) = g_{R_0}(y - x_{R_0})$, for $y \in U_x$. Thus, $\|h_k^{(i)}\|_\infty \leq \sup\{\|g_R^{(k)}\|_\infty : R \in \mathcal{P}_k\} \leq \sup\{4\varepsilon_R : R \in \mathcal{P}_k\} \leq 1/2^k$, for $i = 0, 1, \dots, n-1$. Therefore, the series $\sum_k h_k^{(i)}$ uniformly converges in $X$ for every $i = 0, 1, \dots, n-1$ and $h^{(i)} = \sum_k h_k^{(i)}$.

Now, recall that $h_1^{(n)}(X) \subset U$ and $U$ is bounded, thus $h_1^{(n)}$ is bounded. Also, for $k \geq 2$ and $R = \{0, Q_1, \dots, Q_{k-1}, Q_k\} \in \mathcal{P}_k$, the polygonal curve $P_{Q_{k-1},Q_k}$ is included in a ball of radius $\frac{1}{2^{k+2}}$, and then, $-Q_{k-1} + P_{Q_{k-1},Q_k}$ is included in $\frac{1}{2^{k+1}} B_{\mathcal{L}_s^n(X,Y)}$. Since $g_R^{(n)}(X) \subset -Q_{k-1} + P_{Q_{k-1},Q_k} + 2\varepsilon_R B_{\mathcal{L}_s^n(X,Y)} \subset \frac{1}{2^k} B_{\mathcal{L}_s^n(X,Y)}$, it follows that $\|h_k^{(n)}\|_\infty \leq \frac{1}{2^k}$. Thus, $\sum_k h_k^{(n)}$ is uniformly convergent on $X$ and so the $n$-th derivative $h^{(n)} = \sum_k h_k^{(n)}$ is a continuous and bounded mapping.

It remains to prove that $h^{(n)}(X) = \overline{U}$, the closure of $U$. Clearly, from the construction of the mappings $g_R$, we know that $h^{(n)}(X) \subset \overline{U}$. If $Q \in \partial U$, there is $P = \{0, Q_1, Q_2, \dots\} \in \mathcal{P}$ so that $Q_i$ converges to $Q$. Consider $R_k = \{0, Q_1, \dots, Q_k\}$, $k = 1, 2, \dots$, and the associated sequence of points in $X$, $(x_{R_k})_k$. Since $R_1 \leq R_2 \leq \cdots \leq R_k \leq R_{k+1} \leq \cdots$, we know that there exists the $\lim x_{R_k} = x$. Finally, $h^{(n)}(x_{R_k}) = \sum_{i=1}^k g_{R_i}^{(n)}(x_{R_k} - x_{R_i}) = \sum_{i=1}^k (Q_i - Q_{i-1}) = Q_k$ and then $h^{(n)}(x) = Q$ which finishes the proof. ■

***Remark 2.6*** When $n = 1$ and $Y = \mathbb{R}$, the condition we require for $U \subset X^*$ in Theorem 2.3 is less restrictive than the condition required in [7]. In every Banach space of dimension bigger than 1 there are examples of open bounded and connected sets containing the origin with the properties required in Theorem 2.5,so that even uncountably many points of the boundary of $U$ cannot be end points of paths of *finite length* within $U$. Theorem 2.5 allows us to enlarge the class of subsets of $X^*$ which are known to be the range of the derivative of a $C^1$-smooth bump.

Theorem 2.5 does not hold true when $X$ is finite-dimensional. Next, we give an example of an open bounded subset $U \subset \mathbb{R}^2$ containing the origin and satisfying the condition given in Theorem 2.5, so that the closure of $U$ cannot be the range of the first derivative of any $C^1$-smooth bump on $\mathbb{R}^2$.

***Example*** Consider the open sets of the plane

$$U_n = \Big\{ (x, y) : 1 - \frac{1}{2n} < |x| < 1 - \frac{1}{2n+1}, |y| < 2 \Big\}, n \in \mathbb{N}$$

and

$$U = \bigcup_{n=1}^{\infty} U_n \cup \big\{ (x, y) : 1 < \max(|x|, |y|) < 2 \big\} \cup \big\{ (x, y) : |x| < \frac{1}{4}, |y| < 2 \big\}.$$

Obviously, the closure of $U$ satisfies the conditions required in Theorem 2.5. Assume that the closure of $U$ is the image of a $C^1$-smooth bump $b\colon \mathbb{R}^2 \to \mathbb{R}$. Let us take points $(a_n, 0) \in U_n$ converging to $(1, 0) \in \partial U$. By the assumption, there is a bounded sequence of points $(x_n, y_n) \in \mathbb{R}^2$ so that $b'(x_n, y_n) = (a_n, 0)$. By compactness, we may assume that the points $(x_n, y_n)$ converge to some $(x, y) \in \mathbb{R}^2$. By continuity, $b'(x, y) = (1, 0)$ and there is some $\delta > 0$ so that $A := b'((x, y) + \delta B) \subset (1, 0) + \frac{1}{2} B_{\mathbb{R}^2}$. Since $b'$ is continuous, the set $A$ should be connected. But this is a contradiction, since $\{(x_n, y_n)\}_{n \geq N} \subset A \subset \overline{U} \cap ((1, 0) + \frac{1}{2} B)$ for some $N \in \mathbb{N}$.

Nevertheless, we next show that if for every $\varepsilon > 0$ there is a finite collection of open and connected subsets of $U$ which cover $U$ and have diameter less than $\varepsilon$, then $\overline{U}$ is the image of a $C^1$-smooth bump. The above example clearly shows that if we drop this condition the conclusion does not necessarily hold.

