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# EXTENSION OF RIESZ HOMOMORPHISMS. I

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**Abstract**

In this paper we characterize boundedly laterally complete Riesz spaces, boundedly laterally complete Riesz spaces with the lateral boundedness property and Riesz spaces in which every principal ideal is finite dimensional. The characterizations are given in terms of extension properties of certain Riesz homomorphisms.

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## 1. Introduction

In the theory of Riesz spaces and in particular in the theory of Banach lattices much has been done in the area of extending positive linear maps (see for instance [4], [5], [7], [9], [12], [14] and [19]). In contrast, relatively little is known about extending Riesz homomorphisms. One reason for this is that Riesz homomorphisms simply do not occur as often as positive linear maps.

Results about extending Riesz homomorphisms can be found in [1] (Theorem 23.16), [6] (Theorems 17B and 17C), [13], [16], [17] (Theorem 19.9) and [21].

In [7] it is proved that weak $\sigma$-distributivity of $\sigma$-Dedekind complete Riesz spaces can be characterized completely by means of extension of certain positive linear maps. An inspection of the methods in [7] yields that we need only consider Riesz homomorphisms for the characterization of weak $\sigma$-distributivity.

In this paper we will characterize other classes of Riesz spaces in terms of extension properties of Riesz homomorphisms. Where in [7] the main difficulty is

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the preservation of a continuity condition which is natural with respect to the measure theory behind it, in this paper the continuity conditions are mild and in fact necessary to get any Riesz homomorphic extension at all. In this way quite different results are obtained.

Our standard reference books for the general theory of vector lattices will be [11] and [15]. However, we will assume that all Riesz spaces considered are Archimedean.

This paper is part of a thesis [3] written by the author under supervision of Professor Dr. A. C. M. Van Rooij.

## 2. The extension problem

The extension problem considered in this paper is the following.

> Suppose $E$, $F$ are Riesz spaces, $I \subset E$ is an ideal and $\varphi\colon I \to F$ is a Riesz homomorphism. Does there exist a Riesz homomorphism $\Phi\colon E \to F$ such that $\Phi|_I = \varphi$?

An investigation of Theorem 61.4 in [15] tells us that, without restrictions on $\varphi$, even if $F = \mathbf{R}$ the answer may be 'no', unless $E$ is of a particularly simple form (Theorem 0.8.4 in [3]). Some continuity condition is necessary to get more positive answers. Basically we will consider two different continuity restrictions on $\varphi$ which are defined next.

DEFINITION 2.1. Suppose we have three Riesz spaces $I$, $E$ and $F$ and $I$ is an ideal in $E$. Assume furthermore that $\varphi\colon I \to F$ is a Riesz homomorphism. We say that $\varphi$ is *$o(I, E, F)$-continuous* (or in short, *$o$-continuous*) if for all $f \in E^+$ $\{\varphi(g) \mid g \in I, 0 \leqslant g \leqslant f\}$ is order bounded.

DEFINITION 2.2. Suppose we have three Riesz spaces $I$, $E$ and $F$ and $I$ is an ideal in $E$. Assume furthermore that $\varphi\colon I \to F$ is a Riesz homomorphism. We say that $\varphi$ is *$c(I, E, F)$-continuous* (or in short, *$c$-continuous*) if for all sequences $(f_n)_{n\in\mathbf{N}}$ in $I$ with $f_n \to 0$ relatively uniformly in $E$ we have $\varphi(f_n) \to 0$ relatively uniformly in $F$.

It is not difficult to prove that each $o$-continuous map is $c$-continuous and that not every $c$-continuous map is $o$-continuous. We will say that a pair of Riesz spaces $(E, F)$ has property $(oI)$ (respectively $(cI)$) if for every ideal $I \subset E$ and

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every $o$-continuous (respectively $c$-continuous) Riesz homomorphism $\varphi\colon I \to F$ there exists a Riesz homomorphism $\Phi\colon E \to F$ such that $\Phi|_I = \varphi$.

We now come to the key definition in this paper.

DEFINITION 2.3. A Riesz space $F$ is said to be an *o-extensor* if $(E, F)$ has property $(oI)$ for all Riesz spaces $E$; $F$ is said to be a *c-extensor* if $(E, F)$ has property $(cI)$ for all Riesz spaces $E$.

## 3. $o$-Extensors

We recall from [1] that a Riesz space is said to be laterally complete if the supremum of every disjoint subset of its positive cone exists. Similarly, a Riesz space $F$ is said to be boundedly laterally complete if the supremum of every order bounded disjoint subset of $F^+$ exists. In this section we will prove that the class of all boundedly laterally complete Riesz spaces coincides with the class of all $o$-extensors. Before stating and proving that $o$-extensors are boundedly laterally complete we introduce some notations.

For any infinite set $\Omega$ we define $k(\Omega)$ to be the space of those functions $f\colon \Omega \to \mathbf{R}$ for which there are a finite subset $\Delta$ of $\Omega$ and a real number $c$ such that $f|_{\Omega\setminus\Delta} = c$. The number $c$ naturally belonging to an element $f$ of $k(\Omega)$ will be written as $f(\infty)$. Furthermore $c_{00}(\Omega) = \{ f \in k(\Omega) \mid f \equiv 0 \text{ outside a finite set}\}$.

PROPOSITION 3.1. *Every o-extensor is boundedly laterally complete.*

PROOF. Suppose $F$ is an $o$-extensor. Suppose furthermore, that $A = \{ f_\lambda \mid \lambda \in \Lambda \}$ is a disjoint set in $F^+$ majorized by $g \in F^+$. If $\Lambda$ is a finite set, clearly $\sup A$ exists. Therefore, assume $\Lambda$ is infinite. The set of disjoint subsets of $F^+$ which are disjoint with $A$ is a partially ordered set in which each chain has an upper bound. According to Zorn's lemma there exists a set $B = \{ f_\mu \mid \mu \in \Gamma \}$ which is maximal with the property of being disjoint with $A$.

