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CONVOLUTION THEOREMS OF TITCHMARSH TYPE ON DISCRETE $\mathbf{R}^n$

by YNGVE DOMAR

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0. Summary

This paper contains results related to Titchmarsh's convolution theorem and valid for $\mathbf{R}_d^n$, the additive group of $\mathbf{R}^n$ with the discrete topology. The method of proof consists in transferring the problem to $\mathbf{R}^n$ with the usual topology by a procedure which has been used earlier, for instance in Helson [3].

In Section 1, the classical support theorems are generalized to $\mathbf{R}_d^n$. In [1], Titchmarsh's convolution theorem [6] on $\mathbf{R}$ was generalized to convolutions of functions belonging to certain weighted $\mathbf{L}^p$-spaces on $\mathbf{R}$. Section 2 contains a corresponding generalization to weighted $l^2(\mathbf{R}_d)$.

It should be observed that convolutions of elements f and g in $l^1(\mathbf{R}_d^n)$ can be interpreted as convolutions of bounded discrete measures on $\mathbf{R}^n$. Hence, in that case the support theorem (Theorem 4.33 of Hörmander [5]) is directly applicable to give the results of our Theorems 1 and 3. So the novelty in our theorems lies in the fact that they apply for instance to the case when it is only assumed $f,g \in l^2(\mathbf{R}_d^n)$, together with support conditions. It is not known whether it suffices to assume $f \in l^1(\mathbf{R}_d^n), g \in l^p(\mathbf{R}_d^n)$, when $p > 2$.

1. Theorems on the convex hull of the support

The points on the vector space $\mathbf{R}^n$ are denoted $\lambda = (\lambda_1, \dots, \lambda_n)$. On $\mathbf{R}^n$ we will alternate between the usual and the discrete topology. In the latter case the space is denoted $\mathbf{R}_d^n$. The space $\mathbf{R}_d^n$ is a discrete group under addition, and its dual group is $b\mathbf{R}^n$, the Bohr compactification of $\mathbf{R}^n$. For $t \in \mathbf{R}^n$, e_t denotes the element of $b\mathbf{R}^n$, which corresponds to the character $\lambda \mapsto \exp(i\lambda t)$ on $\mathbf{R}_d^n$. The set of elements e_t forms a Borel measurable subgroup $\mathbf{R}_o^n$ of $b\mathbf{R}^n$, since $t \mapsto e_t$ is continuous. The Fourier transformation $\mathscr{F}$ from $\mathbf{R}_d^n$ to $b\mathbf{R}^n$ is formally defined by

$$\hat{f}(x) = \mathscr{F} f(x) = \sum_{\lambda \in \mathbf{R}_d^n} f(\lambda)\langle x, \lambda \rangle, \; x \in b\mathbf{R}^n,$$

with the inverse

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$$f(\lambda)=\mathscr{F}^{-1}\hat{f}(\lambda)=\int_{b\mathbf{R}^n}\hat{f}(x)\langle x,-\lambda\rangle dm(x),\ \lambda\in\mathbf{R}_d^n,$$

where m is the normalized Haar measure on the compact group $b\mathbf{R}^n$. For convenience we assume in the sequel that all functions in $L^1(b\mathbf{R}^n)$ are chosen Borel measurable.

Let $\hat{f}\in L^1(b\mathbf{R}^n)$. Then $(x,t)\mapsto\hat{f}(x+e_t)$ is a Borel function on $b\mathbf{R}^n\times\mathbf{R}^n$. We put

$$\hat{f}(x+e_t)=\hat{f}_x(t),$$

and observe that $\hat{f}_x$ is Borel measurable on $\mathbf{R}^n$ for every $x\in b\mathbf{R}^n$. Taking into account the invariance of m, Fubini's theorem gives

$$\int_{b\mathbf{R}^n}\int_{\mathbf{R}^n}|\hat{f}_x(t)|(1+|t|)^{-n-1}\,dt\,dm(x)=\int_{b\mathbf{R}^n}|\hat{f}(x)|\,dm(x)\int_{\mathbf{R}^n}(1+|t|)^{-n-1}\,dt<\infty, \tag{1}$$