***Theorem 2.7*** *Let us consider $n, m, p \in \mathbb{N}$ and an open bounded and connected subset $U \subset \mathcal{L}_s^p(\mathbb{R}^n, \mathbb{R}^m)$ containing the origin. Assume that for every $\varepsilon > 0$ there is a finite family $\mathcal{F}_\varepsilon$ of open (non-empty) subsets of $U$ which cover $U$ and are such that every $V \in \mathcal{F}_\varepsilon$ is connected and has diameter less than $\varepsilon$. Then there is a $C^p$-smooth mapping $b\colon \mathbb{R}^n \to \mathbb{R}^m$ with bounded support, so that $b^{(p)}(\mathbb{R}^n) = \overline{U}$.*

**Proof** We will use the following fact.

***Lemma 2.8*** *Let $n, m, p \in \mathbb{N}$, and consider a polygonal curve $P$ in $\mathcal{L}_s^p(\mathbb{R}^n, \mathbb{R}^m)$ from $0$ to a point $Q$. Then there is a constant $M > 1$ (which does not depend on the polygonal curve), so that for any $\varepsilon > 0$ there exists a $C^p$-smooth mapping $g\colon \mathbb{R}^n \to \mathbb{R}^m$, with support in $B_X$ and with bounded derivatives, such that*

$$\|g^{(k)}\|_\infty := \sup\{\|g^{(k)}(x)\| : x \in X\} \leq 4\varepsilon, \quad \textit{for } k = 0, 1, \dots, p-1,$$

$$g^{(p)}(X) \subset P + 2\varepsilon\, B_{\mathcal{L}_s^p(\mathbb{R}^n, \mathbb{R}^m)},$$

$$g^{(p)}\,|_{\delta B_X} \equiv Q \quad \textit{for some} \quad \delta > 0.$$

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We omit the proof of this Lemma, since it is almost identical to that of Lemma 2.1. The only difference is that in this case we do not have the mapping $r$ at our disposal (nor do we need it), so the definition of $h$ in the second step must be changed for $h(x) = b(x)R(x)$; consequently we only obtain the inclusion $g^{(p)}(X) \subset P + 2\varepsilon B_{\mathcal{L}_s^p(\mathbb{R}^n,\mathbb{R}^m)}$.

Let $(\varepsilon_k)_k$ be a summable sequence of positive numbers. We denote by $\mathcal{F}_k$ the finite open covering of $U$ given in the hypothesis for $\varepsilon = \varepsilon_k$. For every open subset $V \in \mathcal{F}_k$, we select a point $T \in V$, and denote the set consisting of all the points obtained in this way by $F_k$. In order to avoid problems of notation, we may consider that the selected points are all different and that $F_k \cap F_j = \varnothing$ whenever $k \neq j$. Notice that, for every $k$, the finite set $F_k$ is an $\varepsilon_k$-net of $\overline{U}$.

By induction on $k \in \mathbb{N}$, we are going to construct a sequence $(h_k)_k$ of mappings from $X = \mathbb{R}^n$ to $\mathbb{R}^m$ such that:

$$\text{for each } T \in F_k, \quad h_k^{(p)} \text{ is constant equal to } T \text{ on a nonempty open ball,}$$

$$\text{for each } k \geq 2, \quad \|(h_k - h_{k-1})^{(p)}\|_\infty \leq \varepsilon_{k-1} \text{ and } h_k^{(p)}(X) \subset U.$$

*Construction of* $h_1$. Since U is connected, for every $T \in F_1$, according to Lemma 2.8 there is a $C^p$-smooth mapping $g_T \colon \mathbb{R}^n \to \mathbb{R}^m$, with support in $B_{\mathbb{R}^n}$, so that there is $0 < \delta_T < 1$ with $g_T^{(p)} |_{\delta_T B_{\mathbb{R}^n}} \equiv T$ and $g_T^{(p)}(\mathbb{R}^n) \subset U$. Now we fix points $\{x_T : T \in F_1\}$ in $\mathbb{R}^n$ with the property that

$$\|x_R - x_S\| > 2, \quad \text{if } R, S \in F_1,\ R \neq S.$$

We then define

$$h_1(x) = \sum_{T \in \mathcal{F}_1} g_T(x - x_T), \quad x \in \mathbb{R}^n,$$

If $\delta_1 = \inf\{\delta_T \ ;\ T \in F_1\}$, for each $T \in F_1$, $h_1^{(p)}$ is constant equal to $T$ on the ball $B_{\mathbb{R}^n}(x_T, \delta_1)$. Since the mappings in the summand defining $h_1$ have disjoint supports, $h_1^{(p)}(\mathbb{R}^n)$ is the union of the sets $g_T^{(p)}(\mathbb{R}^n)$ for $T \in F_1$, and hence it is included in $U$.

*Construction of* $h_k$. Let us fix $k \geq 2$, $V \in \mathcal{F}_{k-1}$ and $S_V$ the associated point in $F_{k-1}$. Let us denote $F_k^V = F_k \cap V$ and consider $T \in F_k^V$. Since $V$ is connected, by Lemma 2.8, there is a $C^p$-smooth mapping $g_{T,V} \colon \mathbb{R}^n \to \mathbb{R}^m$, with support in $B_{\mathbb{R}^n}$, satisfying that $g_{T,V}^{(p)}$ is constant equal to $T - S_V$ on some nonempty open ball $W_{T,V}$ and $S_V + g_{T,V}^{(p)}(\mathbb{R}^n) \subset V$. Now, since $F_k^V$ is finite, by replacing the mappings $g_{T,V}$ with $x \to (\alpha_{T,V})^p g_{T,V}((x - x_{T,V})/\alpha_{T,V})$, for suitable $\alpha_{T,V} \in (0, 1)$ and $x_{T,V} \in \mathbb{R}^n$, we can assume that the supports of the mappings $\{g_{T,V} : T \in F_k^V, V \in \mathcal{F}_{k-1}\}$ are pairwise disjoint and that, for every $V \in \mathcal{F}_{k-1}$, the supports of the mappings $\{g_{T,V} : \ T \in F_k^V\}$ are included in a ball where the mapping $h_{k-1}$ is constant equal to $S_V$. Define

$$g_k(x) = \sum_{V \in \mathcal{F}_{k-1}} \sum_{T \in F_k^V} g_{T,V}(x) \quad \text{and} \quad h_k(x) = h_{k-1}(x) + g_k(x).$$