Define $\varphi\colon c_{00}(\Lambda \cup \Gamma) \to F$ by $\varphi(f) = \Sigma_{\lambda\in\Lambda} f(\lambda) f_\lambda + \Sigma_{\mu\in\Gamma} f(\mu) f_\mu$. Evidently $\varphi$ is a Riesz homomorphism.

Define $S\colon k(\Lambda) \oplus c_{00}(\Gamma) \to F$ by $S(f, f') = \Sigma_{\lambda\in\Lambda}(f(\lambda) - f(\infty)) f_\lambda + f(\infty) g + \Sigma_{\mu\in\Gamma} f'(\mu) f_\mu$. Because $g \geqslant f_\lambda$ for all $\lambda \in \Gamma$ it follows that $S$ is a positive linear map. Because $S|_{c_{00}(\Lambda\cup\Gamma)} = \varphi$, $\varphi$ is $o(c_{00}(\Lambda \cup \Gamma), k(\Lambda) \oplus c_{00}(\Gamma), F)$-continuous. Therefore, there exists a Riesz homomorphism $\Phi\colon k(\Lambda) \oplus c_{00}(\Lambda) \to F$ which extends $\varphi$.

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Define $f_0 = \Phi(1_\Lambda, 0) \wedge g$. Remark that $f_0 \geqslant f_\lambda$ for all $\lambda \in \Lambda$. Because $\Phi$ is a Riesz homomorphism it follows that for all $\lambda \in \Lambda$ we have $|\Phi(1_\Lambda, 0) - 2f_\lambda| = |\Phi(1_\Lambda - 21_\lambda, 0)| = \Phi(1_\Lambda, 0)$ and moreover $(\Phi(1_\Lambda, 0) - \Phi(1_\lambda, 0)) \wedge f_\lambda = 0$ for all $\lambda \in \Lambda$. We claim that $f_0$ is the supremum of $A$.

For suppose it is not and let $f_1$ be an element of $F$ such that $f_1 \geqslant f_\lambda$ for all $\lambda \in \Lambda$, while $f_0 \wedge f_1 < f_0$. Because $(f_0 - f_1)^+ \wedge f_\lambda \leqslant (f_0 - f_\lambda) \wedge f_\lambda \leqslant (\Phi(1_\Lambda, 0) - \Phi(1_\lambda, 0)) \wedge f_\lambda = 0$ for all $\lambda \in \Lambda$, we find $(f_0 - (f_0 \wedge f_1)) \wedge f_\lambda = 0$ for all $\lambda \in \Lambda$. This contradicts the maximality of $B$ because $f_0 - (f_0 \wedge f_1)$ is not in $B$ (use the fact that $\Phi$ is a Riesz homomorphism) and non-zero.

It is not difficult at all to show that every Dedekind complete Riesz space is an $o$-extensor. In fact the extension formula as used in 17B of [6] yields the result immediately. It is well known that a Riesz space is Dedekind complete if and only if it is boundedly laterally complete as well as uniformly complete (see [2] or [20]). The problem in proving the converse of Proposition 3.1 is the absence of uniform completeness. However, every boundedly laterally complete Riesz space does have the projection property (see [20]) and this turns out to be very helpful.

A Riesz space is said to be *universally complete* if it is both laterally complete and Dedekind complete. A universal completion $G$ of a Riesz space $F$ is a universally complete Riesz space containing $F$ as an order dense Riesz subspace, or more precisely, having an order dense Riesz subspace which is Riesz isomorphic to $F$. For every Riesz space $F$ there exists a universal completion and every two universal completions of $F$ are Riesz isomorphic. (A nice proof of the existence of universal completions can be found in [23].) Suppose $F$ is a Riesz space and $G$ is the universal completion of $F$. By the Maeda-Ogasawara representation theorem there exists an extremally disconnected compact Hausdorff space $X$ such that $G$ and $C^\infty(X)$ are Riesz isomorphic (where $C^\infty(X)$ is the set of all extended real continuous functions $f$ on $X$ with the property that $\{x \in X \mid |f(x)| < \infty\}$ is dense in $X$). We identify the image of $F$ under this isomorphism with $F$. For each $f \in C^\infty(X)$ define $W_f = \{x \in X \mid \text{there exists a neighborhood } u_x \text{ of } x \text{ and } g_x \in F \text{ such that } f|_{u_x} = g_x|_{u_x}\}$. In this situation we have the following lemma.

LEMMA 3.2. *Suppose $F$ is boundedly laterally complete. If $f \in C^\infty(X)^+$ is dominated by $g \in F^+$ and $W_f$ is dense in $X$, then $f \in F$.*

PROOF. Let us assume that $F$ is boundedly laterally complete. We will first prove that, if $g \in F^+$ and $U \neq \varnothing$ is an open subset of $X$, then there exists a clopen subset $V$ of $U$, $V \neq \varnothing$, such that $g1_V \in F$. Let $U \subset X$ be open and $g \in F^+$. Because $F$ is order dense in $C^\infty(X)$ there exists an $h \in F$ such that

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$0 < h \leqslant 1_U$. Define $V = \mathrm{clo}\{y \in X \mid h(y) \neq 0\}$ and notice that $g1_V$ is the image of $g$ under the projection onto the band generated by $h$ in $C^\infty(X)$. From the fact that $f$ itself has the projection property combined with the order denseness of $F$ in $C^\infty(X)$ we infer that $g1_V \in F$.

Now suppose $f \in C^\infty(X)^+, f \leqslant g \in F^+$ and $W_f$ is dense in $X$. Choose $\mathscr{A}$ to be a maximal collection of clopen pairwise disjoint subsets of $X$ such that $f1_U \in F$ for all $U \in \mathscr{A}$. Using the result of the first part of this proof it follows that $\bigcup_{U \in \mathscr{A}} V$ is dense in $X$. Define $h = F\text{-}\sup\{f1_U \mid U \in \mathscr{A}\}$. Because $F$ is a normal sublattice of $C^\infty(X)$ (see Lemma 13.21 of [11]) we find that

$$h = C^\infty(X)\text{-}\sup\{f1_U \mid U \in \mathscr{A}\}.$$

Thus $h = f$ on a dense subset of $X$ and thus $h = f \in F$.