where dt stands for integration with respect to the n-dimensional Lebesgue measure. Hence there is a $\mathbf{R}_0^n$-invariant Borel measurable set E with $mE=1$ such that

$$\int_{\mathbf{R}^n}|\hat{f}_x(t)|(1+|t|)^{-n-1}\,dt<\infty, \tag{2}$$

whenever $x\in E$. Thus, if $x\in E$, then $\hat{f}_x\in\mathscr{S}'(\mathbf{R}^n)$, the Schwartz space of tempered distributions, and $\hat{f}_x$ has an inverse Fourier transform $f_x\in\mathscr{S}'(\mathbf{R}^n)$. In this paper it is convenient to use the relation

$$\phi(\lambda)=\int_{\mathbf{R}^n}\hat{\phi}(t)e^{-i\lambda t}\,dt,\ \lambda\in\mathbf{R}^n, \tag{3}$$

as the formal definition of inverse Fourier transformation on $\mathbf{R}^n$.

In the following, for any subset F of $\mathbf{R}^n$ or $\mathbf{R}_d^n$, $\operatorname{ch} F$ denotes its convex hull with respect to the basic vector space $\mathbf{R}^n$ and $\bar{F}$ denotes the closure of F in the topology of $\mathbf{R}^n$. For any complex-valued f on $\mathbf{R}_d^n$,

$$\operatorname{supp} f=\{\lambda\in\mathbf{R}_d^n,\ f(\lambda)\neq 0\},$$

while the support for functions or distributions on $\mathbf{R}^n$ is defined in the sense of distributions.

Lemma 1. *Let* $\hat{f}\in L^1(b\mathbf{R}^n)$, $f=\mathscr{F}^{-1}\hat{f}$. *Then*

$$\operatorname{supp} f_x=\overline{\operatorname{supp} f},$$

for almost every $x\in b\mathbf{R}^n$.

Proof. For any $\hat{\phi}\in\mathscr{S}(\mathbf{R}^n)$, the function

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$$\hat{g}(x)=\int_{\mathbf{R}^n}\hat{\phi}(t)\hat{f}(x-e_t)\,dt \tag{4}$$

is defined on E, the set where (2) holds. Fubini's theorem shows that $\hat{g}\in L^1(b\mathbf{R}^n)$ and

$$g=\mathscr{F}^{-1}\hat{g}=\phi f, \tag{5}$$

with ϕ defined by (3). For $x\in E$, (4) gives

$$\hat{g}_x(s)=\int_{\mathbf{R}^n}\hat{\phi}(t)\hat{f}_x(s-t)\,dt,\ s\in\mathbf{R}^n,$$

and hence $\hat{g}_x$ is continuous, and its distributional inverse Fourier transform g_x satisfies

$$g=\phi f_x. \tag{6}$$

It follows from (1), applied to $\hat{g}$, that $\hat{g}_x=0$ for almost every $x\in E$ if and only if $\hat{g}=0$ almost everywhere on $b\mathbf{R}^n$. Hence (5) and (6) show that

$$\phi f_x=0 \text{ for almost every } x\in E \text{ if and only if } \phi f=0. \tag{7}$$

Let us define $\psi(x)$ as 0, if $g_x=0$, and as 1 elsewhere on $b\mathbf{R}^n$. Then $\psi\in L^1(b\mathbf{R}^n)$, and is constant on the cosets of $\mathbf{R}^n_0$. By a known device (see for instance the proof of Theorem 9 in Helson [3]) this implies that ψ is constant almost everywhere on $b\mathbf{R}^n$. Hence the set where g_x vanishes has either measure 0 or 1, and we can conclude from (6) and (7) that

$$\phi f_x\neq 0 \text{ for almost every } x\in E \text{ if (and only if) } \phi f\neq 0. \tag{8}$$

The lemma follows easily from (7) and (8) by varying ϕ in a suitable denumerable subset of $\mathscr{D}(\mathbf{R}^n)$.