If $x \in W_{T,V}$ then $h_k^{(p)}(x) = h_{k-1}^{(p)}(x) + g_k^{(p)}(x) = S_V + (T - S_V) = T$. Clearly,

$$h_k^{(p)}(\mathbb{R}^n) = h_{k-1}^{(p)}(\mathbb{R}^n) \cup \Big( \bigcup_{V \in \mathcal{F}_{k-1}} \Big( \bigcup_{T \in F_k^V} S_V + g_{T,V}^{(p)}(\mathbb{R}^n) \Big) \Big) \subset U \cup \Big( \bigcup_{V \in \mathcal{F}_{k-1}} V \Big) = U.$$

As the supports of the mappings $g_{T,V}^{(p)}$, $T \in F_k^V$, $V \in \mathcal{F}_{k-1}$ are pairwise disjoint, $(h_k - h_{k-1})^{(p)}(\mathbb{R}^n) = g_k^{(p)}(\mathbb{R}^n)$ is the union of the sets $g_{T,V}^{(p)}(\mathbb{R}^n)$, so it is included in $V - S_V$, which is contained in $2\varepsilon_{k-1} B_{\mathcal{L}_s^p(\mathbb{R}^n,\mathbb{R}^m)}$. Therefore $\|(h_k - h_{k-1})^{(p)}\|_\infty < 2\varepsilon_{k-1}$.

Since the series of $p$-th derivatives $\sum_k \|(h_k - h_{k-1})^{(p)}\|_\infty$ converges and the mapping $h\colon \mathbb{R}^n \to \mathbb{R}^m$ defined as the sum $\sum_{k=1}^{\infty}(h_k - h_{k-1})$ (with $h_0 = 0$) has support within the union of the balls $B_{\mathbb{R}^n}(x_T, 1)$ for $T \in F_1$, it is clear that $h$ is $C^p$-smooth on $\mathbb{R}^n$. On the other hand, $h^{(p)}(\mathbb{R}^n)$ is a closed subset of $\overline{U}$ containing $\bigcup_k F_k$, and $\bigcup_k F_k$ is dense in $\overline{U}$, hence we have that $h^{(p)}(\mathbb{R}^n) = \overline{U}$. ■

***Remark 2.9*** The sufficient condition given in [6] to ensure the existence of a $C^1$-smooth and Lipschitzian bump $b\colon \mathbb{R}^n \to \mathbb{R}$ with $b'(\mathbb{R}^n) = \overline{U}$ implies that every point in the boundary of $U$ is the end point of a path of *finite length* within $U$. However, as we have already pointed out after Theorem 2.5, for every $n > 1$ there are examples of open sets in $\mathbb{R}^n$ satisfying the conditions required in Theorem 2.7 so that even uncountably many points of the boundary of $U$ cannot be end points of paths of finite length within $U$. Thus, Theorem 2.7 enlarges the class from [6] of subsets of $\mathbb{R}^n$ which are known to be the range of the derivative of a $C^1$-smooth bump.

## 3 The Case of Fréchet or Gâteaux Smooth Lipschitzian Mappings with Bounded Support

Let us consider now mappings from $\mathbb{R}^m$ to a separable infinite-dimensional Banach space $Y$. In [2], the authors constructed a Fréchet smooth and Lipschitzian mapping $f\colon \mathbb{R}^m \to Y$ with bounded support so that $f'(\mathbb{R}^m)$ contains $B_{\mathcal{L}_s^1(\mathbb{R}^m,Y)}$. Notice that since the support is compact, we cannot expect $f$ to be $C^1$-smooth. We present here an improvement of the quoted result based on the techniques developed in the preceding section.

***Theorem 3.1*** *Let $Y$ be an infinite-dimensional separable Banach space and let $m \in \mathbb{N}$. Let us consider an open bounded and connected subset $U \subset \mathcal{L}_s^1(\mathbb{R}^m, Y)$ containing the origin so that every point in the boundary of $U$ is the end point of a path within $U$. Then there are Fréchet smooth and Lipschitzian mappings $b, g\colon \mathbb{R}^m \to Y$ with bounded support so that $b'(\mathbb{R}^m) = U$ and $g'(\mathbb{R}^m) = \overline{U}$.*

**Proof** In order to prove the first assertion, notice that $\mathcal{L}_s^1(\mathbb{R}^m, Y)$ is separable. Let us fix a sequence $(\varepsilon_n)_n$ of positive numbers decreasing to 0. We deduce from Remark 2.2(i), following arguments similar to those given in the proof of Theorem 2.3, that there is a countable family $\{U_n\}$ of subsets of $U$ and a sequence of Fréchet

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smooth mappings $h_n\colon \mathbb{R}^m \to Y$ so that

$$\|h_n\|_\infty \le 1, \quad h_n'(\mathbb{R}^m) = U_n, \quad \bigcup_{n=1}^{\infty} U_n = U, \text{ and } \operatorname{supp} h_n \subset B_{\mathbb{R}^m}.$$