We can now prove the converse of Proposition 3.1.

PROPOSITION 3.3. *Every boundedly laterally complete Riesz space is an o-extensor.*

PROOF. Suppose $F$ is bounded laterally complete. Let $E$ be a Riesz space, $I \subset E$ an ideal and $\varphi\colon I \to F$ an $o$-continuous Riesz homomorphism. Let $X$ be an extremally disconnected compact Hausdorff space such that $F$ is an order dense Riesz subspace of $C^\infty(X)$. Take any $f \in E^+$. By the $o$-continuity of $\varphi$ we know that $\{\varphi(j) \mid j \in [0, f] \cap I\}$ is order bounded in $F$ and hence in $C^\infty(X)$. Define for each $x \in X$, $g(x) = \sup\{\varphi(j)(x) \mid j \in [0, f] \cap I\}$. The preceding remark shows that $\{x \in X \mid g(x) < \infty\}$ is the complement of a meagre set and even that almost every point of $X$ has a neighbourhood on which $g$ is bounded. Define $X_1 = \{x \in X \mid \text{there exists a neighbourhood of } x \text{ on which } g \text{ is bounded}\}$ and $X_0 = \{x \in X \mid \text{there exists } j \in I^+ \text{ such that } \varphi(j)(x) > 0\}$.

Clearly, $X_0$ is open and $g \equiv 0$ on $X \setminus \mathrm{clo}(X_0)$. Furthermore $X_0 \cup (X \setminus \mathrm{clo}(X_0))$ is dense in $X$. We will now show that for $\bar{\varphi}(f) = C^\infty(X)\text{-}\sup\{\varphi(j) \mid j \in [0, f] \cap I\}$ we have $W_{\bar{\varphi}(f)} \supset (X_0 \cap (X \setminus \mathrm{clo}(X_0))) \cap X_1$. Therefore, suppose a $X_0 \cap X_1$. Choose $j_0 \in I$, $j_0 \geqslant 0$ such that $\varphi(j_0)(a) \neq 0$. By multiplying $j_0$ with a large enough scalar we may assume that for a clopen neighbourhood $W$ of $a$ we have $\varphi(j_0)1_W > \sup\{g(x) \mid x \in W\}1_W$. Now $f \wedge j_0 \in I \cap [0, f]$. Using this, the particular choice of $j_0$ and because pointwise suprema are less than or equal to suprema, we derive the following:

For all $x \in W$,

$$\begin{aligned} g(x) &\geqslant [\varphi(f \wedge j_0)](x) \geqslant \Big[\sup_{j \in [0,f] \cap 1} \varphi(j \wedge j_0)\Big](x) \\ &\geqslant \varphi(j_0)(x) \wedge \sup_{j \in I \cap [0,f]} [\varphi(j)(x)] \geqslant g(x); \text{ thus } a \in W_{\bar{\varphi}(f)}. \end{aligned}$$

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If we take $a \in X \setminus \operatorname{clo}(X_0) \cap X_1$ then it follows more simply that $a \in W_{\bar{\varphi}(f)}$ because we have $g|_{X \setminus \operatorname{clo}(X_0)} = 0$. Also, $\bar{\varphi}(f)$ is dominated by an upper bound of $\{\varphi(j) \mid j \in I \cap [0, f]\}$ in $F$. Lemma 3.2 tells us now that $\bar{\varphi}(f) \in F$. Because $I \subset E$ is an ideal we find that $f \mapsto \bar{\varphi}(f)$ ($f \in E^+$) is additive. We leave it to the reader to extend $f \mapsto \bar{\varphi}(f)$ ($f \in E^+$) to a Riesz homomorphism $\Phi$: $E \to F$ such that $\Phi|_I = \varphi$ by means of Lemma 2.10 in [11].

We have now proved the following characterization of boundedly laterally complete Riesz spaces.

THEOREM 3.4. *For a Riesz space $F$ the following are equivalent.*
(1) *$F$ is boundedly laterally complete.*
(2) *$F$ is an o-extensor.*

We close this section with the remark that an example of a space which is boundedly laterally complete but not Dedekind complete is the space of all equivalence classes of bounded, countably valued and Lebesgue measurable functions on [0, 1].

## 4. $c$-Extensors

As we remarked before, every $o$-continuous Riesz homomorphism is $c$-continuous. As a result every $c$-extensor is an $o$-extensor and thus boundedly laterally complete. Our first step in the search for $c$-extensors is a lemma proving that every universally complete Riesz space is a $c$-extensor.

LEMMA 4.1. *Suppose $X$ is an extremally disconnected compact Hausdorff space. Then $C^\infty(X)$ is a c-extensor.*

*Furthermore, let $I \subset E$ be an ideal and $\Phi$: $I \to C^\infty(X)$ a $c(I, E, C^\infty(X))$ continuous Riesz homomorphism. Denote $A = \{x \in X \mid$ there exists $h \in I$ such that $\varphi(h)(x) \neq 0\}$. Then there exists a Riesz homomorphism $\Phi$: $E \to C^\infty(X)$ such that $\Phi|_I = \varphi$ and for all $f \in E^+$ there is a collection of clopen subsets $\{V_t \mid t \in T\}$ of such that*
(1) *$V_t \cap V_{t'} = \varnothing$ if $t \neq t'$,*
(2) *For each $t \in T$ there exists $h_t \in I \cap [0, f]$ with $1_{V_t} \leqslant \varphi(h_t)$,*
(3) *For all $t \in T$ there exists $n \in \mathbf{N}$ such that $\Phi(f)|_{V_t} = \varphi(f \wedge nh_t)|_{V_t}$,*
(4) *$\bigcup_{t \in T} V_t$ is a dense subset of $A$,*
(5) *$\Phi(f)|_{\bar{A}^c} = 0$.*