Definition 1. *If $f=\mathscr{F}^{-1}\hat{f}$, $g=\mathscr{F}^{-1}\hat{g}$, with $\hat{f}$, $\hat{g}$, $\hat{f}\hat{g}\in L^1(b\mathbf{R}^n)$, we define convolution $f*g$ of f and g by*

$$f*g=\mathscr{F}^{-1}(\hat{f}\hat{g}).$$

Theorem 1. *Let $f=\mathscr{F}^{-1}\hat{f}$, $g=\mathscr{F}^{-1}\hat{g}$, with $\hat{f}$, $\hat{g}$, $\hat{f}\hat{g}\in L^1(b\mathbf{R}^n)$, and with* $\operatorname{supp} f$ *and* $\operatorname{supp} g$ *bounded. Then*

$$\overline{\operatorname{ch}\operatorname{supp} f*g}=\overline{\operatorname{ch}\operatorname{supp} f}+\overline{\operatorname{ch}\operatorname{supp} g}.$$

Proof. By Lemma 1 we have, for almost every $x\in b\mathbf{R}^n$,

$$\hat{f}_x\in\mathscr{S}'(\mathbf{R}^n),\ \operatorname{supp} f_x=\overline{\operatorname{supp} f},$$

$$\hat{f}_x\in\mathscr{S}'(\mathbf{R}^n),\ \operatorname{supp} g_x=\overline{\operatorname{supp} g},$$

$$\hat{f}_x\hat{g}_x \in \mathscr{S}'(\mathbf{R}^n),\ \operatorname{supp}(f*g)_x = \overline{\operatorname{supp} f*g}.$$

Hence f_x and g_x have compact support, for these values of x. But then the Titchmarsh support theorem in $\mathbf{R}^n$ (see for instance Hörmander [**5**, Theorem 4.3.3]) implies that

$$\text{ch supp}\, f_x * g_x = \text{ch supp}\, f_x + \text{ch supp}\, g_x,$$

and it remains to prove that

$$f_x * g_x = (f*g)_x. \tag{9}$$

Here $f_x * g_x$ is, of course, convolution in ordinary distribution sense. Since f_x and g_x have compact support, $\hat{f}_x$ and $\hat{g}_x$ are continuous almost everywhere on $\mathbf{R}^n$ and

$$\mathscr{F}(f_x * g_x) = \hat{f}_x\hat{g}_x$$

The Fourier transform of the right hand member of (9) is by Definition 1

$$(\hat{f}\hat{g})_x = \hat{f}_x\hat{g}_x,$$

and (9) is proved.

Parseval's relation shows that Definition 1 is applicable in the case when $f, g \in l^2(\mathbf{R}^n_d)$, and that

$$f*g(\lambda) = \sum_{\nu \in \mathbf{R}^n_d} f(\lambda - \nu)g(\nu),\ \lambda \in \mathbf{R}^n_d. \tag{10}$$

We have then the following more precise theorem.

Theorem 2. *Let $f, g \in l^2(\mathbf{R}^n_d)$, with* supp f *and* supp g *bounded. Then*

$$\text{ch supp}\, f*g = \text{ch supp}\, f + \text{ch supp}\, g.$$

Proof. Since Theorem 1 holds, it is enough to discuss the points on the boundary (with respect to the topology of $\mathbf{R}^n$) of ch supp $f*g$. This is done by induction in n. For $n=1$, the theorem is an obvious consequence of (10). So let us assume that $n \geqq 2$ and that the theorem is true for the dimension $n-1$. Let P be any support hyperplane of ch supp$(f*g)$, and let P_1 and P_2 be the corresponding parallel support hyperplanes of ch supp f and ch supp g, respectively, such that $P = P_1 + P_2$. (We have here used Theorem 1.) Denote by f', g' and $(f*g)'$ the functions obtained by multiplying f, g and $f*g$ with the characteristic functions of P_1, P_2 and P, respectively. (10) shows that

$$(f*g)' = f' * g',$$

and the induction assumption gives easily

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$$(\operatorname{ch}\operatorname{supp} f * g) \cap P = \operatorname{ch}\operatorname{supp}(f * g)' = \operatorname{ch}\operatorname{supp} f' + \operatorname{ch}\operatorname{supp} g' = (\operatorname{ch}\operatorname{supp} f + \operatorname{ch}\operatorname{supp} g) \cap P.$$

By varying P we obtain the theorem.