Now, replacing the mappings $h_n$ by $b_n(x) = r_n g_n(\frac{x-a_n}{r_n})$, $x \in X$, with suitable sequences of points $(a_n)_n$ in $\mathbb{R}^m$ and of positive numbers $(r_n)_n$, we may assume that

$$\|b_n\|_\infty \le \varepsilon_{n+1}^2, \quad b_n'(\mathbb{R}^m) = U_n, \quad \bigcup_{n=1}^{\infty} U_n = U, \text{ and } \operatorname{supp} b_n \subset \varepsilon_n B_{\mathbb{R}^m} \setminus \varepsilon_{n+1} B_{\mathbb{R}^m}.$$

Now, we define $b\colon \mathbb{R}^m \to Y$, as $b(x) = \sum_n b_n(x)$. Clearly $b$ is continuous and Fréchet smooth in $\mathbb{R}^m \setminus \{0\}$. An easy calculation shows that $b$ is Fréchet smooth at 0. Indeed, if $\varepsilon_{n+1} < |t| \le \varepsilon_n$ and $v$ is a norm one vector in $\mathbb{R}^m$, we have

$$\frac{|b(tv) - b(0)|}{|t|} = \frac{|b_n(tv)|}{|t|} \le \varepsilon_{n+1},$$

and this implies that $b$ is Fréchet smooth at 0 and $b'(0) = 0$.

Let us now prove the second assertion. Take $M > 0$ so that $\overline{U} \subset M B_{\mathcal{L}_s^1(\mathbb{R}^m,Y)}$. Take a dense sequence $(T_n)_n$ in $MB_{\mathcal{L}_s^1(\mathbb{R}^m,Y)}$ with $T_1 = 0$ and $T_n \neq 0$ for $n \ge 2$. Fix a summable sequence of positive numbers $(\varepsilon_n)_n$. We define $\mathcal{P}$ as the family of sequences $R = \{Q_1, Q_2, \dots\}$ satisfying

(a) $Q_i \in \{0, T_i\}$ and $\sum_{i\le n} Q_i \in U$ for every $n \in \mathbb{N}$.

(b) There exists $T \in \overline{U}$ so that $\|T - \sum_{i\le m_k} Q_i\| \le \varepsilon_k$, for some sequence $1 < m_1 < m_2 < m_3 < \cdots$ and $Q_i = 0$ whenever $i \notin \{m_1, m_2, m_3, \dots\}$. Moreover, we may assume that there is a path from $\sum_{i\le m_k} Q_i$ to $\sum_{i\le m_{k+1}} Q_i$ within $U \cap (T + \varepsilon_k B_{\mathcal{L}(\mathbb{R}^m,Y)})$.

Define

$$\mathcal{P}_n = \Big\{ R = \{Q_1, \dots, Q_n\} : \text{ there exists } P \in \mathcal{P} \text{ whose first } n \text{ points are } Q_1, \dots, Q_n \Big\}.$$

Notice that each set $\mathcal{P}_n$ is finite.

We obtain from Remark 2.2(i), as in the proof of Theorem 2.5, a family of Fréchet smooth mappings $h_R\colon \mathbb{R}^m \to Y$, $R \in \mathcal{P}_n, n \ge 2$, and a bounded family $\{x_R, R \in \mathcal{P}_n, n \in \mathbb{N}\}$ of points in $\mathbb{R}^m$ satisfying the following conditions:

(c) If $R \in \mathcal{P}_n$, then $\operatorname{supp} h_R \subset x_R + s_n B_{\mathbb{R}^m}$ and $\|h_R\|_\infty \le c_n$. If $R, S \in \mathcal{P}_n$ and $R \neq S$, the supports of $h_R$ and $h_S$ are disjoint. The sequences $(s_n)_n$ and $(c_n)_n$ of positive numbers decrease to 0 and satisfy the additional conditions which will be deduced later from inequalities (3.2) and (3.3).

(d) If $R = \{Q_1, \dots, Q_n\}$ and there are $k$ non-zero elements with $Q_n \neq 0$, then $\sup\{\|h_R'(x)\| : x \in \mathbb{R}^m\} \le \varepsilon_{k-1}$ (when $k \ge 2$), $\sum_{i\le n-1} Q_i + h_R'(\mathbb{R}^m) \subset U$ and $h_R(x) = Q_n(x - x_R)$ whenever $x \in x_R + \delta_n B_{\mathbb{R}^m}$. If $Q_n = 0$ then we assume that $g_R \equiv 0$.

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(e) If $R \in \mathcal{P}_{n+1}$, $S \in \mathcal{P}_n$ and $S < R$, then $\operatorname{supp} h_R \subset x_S + \delta_n B_{\mathbb{R}^m}$. (Here, and below, $S < R$ means that $R$ is a "right" extension of $S$.) The sequence $(\delta_n)_n$ satisfies that $\delta_n < \frac{s_n}{2}$ and the additional conditions which will be deduced below from inequalities (3.2) and (3.3).

(f) If $R, S \in \mathcal{P}_{n+1}$ and $R \neq S$, then the balls $B(x_R, s_{n+1})$ and $B(x_S, s_{n+1})$ are separated by the distance $\frac{\delta_n}{2}$.