PROOF. Suppose $I \subset E$ is an ideal and $\varphi: I \to C^\infty(X)$ is a $c(I, E, C^\infty(X))$-continuous Riesz homomorphism. We define $A = \{x \in X \mid \text{there exists } g \in I \text{ such that } \varphi(g)(x) \neq 0\}$. The collection of all sets of characteristic functions in the ideal generated by $\varphi(I)$ in $C^\infty(X)$ is partially ordered by inclusion. Every chain in it has a maximal element. Choose, by Zorn's lemma, a maximal disjoint set $\{1_{U_s} \mid s \in S\}$ in the ideal generated by $\varphi(I)$ in $C^\infty(X)$.

Suppose $x \in A \setminus \bigcup_{s \in S} U_s$ and $V$ is an open subset of $X$ containing $x$ but disjoint from $\bigcup_{s \in S} U_s$. Then we can find $g \in I$ such that $\varphi(g)(x) > 1$ which easily leads to a contradiction with the maximality of $\{1_{U_s} \mid s \in S\}$. This shows that $\bigcup_{s \in S} U_s$ is a dense subset of $A$ and hence $\overline{\bigcup_{s \in S} U_s} = \bar{A}$.

For every $s \in S$ choose $h_s \in I^+$ such that $1_{U_s} \leqslant \varphi(h_s)$. If $f \in E^+$ and $x \in X$ we define $l_f(x) = \sup\{\varphi(g)(x) \mid g \in [0, f] \cap I\}$ and we remark that $l_f(x) \in [0, \infty]$.

Let $s \in S$. On $U_s$ we can give a somewhat more tractable formula for $l_f$:

$$l_f(x) = \sup\{\varphi(f \wedge nh_s)(x) \mid n \in \mathbf{N}\} \quad \text{for all } x \in U_s.$$

Indeed, take $x \in U_s$. If $l_f(x) = \infty$ then

$$l_f(x) \wedge \sup\{n\varphi(h_s)(x) \mid n \in \mathbf{N}\} = \sup\{n\varphi(h_s)(x) \mid n \in \mathbf{N}\} = \infty = l_f(x).$$

If $l_f(x) < \infty$ then $l_f(x) \wedge \sup\{n\varphi(h_x)(x) \mid n \in \mathbf{N}\} = l_f(x)$ also. This means that in either case

$$\begin{aligned} l_f(x) &= l_f(x) \wedge \sup\{n\varphi(h_s)(x) \mid n \in \mathbf{N}\} \\ &= \sup\{\varphi(g)(x) \wedge n\varphi(h_s)(x) \mid n \in \mathbf{N}, g \in I \cap [0, f]\} \\ &= \sup\{\varphi(g \wedge nh_x)(x) \mid n \in \mathbf{N}, g \in I \cap [0, f]\} \\ &\leqslant \sup\{\varphi(f \wedge nh_s)(x) \mid n \in \mathbf{N}\} \leqslant l_f(x) \end{aligned}$$

for all $x \in U_s$.

We know that $n^{-1/2}(f \wedge nh_s) \to 0$ relatively uniformly with respect to $f$. Because $\varphi$ is $c$-continuous it follows that $n^{-1/2}\varphi(f \wedge nh_s) \leqslant \varepsilon_n F_s$ with $\varepsilon_n \to 0$ and $F_s \in C^\infty(X)$. We will now prove that

$$l_f(y) < \infty \text{ and } l_f \text{ is continuous at } y \text{ if } y \in U_s \text{ and } F_s(y) < \infty.$$

Therefore, take $y \in U_s$ such that $F_s(y) < \infty$. Then for all $g \in [0, f] \cap I$ we see that $[n^{-1/2}\varphi(g) \wedge n^{1/2}](y) \leqslant [n^{-1/2}\varphi(g) \wedge n^{1/2}\varphi(h_s)](y) \leqslant \varepsilon_n F_s(y) \to 0$. In this manner we find an $n_0 \in \mathbf{N}$ such that for all $g \in [0, f] \cap I$ we have $n_0^{-1/2}\varphi(g)(y) \leqslant n_0^{1/2}$, i.e. $\varphi(g)(y) \leqslant n_0$ and $l_f(y) < \infty$. Instead of doing this work at the point $y$ only, we can do the same on a neighbourhood $V$ of $y$ on which $F_s$ is bounded and which contains points of $U_s$ only. Suppose $V$ is a neighbourhood of $y$ as in the preceding line. We then find an $n_0 \in \mathbf{N}$ such that for all $g \in I \cap [0, f]$ and for all $y' \in V$, $n_0^{-1/2}\varphi(g)(y') \leqslant n_0^{1/2}$. Thus

$$(**)\qquad \begin{aligned} l_f(y') &= \sup\{\varphi(g)(y') \wedge k\varphi(h_s)(y') \mid k \in \mathbf{N}, g \in I \cap [0, f]\} \\ &= \sup\{[\varphi(g) \wedge n_0\varphi(h_s)](y') \mid g \in I \cap [0, f]\} \\ &\leqslant \varphi(f \wedge n_0 h_s)(y') \leqslant l_f(y'). \end{aligned}$$

We infer that $l_f\mid_V = \varphi(f \wedge nh_s)\mid_V$ for some $n \in \mathbf{N}$. In particular, $l_f$ is continuous at $y$. We now define $B = \{x \in X \mid \text{there exists } s \in S \text{ such that } F_s(x) < \infty \text{ and } x \in U_s\}$.