In the case $n=1$, we have the following theorem, which is slightly more general than Theorem 1.

Theorem 3. *Let $f=\mathscr{F}^{-1}\hat{f}$, $g=\mathscr{F}^{-1}\hat{g}$, with $\hat{f},\hat{g},\hat{f}\hat{g}\in L^1(b\mathbf{R})$, and with* $\operatorname{supp} f$ *and* $\operatorname{supp} g$ *bounded from below. Then*

$$\inf\operatorname{supp} f * g = \inf\operatorname{supp} f + \inf\operatorname{supp} g.$$

Proof. It is a known fact (Hoffman [**4**, pp. 132–133]), that for a function $\hat{k}$ in $H^1(\mathbf{R})$, the exponential function in the product representation of the extension of $\hat{k}$ to the upper half-plane, determines $\inf\operatorname{supp} k$, with k defined in accordance with (3). It follows easily from this that if $\hat{\alpha}, \hat{\beta}, \hat{\gamma}=\hat{\alpha}\hat{\beta}\in L^1(\mathbf{R})$, then

$$\inf\operatorname{supp}\gamma = \inf\operatorname{supp}\alpha + \inf\operatorname{supp}\beta, \tag{11}$$

if the terms to the right are $>-\infty$. By (2) and Lemma 1, the assumptions of the theorem imply that (11) holds with

$$\hat{\alpha}(t)=\hat{f}_x(t)(i+t)^{-2},\ \hat{\beta}(t)=\hat{g}_x(t)(i+t)^{-2},$$

for almost every x. Easy considerations show that this implies

$$\inf\operatorname{supp}(f * g)_x = \inf\operatorname{supp} f_x + \inf\operatorname{supp} g_x,$$

for almost every x, and then Lemma 1 gives the desired result.

Remark. In the case $f\in l^2(\mathbf{R}_d), g\in l^2(\mathbf{R}_d)$ Theorem 1 is a consequence of Helson's theory of cocycles [3]. (See Helson [**2**, p. 480].)

2. Generalized Titchmarsh theorems

Let $\Omega\subseteq\mathbf{R}^n_d$, with Ω open in the topology of $\mathbf{R}^n$. If f is a function on Ω such that, for every $K\subseteq\Omega$ with K compact in $\mathbf{R}^n$, f coincides on K with a function in $\mathscr{F}^{-1}L^1$, we say with a slight abuse of language that f is in $\mathscr{F}^{-1}L^1$ locally on Ω. If $g^1, g^2\in\mathscr{F}^{-1}L^1$, and if both g^1 and g^2 coincide with f on K, Lemma 1 applied to g^1-g^2 shows that g^1_x and g^2_x coincide on the interior of K, for almost every x. Hence it is possible to extend the mappings $f\to f_x$, for almost every x, in Section 1, to mappings from the family of functions locally in $\mathscr{F}^{-1}L^1$ on Ω to $\mathscr{D}'(\Omega)$ in such a way that the relation $g=f$ on an open set $\Omega'\subseteq\Omega$, implies that $f_x=g_x$ on Ω', for almost every x. The following lemma is then an obvious extension of Lemma 1.

Lemma 2. *Let f be locally $\mathscr{F}^{-1}L^1$ on Ω. Then*

$$\operatorname{supp} f_x = \overline{\operatorname{supp} f},$$

for almost every $x \in b\mathbf{R}^n$.

We will in the following assume that $n=1$. Let w be a decreasing positive function on $\mathbf{R}_d$. We define

$$w_1(\lambda) = 1/w(-\lambda),\ \lambda \in \mathbf{R}_d.$$

$l^2_w(\mathbf{R}_d)$ is the space of all f with $fw \in l^2(\mathbf{R}_d)$. For every complex-valued f on $\mathbf{R}_d, f^n$ denotes the product of f and the characteristic function of $(n, n+1]$.