Then we define for every $n \in \mathbb{N}$,

$$b_n = \sum_{R \in \mathcal{P}_n} h_R \quad \text{and} \quad b = \sum_n b_n.$$

Clearly $b$ is continuous and Fréchet smooth at $x \in \mathbb{R}^m$ whenever $x$ is not a point of accumulation of the set $\{x_R : R \in \mathcal{P}_n, n \in \mathbb{N}\}$. Let us check that $b$ is Fréchet smooth at any point $x \in \mathbb{R}^m$ and that $b'(\mathbb{R}^m) = \overline{U}$.

If $x = \lim_n x_{R_n}$, where $R_n = \{Q_1, \dots, Q_n\} \in \mathcal{P}_n$ and $R_1 < R_2 < R_3 < \cdots$, take $v \in \mathbb{R}^m$ of norm one, $N \in \mathbb{N}$ and $x + tv \in (x_{R_N} + s_N B_{\mathbb{R}^m}) \setminus (x_{R_{N+1}} + s_{N+1} B_{\mathbb{R}^m})$. Then

$$\Big\| \frac{b(x+tv) - b(x)}{t} - \sum_n Q_n(v) \Big\| = \Big\| \sum_{n \geq N} \frac{(h_{R'_n}(x+tv) - h_{R_n}(x))}{t} - \sum_{n \geq N} Q_n(v) \Big\|,$$

for suitable $R'_N < R'_{N+1} < \cdots$, $R'_i \in \mathcal{P}_i$ and $R_N = R'_N$. Thus,

$$\begin{aligned} (3.1) \quad & \Big\| \frac{b(x+tv) - b(x)}{t} - \sum_n Q_n(v) \Big\| \\ & \leq \Big\| \frac{h_{R_N}(x+tv) - h_{R_N}(x)}{t} - Q_N(v) \Big\| + \frac{1}{t} \sum_{n \geq N+1} (\|h_{R'_n}(x+tv)\| + \|h_{R_n}(x)\|). \end{aligned}$$

If $h_{R_N}$ is not identically zero, this implies that $Q_N \neq 0$. Assume that $Q_N$ is the $k$-th non-zero element of $R_N$. Since $\|h'_{R_N}\|_\infty \leq \varepsilon_{k-1}$, where $\varepsilon_0 = M$, we have that

$$\Big\| \frac{h_{R_N}(x+tv) - h_{R_N}(x)}{t} - Q_N(v) \Big\| \leq 2\varepsilon_{k-1}.$$

If $Q_N = 0$ the above summand is zero. Now, let us find a suitable upper bound for the second summand of (3.1). Since $x \in x_{R_n} + \delta_n B_{\mathbb{R}^m}$, we have $h_{R_n}(x) = Q_n(x - x_{R_n})$ for every $n \in \mathbb{N}$, and then

$$\frac{1}{t} \sum_{n \geq N+1} \|h_{R_n}(x)\| \leq \frac{1}{t} \sum_{n \geq N+1} \|Q_n\| \|x - x_{R_n}\| \leq \frac{M}{t} \sum_{n \geq N+1} \|x - x_{R_n}\|.$$

Since we are assuming that $x + tv \notin x_{R_{N+1}} + s_{N+1} B_{\mathbb{R}^m}$, we deduce from (e) that $|t| \geq s_{N+1} - \delta_{N+1} \geq s_{N+1} - \frac{s_{N+1}}{2} = \frac{s_{N+1}}{2}$ and thus

$$(3.2) \qquad \frac{2M}{s_{N+1}} \sum_{n \geq N+1} \|x - x_{R_n}\| \leq \frac{2M}{s_{N+1}} \sum_{n \geq N+1} \delta_n = \gamma(N+1).$$

Now it is possible to assume that we have selected the sequences $(s_n)_n$ and $(\delta_n)_n$ so that they satisfy $\lim_n \gamma(n) = 0$.

Finally, if the summand $\frac{1}{t}\sum_{n\geq N+1} \|h_{R'_n}(x+tv)\|$ is not zero, then $x+tv \in x_{R'_{N+1}} + s_{N+1}B_{\mathbb{R}^m}$. Since $R_{N+1} \neq R'_{N+1}$, we deduce from (f) that $|t| \geq \frac{\delta_N}{2}$ and then

$$\frac{1}{t}\sum_{n\geq N+1} \|h_{R'_n}(x+tv)\| \leq \frac{2}{\delta_N}\sum_{n\geq N+1} \|h_{R'_n}(x+tv)\| \leq \frac{2}{\delta_N}\sum_{n\geq N+1} c_n = \mu(N). \tag{3.3}$$

Again, it is possible to assume that we have selected the sequences $(\delta_n)_n$ and $(c_n)_n$ so that they satisfy $\lim_n \mu(n) = 0$. This proves that $b$ is Fréchet differentiable at $x$ and $b'(x) = \sum_i Q_i$.

Clearly, from the construction of $b$ we have that $b'(\mathbb{R}^m) \subset \overline{U}$. Now, if $T \in \overline{U}$ there exists $R = \{Q_1, Q_2, \dots\} \in \mathcal{P}$ so that $\sum_i Q_i = T$. Thus, the above implies that there exists $x \in \mathbb{R}^m$ so that $b'(x) = T$ and then $b'(\mathbb{R}^m) = \overline{U}$. ■

Finally, let us consider Gâteaux smooth mappings between two separable Banach spaces $X$ and $Y$. It was proved in [2] that when $X$ is infinite-dimensional, there exists a uniformly Gâteaux smooth Lipschitzian mapping $f\colon X \to Y$ with bounded support so that $f(X)$ contains $B_Y$ and $f'(X)$ contains $B_{\mathcal{L}^1_s(X,Y)}$. Next we construct $f$ so that the images of $f$ and $f'$ are exactly these two sets, that is to say, $f(X) = B_Y$ and $f'(X) = B_{\mathcal{L}^1_s(X,Y)}$.