$B$ is an open dense subset of $A$ and by the preceding arguments $l_f$ is continuous on $B$. Thus we can extend $l_f$ to a continuous function $f^* \in C^\infty(X)$ such that $f^*\mid_{\bar{A}^c} = 0$. Because $\varphi(g)\mid_B \leqslant f^*\mid_B = l_f\mid_B$ for $g \in I \cap [0, f]$, it follows that $f^* \geqslant \varphi(g)$ for all $g \in I \cap [0, f]$. Also, if $h \geqslant \varphi(g)$ for all $g \in I \cap [0, f]$, then $h\mid_B \geqslant f^*\mid_B$ and hence $h \geqslant f^*$. Therefore $f^* = \sup\{\varphi(G) \mid g \in I \cap [0, f]\}$. By applying Lemma 2.10 in [11] again, we extend the map $f \mapsto f^*$ $(f \in E^+)$ to a Riesz homomorphism $\Phi$: $E \to C\infty(X)$ such that $\Phi\mid_I = \varphi$. So far we have proved that $C^\infty(X)$ is a $c$-extensor.

For the second part of the lemma, we take the Riesz homomorphism $\Phi$ which has been constructed above. Suppose $f \in E^+$. To prove (5) we refer to the construction. With the aid of Zorn's lemma once more, we take a maximal collection of clopen subsets $\{V_t \mid t \in T\}$ of $X$ with (1), (2) and (3). The proof of (4), in using the arguments that led to the statement $(**)$ above and the maximality $\{V_t \mid t \in T\}$, is left to the reader.

The important definition in this section is the following.

Definition 4.2. A Riesz space $F$ is said to have the *lateral boundedness property* if a disjoint set $B \subset F^+$ is order bounded whenever for any sequence $(f_n)_{n \in \mathbf{N}}$ of elements of $B$ and any sequence $(a_n)_{n \in \mathbf{N}}$ of positive real numbers decreasing to zero, $a_n f_n \to 0$ relatively uniformly.

Certainly, $C^\infty(X)$ has the lateral boundedness property. Therefore, the following proposition is a generalization of Lemma 4.1.

Proposition 4.3. *If a Riesz space $F$ is boundedly laterally complete and has the lateral boundedness property, then $F$ is a c-extensor.*

PROOF. Suppose we are dealing with the following instance of the extension problem:

$$\begin{array}{ccc} I & \subset & E \\ \varphi \downarrow & & \downarrow \Phi \\ F & \subset & C^\infty(X) \end{array}$$

In this diagram $F$ is a Riesz space which is boundedly laterally complete and has the lateral boundedness property, $C^\infty(X)$ is its universal completion, $I$ and $E$ are Riesz spaces, $I$ is an ideal in $E$ and $\varphi$ is a Riesz homomorphism. Denote again $A = \{x \in X \,|\, \text{there exists } f \in I \text{ such that } \varphi(f)(x) \neq 0\}$. Let $\Phi$ be the extension of $\varphi$ which was produced in Lemma 4.1, such that for all $f \in E$ we have $\Phi(f)|_{\bar{A}^c} = 0$. Take any $f \in E^+$. Choose a collection of clopen subsets $\{V_t|: t \in T\}$ of $X$ and a set of functions $\{h_t \,|\, t \in T\} \subset I^+$ such that (1)–(5) of Lemma 4.1 are valid.

For all $t \in T$ we define $f_t = f^* 1_{V_t}$ (where $f^* = \Phi(f)$). First of all, for all $t \in T$, $f_t \in F$ by the following reasoning. If $y \in \operatorname{clo}\{x \,|\, f_t(x) > 0\}$ then, according to (**) above there are a neighbourhood $U_y$ of $y$ and $g_y \in F$ such that $f_t(y)|_{U_y} = g_y|_{U_y}$. Moreover, if $y \in (\operatorname{clo}\{x \,|\, f_t(x) > 0\})^c$ then more easily we can find such $U_y$ and $g_y$. According to Lemma 3.2, $f_t \in F$. We will now apply the lateral boundedness property to show that $\{f_t \,|\, t \in T\}$ is order bounded. Therefore, suppose $(\alpha_i)_{i \in \mathbf{N}}$ is a sequence of real numbbers decreasing to zero. Let $(f_{t_i})_{i \in \mathbf{N}}$ be a sequence of elements of $\{f_t \,|\, t \in T\}$.

$$(*) \qquad \alpha_i f_{t_i} = \alpha_i f^* 1_{V_{t_i}} = \alpha_i \varphi\big(f \wedge n_{t_i} h_{t_i}\big) 1_{V_{t_i}} \leqslant \varphi\big(\alpha_i\big(f \wedge n_{t_i} h_{t_i}\big)\big).$$

As $\alpha_i(f \wedge n_{t_i} h_{t_i}) \to 0$ relatively uniformly in $E$ (with respect to $f$), we know by $c$-continuity of $\varphi$ that $\varphi(\alpha_i(f \wedge n_{t_i} h_{t_i})) \to 0$ relatively uniformly in $F$, and because of (*), $\alpha_i f_{t_i} \to 0$ relatively uniformly in $F$. The lateral boundedness property now shows that $\{f_t \,|\, t \in T\}$ is order bounded in $F$ and because $F$ is boundedly laterally complete $f^{**} = \sup\{f_t \,|\, t \in T\}$ exists in $F$. Certainly $f^{**}|_B \geqslant f^*|_B$ where $B$ is as in the proof of Lemma 4.1. We will not repeat the calculations of Lemma 4.1 to show that $W_{f^*}$ is dense in $X$. It follows that $f^*$ is in $F$ by Lemma 3.2, because $f^*$ is dominated by $f^{**}$. Thus, $\Phi(E) \subset F$.

We will now prove the main theorem of this section.