Let $f \in l^2_w(\mathbf{R}_d)$, $g \in l^2_{w_1}(\mathbf{R}_d)$. Then, by Parseval's relation,

$$\sum_{n\in\mathbf{Z}} \int_{b\mathbf{R}} |\widehat{f^n}(x)|^2\, dx\, w(n+1)^2 = \sum_{n\in\mathbf{Z}} \sum_{\lambda\in\mathbf{R}_d} |f^n(\lambda)|^2 w(n+1)^2 < \infty, \tag{12}$$

and

$$\sum_{n\in\mathbf{Z}} \int_{b\mathbf{R}} |\widehat{g^n}(x)|^2\, dx\, w_1(n+1)^2 = \sum_{n\in\mathbf{Z}} \sum_{\lambda\in\mathbf{R}_d} |g^n(\lambda)|^2 w_1(n+1)^2 < \infty, \tag{13}$$

(12) and (13) show that (2) holds with $\hat{f}$ replaced by any of the functions

$$\sum_{n\in\mathbf{Z}} |\widehat{f^n}|^2 w(n+1)^2 \text{ and } \sum_{n\in\mathbf{Z}} |\widehat{g^n}|^2 w_1(n+1)^2,$$

and we obtain, for almost every x,

$$\sum_{n\in\mathbf{Z}} \int_{\mathbf{R}} |(\widehat{f^n})_x(t)|^2 (1+|t|)^{-2}\, dt\, w(n+1)^2 < \infty, \tag{14}$$

$$\sum_{n\in\mathbf{Z}} \int_{\mathbf{R}} |(\widehat{g^n})_x(t)|^2 (1+|t|)^{-2}\, dt\, w_1(n+1)^2 < \infty. \tag{15}$$

The Schwartz inequality shows that $(f*g)(\lambda)$ is well defined by (10), if $\lambda \geqq 0$. Let $N \in \mathbf{Z}$, $N \geqq 4$. (12) and (13) show that

$$h = \sum_{n\in\mathbf{Z}} (f^{n-2} * g^{N-n} + f^{n-1} * g^{N-n} + f^n * g^{N-n})$$

belongs to $\mathscr{F}^{-1}L^1$, and

$$h = f * g, \text{ on } (N-1, N+1]. \tag{16}$$

By (14) and (15) we have, for almost every x,

$$h_x = \sum_{n\in\mathbf{Z}} \{(f^{n-2})_x * (g^{N-n})_x + (f^{n-1})_x * (g^{N-n})_x + (f^n)_x * (g^{N-n})_x\},$$

with convergence in distribution sense. Note that Lemma 1 shows that

$$\operatorname{supp}(f^m)_x \subseteq [m, m+1], \operatorname{supp}(g^m)_x \subseteq [m, m+1],$$

for every $m \in \mathbf{Z}$ and almost every x.

Let $\phi \in \mathscr{D}(\mathbf{R})$, with $\operatorname{supp}\phi \leqq [0, 1/3]$. Then an easy calculation, using (16), shows that, for almost every x,

$$h_x * \phi * \phi = (f_x * \phi) * (g_x * \phi), \tag{17}$$

on $(N-1/3, N+1)$. The right hand member of this equality is well defined on $[2, \infty)$, since (14) and (15) show that

$$\int_{\mathbf{R}} |f_x * \phi(\lambda)|^2 w(\lambda+1)^2 \, d\lambda < \infty \tag{18}$$

and

$$\int_{\mathbf{R}} |g_x * \phi(\lambda)|^2 w_1(\lambda+1)^2 \, d\lambda < \infty. \tag{19}$$

Since $N \geqq 4$ was arbitrary, we have the following conclusion of (17) and Lemma 2.

Lemma 3. *Let $f \in l^2_w(\mathbf{R}_d)$, $g \in l^2_{w_1}(\mathbf{R}_d)$, and*

$$f * g(\lambda) = 0, \ \lambda \geqq 0.$$

Then, for almost every x, (18) and (19) hold, and for every $\phi \in \mathscr{D}(\mathbf{R})$, with $\operatorname{supp}\phi \subseteq [0, 1/3]$,

$$(f_x * \phi) * (g_x * \phi)(\lambda) = 0, \text{ for } \lambda \geqq 4.$$

We are now in a position to prove the following theorem.