Consider on $\mathcal{L}^1_s(X,Y)$ the topology $\tau$ of the pointwise convergence on $X$. It is well known that if $X$ and $Y$ are separable, then the topological space $(B_{\mathcal{L}^1_s(X,Y)}, \tau)$ is separable and every element of $(B_{\mathcal{L}^1_s(X,Y)}, \tau)$ has a countable basis of neighborhoods. We shall use the following lemma which is a slight modification of a result from [2]. We omit the proof of this lemma since it is straightforward.

***Lemma 3.2*** ([2]) *Let $X$ and $Y$ be separable Banach spaces and $(V_n)_n$ be a decreasing family of $\tau$-closed, convex and symmetric subsets of $2B_{\mathcal{L}^1_s(X,Y)}$ which is a base of neighborhoods of $0$ in $(2B_{\mathcal{L}^1_s(X,Y)}, \tau)$. Then, there is a sequence $(T_m)_m \subset 2B_{\mathcal{L}^1_s(X,Y)}$, and an increasing sequence of positive numbers $(\varepsilon_n)_n$ converging to $1$ with the property that for any $T \in B_{\mathcal{L}^1_s(X,Y)}$,*

$$T = \tau\text{-sum}\sum_k T_{m_k}, \text{ for some subsequence } (T_{m_k})_k. \tag{3.4}$$

*Moreover, the partial sums satisfy the stronger condition*

$$T - \sum_{i\leq k} T_{m_i} \in V_k \quad \text{and} \quad \|\sum_{i\leq k} T_{m_i}\| \leq \varepsilon_k, \quad k \in \mathbb{N}. \tag{3.5}$$

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***Proposition 3.3*** *Let $X$ and $Y$ be separable Banach spaces, where $X$ is infinite-dimensional. Then, there is a uniformly Gâteaux smooth and Lipschitzian mapping $f\colon X \to Y$ with bounded support so that $f(X) = B_Y$, $f'(X) = B_{\mathcal{L}^1_s(X,Y)}$ and $f'$ is $\|\cdot\|$-$\tau$ continuous.*

**Proof** We first construct uniformly Gâteaux smooth mappings $g$ and $h$ from $X$ to $Y$ with bounded supports so that $g(X) \subset B_Y$, $g'(X) = B_{\mathcal{L}^1_s(X,Y)}$ and $h(X) = B_Y$, $h'(X) \subset B_{\mathcal{L}^1_s(X,Y)}$. Then, the required mapping $f$ is defined as the sum of suitable translations with disjoint supports of $g$ and $h$, that is to say, $f(x) = g(x-a)+h(x-b)$ where $(a + \operatorname{supp} g) \cap (b + \operatorname{supp} h) = \varnothing$.

Since $X$ is separable, there exists an equivalent uniformly Gâteaux smooth norm $\|\cdot\|$ on $X$ [8, p. 68] and thus, by composing it with a suitable $C^\infty$-smooth bump $\gamma$ on $\mathbb{R}$, we get a uniformly Gâteaux smooth bump $\theta\colon X \to \mathbb{R}$, $\theta(x) = \gamma(\|x\|)$, with the properties $\theta(x) = 1$ for $\|x\| \le 1/2$, $\theta(x) = 0$ for $\|x\| \ge 1$ and $\theta(X) = [0, 1]$. Notice that $\sup_{x\in X} \|x\|\|\theta'(x)\| = M < \infty$ since $\operatorname{supp} \theta' \subset B_X \setminus \frac{1}{2}B_X$.

Let us denote by $\mathcal{P}$ the family of subsequences $P = \{Q_1, Q_2, \dots\}$ of $(T_n)_n$ satisfying conditions (3.4) and (3.5) and $\mathcal{P}_n = \big\{R = \{Q_1, \dots, Q_n\} : \text{ there exists } P \in \mathcal{P} \text{ whose first } n \text{ points are } Q_1, \dots, Q_n\big\}$. By induction on $n$, we shall construct a sequence $(g_n)_n$ of mappings from $X$ to $Y$ so that $g = \sum_n g_n$ as follows.

*First step: Definition of $g_n$ and $g$.* Let us consider for every $R = \{Q_1, \dots, Q_n\} \in \mathcal{P}_n$ the straight line $L$ from $S_{n-1} := \sum_{k\le n-1} Q_k$ to $S_n := \sum_{k\le n} Q_n$ (from 0 to $Q_1$ if $n = 1$). By assumption, there is $0 < s_n < 1$ (for instance $s_n = 1 - \varepsilon_n$) so that $L + s_n B_{\mathcal{L}(X,Y)} \subset B_{\mathcal{L}(X,Y)}$. Take $m_n \in \mathbb{N}$ with $\max\{2^{n+2}, \frac{2}{s_n}(1 + M)\} \le m_n$. Let us define

$$g_R(x) = \sum_{i=1}^{m_n} \frac{Q_n(x)}{m_n}\theta(2^{i-1} x), \quad x \in X.$$

Then, $\|g_R\|_\infty \le \frac{1}{2^n}$. Also, if we put by $g_{R,i}(x) = \frac{Q_n(x)}{m_n}\theta(2^{i-1}x), x \in X,\ i = 1, \dots, m_n$, then

$$\|g'_{R,i}(x)\| \le \frac{\|Q_n\|}{m_n} + \frac{\|Q_n\|}{m_n}\|2^{i-1}x\|\|\theta'(2^{i-1}x)\| \le \frac{2}{m_n}\big(1 + M\big) \le s_n$$

and

$$g'_R(X) \subset \bigcup_{i=1}^{m_n} Q_n \frac{i}{m_n} + s_n B_{\mathcal{L}(X,Y)}. \tag{3.6}$$

Therefore,

$$S_{n-1} + g'_R(X) \subset \bigcup_{i=1}^{m_n}\big(S_{n-1} + Q_n\frac{i}{m_n} + s_n B_{\mathcal{L}(X,Y)}\big) \subset B_{\mathcal{L}^1_s(X,Y)}. \tag{3.7}$$

Notice that $g'_R$ is a constant equal to $\sum_{k\le n} Q_n$ in $2^{-m_n}B_X$, if $R = \{Q_1, Q_2, \dots, Q_n\}$.