THEOREM 4.4. *Suppose $F$ is any Riesz space. Then the following are equivalent.*
(1) *$F$ is a $c$-extensor.*
(2) *$F$ is boundedly laterally complete and has the lateral boundedness property.*

PROOF. (2) $\Rightarrow$ (1) has been proved in Proposition 4.3. Conversely, suppose $F$ is a $c$-extensor. It follows that $F$ is an $o$-extensor and thus boundedly laterally complete by Proposition 3.1. The remaining thing to do is to prove that $F$ has the lateral boundedness property. Therefore, suppose that $\{f_s \,|\, s \in S\} \subset F^+$ is a disjoint set such that for every sequence $(f_n)_{n\in\mathbf{N}}$ of elements of $\{f_s \mid s \in S\}$ and for any sequence $(\alpha_n)_{n\in\mathbf{N}}$ of positive real numbers decreasing to zero, $\alpha_n f_n \to 0$ relatively uniformly.

Define $c_0(S) = \{f\colon S \to \mathbf{R} \mid \text{ for each } \varepsilon > 0, \{s \in S \mid |f(s)| > \varepsilon\} \text{ is finite}\}$. Suppose $a \in c_0(S)^+$ and $\operatorname{supp}(a)(= \{s \in S \mid a(s) \neq 0\}) = \{s_1, s_2, \ldots\}$. Let $\alpha_i = a(s_i)$ $(i \in \mathbf{N})$,. Because $\alpha_i f_{s_i} \to 0$ relatively uniformly in $F$, $\{\alpha_i f_{s_i} \mid i \in \mathbf{N}\}$ is order bounded. Because $F$ is boundedly laterally complete, $\sup\{\alpha_i f_{s_i} \mid i \in \mathbf{N}\}$ exists.

Therefore, it is easy to define a Riesz homomorphism $\varphi\colon c_0(S) \to F$ such that $\varphi(a) = \sup\{a(s)f_s \mid s \in S\}$ $(a \in c_0(S)^+)$. Suppose $a_n \in c_0(S)$ for all $n \in \mathbf{N}$ and $a_n \to 0$ with respect to $b \in l^\infty(S)$. Then there exist a sequence of positive numbers $(\varepsilon_n)_{n\in\mathbf{N}}$ which converges to zero such that $|a_n| \leqslant \varepsilon_n b$. Now define $a$: $S \to \mathbf{R}$ by $a(s) = \sup_{n\in\mathbf{N}}(1/\sqrt{\varepsilon_n})a_n(s)$ and prove by the pigeon hole principle that $a \in c_0(S)$. It follows that $a_n \to 0$ relatively uniformly with respect to $a$. Thus $\varphi$ is $c(c_0(S), l^\infty(S), F)$-continuous and can be extended to a Riesz homomorphism $\Phi\colon l^\infty(S) \to F$. Surely, $f_s \leqslant \Phi(1_S)$ for all $s \in S$. Thus, $\{f_s \mid s \in S\}$ is order bounded in $F$.

Of course, the lateral boundedness property is the lateral version of the well known boundedness property (see page 51 in [18]). For that reason, as every Riesz space with the boundedness property also has the lateral boundedness property, it is a corollary of the Propositions 5.13 and 5.14 in [18] that some special classes of Riesz spaces can be identified as being $c$-extensors.

PROPOSITION 4.5. *Each of the following conditions on $E$ implies that $E$ is a $c$-extensor.*

(1) *$E$ contains an order unit and is Dedekind complete.*

(2) *The positive cone in $E$ is countably generated and $E$ is Dedekind complete.*

(3) *$E$ is a Dedekind complete perfect sequence space.*

Thus, the following spaces are $c$-extensors: $\mathbf{R}^{\mathbf{N}}$; $l^\infty$; $C(X)$ if $X$ is compact and extremally disconnected. It is interesting to observe that not every boundedly laterally complete Riesz space with the lateral boundedness property has the boundedness property.

EXAMPLE 4.6. An example of a boundedly laterally complete Riesz space with the lateral boundedness property but without the boundedness property.

For the moment, in this example, we think of $\mathbf{R}$ as equipped with the discrete topology. On $\mathbf{R}^{\mathbf{N}}$ we consider the product topology. For $u \in \mathbf{R}^n$ we write $V_u = \{x \in \mathbf{R}^{\mathbf{N}} \mid x_i = u_i \text{ if } i \leqslant n\}$; $\mathscr{B}$ is the space of all Borel measurable functions on $\mathbf{R}^{\mathbf{N}}$. As usual, a set $A \subset \mathbf{R}^{\mathbf{N}}$ is called meagre if there exist countably many closed sets $A_n \subset \mathbf{R}^{\mathbf{N}}$ such that $A \subset \bigcup_{n \in \mathbf{N}} A_n$ and for all $n \in \mathbf{N}$, $A_n$ has empty interior. Because $\mathbf{R}^{\mathbf{N}}$ is completely metrizable, the Baire theorem yields that $\mathbf{R}^{\mathbf{N}}$ is not meagre (Corollary 25.4 in [22]) and also that (for every $u \in \mathbf{R}^n$, $n \in \mathbf{N}$) $V_u$ is not meagre because $V_u$ is open and closed. $(*)$

Furthermore, we will need the fact that $\{V_u | u \in \mathbf{R}^n, n \in \mathbf{N}\}$ forms a base for the topology on $\mathbf{R}^{\mathbf{N}}$. Denote $M = \{f \in \mathscr{B} | \operatorname{supp}(f) \text{ is meagre}\}$. It is easily seen that $M$ is a $\sigma$-ideal in $\mathscr{B}$. Thus, $F := \mathscr{B}/M$ is Archimedean and the natural Riesz homomorphism $\pi\colon \mathscr{B} \to F$ is a $\sigma$-homomorphism (i.e. preserves countable suprema).

Remark $(*)$ tells us that $C(\mathbf{R}^{\mathbf{N}})$ can be identified naturally as a Riesz subspace of $F$. In fact, $C(\mathbf{R}^{\mathbf{N}})$ is an order dense Riesz subspace of $F$. The latter can be proved along the lines of Theorem 14.9 and page 112, $d \Rightarrow c$, in [11]. We will refer to it as $(**)$. From [11] we also adopt the convention on the use of the term 'almost everywhere'. Though our example is not going to be $F$, but the universal completion of $F$, we start with giving a proof of the fact that $F$ does not have the boundedness property.