Theorem 4. *Suppose that $\log w$ is convex in $(-\infty, 0]$ and concave in $[0, \infty)$, and that*

$$\varliminf_{\lambda \to -\infty} \frac{\log w(\lambda)}{|\lambda|^a} > 0, \ \varlimsup_{\lambda \to \infty} \frac{\log w(\lambda)}{\lambda^b} < 0, \tag{20}$$

where $a > 1$, $b > 1$, $1/a + 1/b = 1$, and where at least one of the limits is infinite. Let $f \in l^2_w(\mathbf{R}_d)$, $g \in l^2_{w_1}(\mathbf{R}_d)$, both not identically vanishing. If

$$f * g(\lambda) = 0, \quad \text{for all} \quad \lambda \leqq 0,$$

then $\inf \operatorname{supp} f > -\infty$, $\inf \operatorname{supp} g > -\infty$.

Proof. By Lemma 3 we obtain, for almost every x, that

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$$(f_x * \phi) * (g_x * \phi)(\lambda) = 0,$$

for $\lambda \geqq 4$, if $\phi \in \mathscr{D}(\mathbf{R})$, $\operatorname{supp}\phi \leqq [0, 1/3]$, and that (18) and (19) hold. By Theorem 1 of [**1**],

$$\operatorname{supp}(f_x * \phi) \text{ and } \operatorname{supp}(g_x * \phi)$$

are bounded from below unless one of the sets is empty. This implies, by varying ϕ, that $\inf\operatorname{supp} f_x$ and $\inf\operatorname{supp} g_x$ are finite, for almost every x. Hence the same holds, by Lemma 2, for $\inf\operatorname{supp} f$ and $\inf\operatorname{supp} g$.

Theorem 5. *If* $\inf\operatorname{supp} f > -\infty$, *the conclusion of Theorem 4 holds with (20) changed to the weaker condition*

$$\lim_{\lambda\to\infty} \frac{\log|\log w(\lambda)| - \log\lambda}{\sqrt{\log\lambda}} = \infty.$$

Proof. Here we apply instead Theorem 2 of [**1**] in the preceding proof.

Let us form the space $l_w^2(\mathbf{R}_d^+)$ of all f on $\mathbf{R}_d^+$ with $fw \in l^2(\mathbf{R}_d^+)$. Both $l_w^2(\mathbf{R}_d)$ and $l_w^2(\mathbf{R}_d^+)$ are Hilbert spaces. For $a \geqq 0$, (right) translation T_a is defined by

$$T_a f(\lambda) = f(\lambda - a),$$

if $f \in l^2(\mathbf{R}_d)$, while

$$T_a f(\lambda) = \begin{cases} f(\lambda - a), \ \lambda \geq a, \\ 0, \ 0 \leqq \lambda < a, \end{cases}$$

if $f \in l^2(\mathbf{R}_d^+)$. T_a is a contraction, if we assume that w decreases. $l_w^2(\mathbf{R}_d)$ or $l_w^2(\mathbf{R}_d^+)$ is called *unicellular*, if all closed translation-invariant subspaces are of the form

$$\{f : f(x) = 0, \text{ if } x \leqq b\} \text{ or } \{f : f(x) = 0, \text{ if } x < b\}.$$

Theorem 3 is trivially extendable to arbitrary functions which belong locally to l^2 and have supports bounded from below. By this and Theorems 4 and 5 (cf. the discussion on p. 299 of [**1**]) we find easily the following.

Theorem 6. $l_w^2(\mathbf{R})$ *and* $l_w^2(\mathbf{R}_d^+)$ *are unicellular, if* w *satisfies the assumptions of Theorem 4 and Theorem 5, respectively.*

Remark. It is not known whether the results in this paper can be extended to convolutions of functions which are locally in l^p and l^q, where $p \neq 2$, and p and q are conjugate exponents. It would be of particular interest to know whether Theorem 3 holds if the assumption $\hat{f}, \hat{g}, \hat{f}\hat{g} \in l^1(b\mathbf{R})$ is changed to

$$f \in l^1(\mathbf{R}_d),\ g \in l^\infty(\mathbf{R}_d),$$

or to the stronger assumption

$$f \in l^1(\mathbf{R}_d),\ g \in c_0(\mathbf{R}_d).$$

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Department of Mathematics
Uppsala University
Thunbergsvagen 3, S-752 38
Uppsala, Sweden