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Let us consider a bounded family of points $\{x_R : R \in \mathcal{P}_1\}$ with $\|x_R - x_S\| > 2$ whenever $R \neq S$ and replace $g_R$ by $x \mapsto g_R(x - x_R)$. Define

$$g_1(x) = \sum_{R \in \mathcal{P}_1} g_R(x), \quad x \in X.$$

Let us define $r_1 = 1$ and $\delta_1 = \frac{1}{2^{m_1}}$. Then, supp $g_1$ is included in the union of the disjoint balls $\{B(x_R, r_1) : R \in \mathcal{P}_1\}$ and $g_1'$ is constant equal to $Q_1$ in $B(x_R, \delta_1)$, if $R = \{Q_1\} \in \mathcal{P}_1$.

In general, for $n \geq 2$, we consider $0 < \delta_n < r_n < 1$ and points $\{x_R : R \in \mathcal{P}_n\}$ so that replacing the mappings $g_R$ by $x \to r_n g_R(\frac{x - x_R}{r_n})$, for every $R \in \mathcal{P}_n$, we can assume that the support of the mapping $g_R$ is included in $x_S + \delta_{n-1} B_X$, if $S < R$ and $S \in \mathcal{P}_{n-1}$, that is to say $x_R + r_n B_X \subset x_S + \delta_{n-1} B_X$, and also $g_R'$ is constant equal $Q_n$ in $B(x_R, \delta_n)$ if $R = \{Q_1, \dots, Q_n\} \in \mathcal{P}_n$. We also assume that $B(x_R, r_n) \cap B(x_{R'}, r_n)$, if $R \neq R'$ and $R, R' \in \mathcal{P}_n$. Let us define for $n \geq 2$

$$g_n(x) = \sum_{R \in \mathcal{P}_n} g_R(x) \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} g_n(x), \quad x \in X.$$

Obviously, $g$ is continuous since the series $\sum_{n=1}^{\infty} g_n$ converges uniformly in $X$.

*Second step: The mapping g is uniformly Gâteaux smooth.* Let us check first that $g$ is Gâteaux smooth. Fix $x \in X$, $v \in B_X$ and consider

$$\varphi \colon [-1, 1] \to Y, \quad \varphi(t) = g(x + tv),$$
$$\varphi_n \colon [-1, 1] \to Y, \quad \varphi_n(t) = g_n(x + tv).$$

Then $\varphi(t) = \sum_n \varphi_n(t)$, for $|t| \leq 1$. Let us prove that the series of gradients $\sum_n \varphi_n'(t)$ is uniformly convergent for $|t| \leq 1$. If $y = x + tv = \lim_n x_{R_n}$ for some sequence $R_1 < R_2 < \cdots < R_n \in \mathcal{P}_n$, then $g_n'$ is constant equal $Q_n$ in a neighborhood of $y$ and we have

$$\sum_{n \geq 1} \varphi_n'(t) = \sum_{n \geq 1} Q_n(v). \tag{3.8}$$

Otherwise, there is $k \in \mathbb{N}$ and $R_1' < \cdots < R_k' = \{Q_1', \dots, Q_n'\}$, where $R_i' \in \mathcal{P}_i$ so that

$$\sum_{n \geq 1} \varphi_n'(t) = \sum_{n=1}^{k-1} Q_n'(v) + g_k'(x + tv)(v) \tag{3.9}$$

Therefore for $N \in \mathbb{N}$ we deduce from inclusion (3.6) that,

$$(3.10)\quad \Big\|\sum_{n\geq N}\varphi_n'(t)\Big\| = \Big\|\sum_{n\geq N} Q_n(v)\Big\|, \qquad \text{in the case (3.8),}$$

$$(3.11)\quad \Big\|\sum_{n\geq N}\varphi_n'(t)\Big\| \leq \Big\|\sum_{n=N}^{k-1} Q_k'(v)\Big\| + \|Q_k'(v)\| + s_k, \qquad \text{in the case (3.9)}$$

(3.12) (assuming that $k \geq N$, and 0 otherwise.)

Inequalities (3.10) and (3.11) imply that $\sum_{n\geq N}\varphi_n'(t)$ tends to 0 as $N \to \infty$, uniformly on $[-1, 1]$. Thus $\varphi$ is differentiable and $\varphi'(t) = \sum_n \varphi_n'(t)$. In particular, $\varphi'(0) = \sum_n g_n'(x)(v)$. Since this can be done for every $v$ in $B_X$, we deduce that $g$ is Gâteaux smooth at $x$ and $g'(x) = \sum_{n\geq 1} g_n'(x)$, where this sum is considered in the $\tau$ topology. It is easy to check that the derivative $g'$ is $\|\cdot\|$-$\tau$ continuous as well.