Therefore, consider $B = \{n1_{V_u} | u \in \mathbf{R}, n \in \mathbf{N}\}$. Certainly, $\sup\{f(t) | f \in B\} = \infty$ for all $t \in \mathbf{R}^{\mathbf{N}}$. We are going to prove that $\pi(B)$ is not order bounded. Suppose $\pi(B)$ is order bounded, i.e. there exists $g \in \mathscr{B}$ such that for all $f \in \mathscr{B}$, $g \geqslant f$ almost everywhere. Let $n \in \mathbf{N}$. Because $V_u \cap [g < n]$ is meagre for all $u \in \mathbf{R}^{\mathbf{N}}$, we can find closed sets $A_{u,j}$ $(j \in \mathbf{N})$ with empty interior such that $V_u \cap [g < n] \subset \bigcup_{j=1}^{\infty} A_{u,j}$ and $A_{u,j} \subset V_u$. Now $\bigcup_{u \in \mathbf{R}^n} A_{u,j}$ is closed for each $j \in \mathbf{N}$ and $V_u \cap [g < n] \subset \bigcup_{j=1} \bigcup_{u \in \mathbf{R}^n} A_{u,j}$. So, trying to prove that $[g < n]$ is meagre, we are done if $\bigcup_{u \in \mathbf{R}^n} A_{u,j}$ has empty interior. If $\bigcup_{u \in \mathbf{R}} A_{u,j}$ did *not* have empty interior we could find $V_t$ with $t \in \mathbf{R}^m$ for some $m \geqslant n$ such that $V_t \subset \bigcup_{u \in \mathbf{R}^n} A_{u,j}$. Thus, $V_t \subset A_{u,j}$ for some $u \in \mathbf{R}^n$, which is impossible. The result is now that $[g < \infty]$ is meagre which is impossible also. Thus, our assumption on $\pi(B)$ was not correct, i.e. $\pi(B)$ is not order bounded.

However, every countable subset of $\pi(B)$ *is* order bounded as is shown next. Suppose $B_1 \subset B$ is countable and $n_0 \in \mathbf{N}$. Define $\mathscr{W} = \{u \in \mathbf{R}^{n_0} |$ there exists $n \in \mathbf{N}$ and $w \in \mathbf{R}^n$ such that $n1_{V_w} \in B_1$ and $n \geqslant n_0$ and $u(k) = w(k)$ for all $k \leqslant n_0\}$. Then $\{n1_{V_u} | n1_{V_u} \in B_1 \text{ and } n \geqslant n_0\}$ is a countable set so $\mathscr{W}$ is countable. Let $x \notin \bigcup_{w \in \mathscr{W}} V_w$. Assume furthermore that $n1_{V_u} \in B_1$ and $n \geqslant n_0$. Then $n1_{V_u}(x) = 0$. Thus outside $\bigcup_{w \in \mathscr{W}} V_w$, $\sup_{f \in B_1} f \leqslant n_0$ pointwise, i.e. the pointwise supremum is finite outside a small set. (A set $A$ is called small here if for every $n \in \mathbf{N}$ there exist $u_1, u_2, \ldots \in \mathbf{R}^n$ such that $A \subset \bigcup_{i=1}^{\infty} V_{u_i}$.) This implies that the pointwise supremum is finite almost everywhere, because a small set, being contained in a closed set with empty interior is meagre.

Let $G$ be the universal completion of $F$. Every universally complete Riesz space has the lateral boundedness property. However, does $G$ have the boundedness property? Again we consider the set $\pi(B)$. To check whether $\pi(B)$ is order bounded we will use a theorem by Fremlin [8]. The latter theorem states that $\pi(B)$ is order bounded in $G$ if and only if it is a dominable subset of $F$, i.e. for every $f \in F^+\backslash\{0\}$ there exists $0 < g \in F$ and a positive integer $k$ satisfying $(kf - h)^+ \geqslant g$ for all $h \in \pi(B)$.

Suppose $\pi(B)$ is a dominable set of $F$. In particular, we can find $0 < g \in \mathscr{B}$ and $k \in \mathbf{N}$ such that for all $n \in N$ and all $u \in \mathbf{R}^n$,

$$(***) \qquad \left(k1_{\mathbf{R}^{\mathbf{N}}} - n1_{V_u}\right)^+ \geqslant g \quad \text{almost everywhere.}$$

By $(**)$ we can even choose $g \in C(\mathbf{R}^{\mathbf{N}})$. This implies that we can find $\varepsilon > 0$ and $u \in \mathbf{R}^n$ for some $n \in \mathbf{N}$ such that $g \geqslant \varepsilon 1_{V_u}$. Choose $n' \geqslant k$ and $u' \in \mathbf{R}^{n'}$ such that $n' > n$ and $u(j) = u'(j)$ for all $j \leqslant n$. It follows that $(k1_{\mathbf{R}^{\mathbf{N}}} - n'1_{V_{u'}})^+|_{V_{u'}} = 0$ while $g|_{V_{u'}} \geqslant \varepsilon$. Because $V_{u'}$ is not meagre this is in contradiction with $(***)$.

This means that $\pi(B)$ is not a dominable subset of $F$ and neither order bounded in $G$. However, every countable subset of $\pi(B)$ has a supremum in $G$, because even in $F$ every countable subset of $\pi(B)$ has a supremum. Hence $G$ does not have the boundedness property.

## 5. Riesz spaces in which every principal ideal is finite

In this final section we will characterize Riesz spaces in which every principal ideal is finite dimensional. We refer the interested reader for other characterizations to [10].

To prove our result here, we need the following easy lemma, whose proof we omit.