The uniform Gâteaux smoothness of $g$ can be proved as in [2]. Let us give here the proof for completeness. Fix $v \in B_X$. Consider for every $n \in \mathbb{N}$ the mapping $G_n\colon X \to Y$ defined as

$$G_n(x) = g_n'(x)(v), \quad x \in X.$$

On the one hand, it is straightforward to verify that the mapping $G_n$ is uniformly continuous (this is a consequence of the uniform Gâteaux smoothness of the norm $\|\cdot\|$ since then $x \to \|\cdot\|'(x)(v)$ is uniformly continuous outside a ball containing the origin [8, p. 61]). On the other hand, it can be deduced from inequalities (3.10), (3.11) and the strong property (3.5) related to the "directional uniform" convergence of the series $\sum_n Q_n$, that the series $\sum_n G_n$ is uniformly convergent on $X$. In particular, this implies that the limit mapping of the series $G\colon X \to Y$, $G(x) = \sum_n G_n(x) = g'(x)(v)$ is uniformly continuous on $X$. If $v$ ranges over all elements of $B_X$, we obtain that $g$ is uniformly Gâteaux smooth.

From the inclusion (3.7) and the expression of $g'$ we have that $g'(X) \subset B_{\mathcal{L}_s^1(X,Y)}$. Since for any $T \in \mathcal{B}_{\mathcal{L}_s^1(X,Y)}$ there is a sequence $(R_n)_n$, $R_n = \{Q_1, \dots, Q_n\} \in \mathcal{P}_n$ with $\sum_{n\geq 1} Q_n = T$, we obtain that $g'(X)$ fills in $B_{\mathcal{L}_s^1(X,Y)}$. Moreover, since the image of $g$ is bounded, we may assume, replacing $g$ by $x \to rg(x/r)$ that $g(X) \subset B_Y$.

*Third step: Construction of the mapping h.* Consider a dense sequence $(y_n)_n$ in the unit sphere of $Y$ and for every $n \in \mathbb{N}$ the family

$$(3.13)\quad \mathcal{P}_n = \Big\{\sigma = (\sigma(1), \dots, \sigma(n)) \in \mathbb{N}^n : \Big\|\sum_{i=1}^{k} \varepsilon^{2(i-1)} y_{\sigma(i)}\Big\| \leq 1, \quad \text{for } k = 1, \dots, n\Big\}.$$

Fix $0 < \varepsilon < \frac{1}{12}$ and select a bounded family $\mathcal{F} = \{x_\sigma : \sigma \in \mathcal{P}_n, n \in \mathbb{N}\}$ in $X$ satisfying the following conditions:

(1) if $\sigma \neq \sigma'$, $\sigma, \sigma' \in \mathbb{N}^n$, then $\|x_\sigma - x_{\sigma'}\| > 3\,\varepsilon^{n-1}$,

(2) if $\sigma \in \mathbb{N}^n$, $\sigma' \in \mathbb{N}^{n+1}$ and $\sigma < \sigma'$, then $\|x_\sigma - x_{\sigma'}\| = \frac{\varepsilon^{n-1}}{4}$.

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(Here $\sigma < \sigma'$ means that $\sigma'$ is a right extension of $\sigma$.)

Then, for every $n$ and $x \in X$, we define $h_n \colon X \to Y$

$$h_n(x) = \sum_{\sigma \in \mathcal{P}_n} \varepsilon^{2(n-1)} y_{\sigma(n)} \theta\Big(\frac{x - x_\sigma}{\varepsilon^{n-1}}\Big)$$

and $h \colon X \to Y$, $h(x) = \sum_n h_n(x)$. It is easy to check that the mapping $h$ is continuous, has bounded support, is uniformly Gâteaux smooth and has bounded derivative. Moreover, $h(X) = B_Y$. Indeed, for every $y \in B_Y$ there is a sequence $(\sigma_n)_n$ where $\sigma_n \in \mathcal{P}_n$ and $\sigma_n < \sigma_{n+1}$ so that $y = \sum_{n \geq 1} \varepsilon^{2(n-1)} y_{\sigma(n)}$. Then, if $x = \lim_n x_{\sigma_n}$, we have $h(x) = y$. Also from condition (3.13) we have that $h_n(X) \subset B_Y$. Finally, if we replace $h$ by $x \mapsto h(rx)$ we may assume that $h(X) = B_Y$ and $h'(X) \subset B_{\mathcal{L}_s^1(X,Y)}$. ■

***Remark 3.4*** It is worth observing that, by a slight modification of the proofs of Propositions 3.1 and 3.3, we obtain the following result:

*Let $X$ and $Y$ be separable Banach spaces and $X$ infinite-dimensional. Let us consider a bounded open subset $U \subset \mathcal{L}_s^1(X, Y)$, $M > 0$ so that $U \subset MB_{\mathcal{L}_s^1(X,Y)}$, and let $(V_n)_n$ be a decreasing family of $\tau$-closed, convex and symmetric subsets of $2MB_{\mathcal{L}_s^1(X,Y)}$ which form a basis of neighborhoods of $0$ in $(MB_{\mathcal{L}_s^1(X,Y)}, \tau)$. Assume that there exists a sequence $(Q_n)_n$ in $U$ so that for every $T \in \overline{U}$, there is a subsequence $(Q_{n_i})_i$ with*

(1) *$T - \sum_{i=1}^{k} Q_{n_i} \in V_k$, for every $k \in \mathbb{N}$,*

(2) *the linear segment between $S_k := \sum_{i=1}^{k} Q_{n_i}$ and $S_{k+1} := \sum_{i=1}^{k+1} Q_{n_i}$ (between $0$ and $Q_{n_1}$ if $k = 0$) is included in $U$, for every $k = 0, 1, 2, \ldots$.*

*Then, there is a continuous and Gâteaux smooth mapping $f \colon X \to Y$ with bounded support so that $f'(X) = \overline{U}$.*

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