LEMMA 5.1. *Suppose $E$ is a Riesz space, $I \subset E$ is an ideal and $\varphi\colon I \to \mathbf{R}$ is a non-zero Riesz homomorphism. If $I' \supseteq I$ and $\psi\colon I' \to \mathbf{R}$ is a Riesz homomorphism that extends $\varphi$ then $\psi(e) = \sup\{\varphi(i) \mid i \in [0, e] \cap I\}$ for all $e \in I'^+$.*
*In particular, any extension $I' \to \mathbf{R}$ of $\varphi$ coincides with $\psi$.*

THEOREM 5.2. *For a Riesz space $E$ the following are equivalent.*
(1) *$(E, F)$ has property $(oI)$ for all Riesz spaces $F$.*
(2) *$(E, F)$ has property $(cI)$ for all Riesz spaces $F$.*
(3) *There exists a set $S$ such that $E$ and $c_{00}(S)$ are Riesz isomorphic.*
(4) *Every principal ideal in $E$ is finite dimensional.*

PROOF. The equivalence of (3) and (4) is part of Theorem 6.14 in [15]; (3) $\Rightarrow$ (2) is left to the reader and (2) $\Rightarrow$ (1) is trivial. We will prove (1) $\Rightarrow$ (4) by contradiction.

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Suppose $(E, F)$ has property $(oI)$ for all Riesz spaces $F$, though for some $f \in E^+$, $(f)$ (the principal ideal generated by $f$) is not finite dimensional. By the Maeda Ogawasara representation theorem there exists an extremally disconnected compact Hausdorff space $X$, such that $E$ is an order dense Riesz subspace of $C^\infty(X)$. As $(f)$ is not finite dimensional there exists an infintie set $A \subset X$ such that for all $a \in A$, $0 < f(a) < \infty$. Choose a subset $\{y_n | n \in \mathbf{N}\} \subset A$ and a disjoint set of functions $\{g_m | m \in \mathbf{N}\} \subset C^\infty(X)^+$ such that $g_m(y_n) = \delta_{m,n}$ for all $m, n \in \mathbf{N}$. Because $E$ is an order dense subset of $C^\infty(X)$, for each $n \in \mathbf{N}$ we can find an $f_n' \in E$ such that $0 < f_n' \leqslant g_n$. Define $f_n = f_n' \wedge f \in E$ for all $n \in \mathbf{N}$. Choose for each $n \in \mathbf{N}$, $x_n \in X$ such that $0 < f(x_n) < \infty$, and a clopen subset $U_n \subset X$ such that $x_n \cap U_n$ and $U_i \cap U_j = \varnothing$ if $i \neq j$.

Define $F = \{(h, g) | h \in E, g \in E$ and $h|_{U_n} = g|_{U_n}$ for all but at most finitely many $n \in \mathbf{N}\}$. Let $(f_1, f_2, \ldots)$ be the ideal generated by $\{f_n | n \in \mathbf{N}\}$ in $E$. Define $\varphi$: $(f_1, f_2, \ldots) \to F$ by $\varphi(g) = (g, \frac{1}{2}g)$ $(g \in (f_1, f_2, \ldots))$. Remark that $F$ is a (non-uniformly complete) Riesz space and $\varphi$ is an $o$-continuous Riesz homomorphism. Thus we can find a Riesz homomorphism $\Phi$: $E \to F$ such that $\Phi|_{(f_1, f_2, \ldots)} = \varphi$. Define for each $(h, g) \in F$, $(h, g)_1 = h$ and $(h, g)_2 = g$. Define furthermore $\psi_{n,1}$: $(f) \to \mathbf{R}$ and $\psi_{n,2}$: $(f) \to \mathbf{R}$ by $\psi_{n,1}(g) = (\Phi(g))_1(x_n)$ $(g \in (f))$, respectively $\psi_{n,2} = (\Phi(g))_2(x_n)$ $(g \in (f))$.

In this situation it follows by Lemma 5.1 that $\psi_{n,1}(f) = 2\psi_{n,2}(f)$. It therefore follows that $\Phi(f)$ cannot be an element of $F$.

## References


[1] C. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces*, (Academic Press, New York, San Francisco, London, 1978).

[2] S. J. Bernau, 'Lateral and Dedekind completion of Archimedean lattice groups', *J. London Math. Soc.* **12** (1976), 320–322.

[3] Gerard Buskes, *Extension of Riesz homomorphisms*, Thesis 1983, University of Nijmegen, The Netherlands.

[4] D. I. Cartwright, 'Extensions of positive operators between Banach lattices', *Mem. Amer. Math. Soc.* **122** (1966$^{\mathrm{II}}$), 277–288.

[5] Klaus Donner, *Extensions of positive operators and Korovkin theorems*, (Lecture Notes in Math. 904, Springer Verlag, Berlin-Heidelberg-New York, 1982).

[6] D. H. Fremlin, *Topological Riesz spaces and measure theory*, (Cambridge Univ. Press, London, New York, 1974).

[7] D. H. Fremlin, 'A direct proof of the Mathes-Wright integral extension theorem', *J. London Math. Soc.* (2) **11** (1975), 276–284.

[8] D. H. Fremlin, 'Inextensible Riesz spaces', *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 71–89.

[9] R. Haydon, 'Injective Banach lattices', *Math. Z.* **156** (1977), 19–47.

[10] C. B. Huijsmans, 'Riesz spaces for which every ideal is a projection band', *Proc. Netherl. Acad. Sci. A* **79** (1976), 30–35.

[11] E. de Jonge and A. C. M. van Rooij, *Introduction to Riesz spaces*, (Math. Centre Tracts 78, Math. Centrum, Amsterdam, 1977).

[12] L. V. Iantorovitch, 'Concerning the problem of moments for a finite interval', *Dokl. Akad. Nauk SSSR* **14** (1937), 531–536.

[13] Z. Lipecki, 'Extension of vector lattice homomorphisms', *Proc. Amer. Math. Soc.* **79** (1980), 247–248.

[14] H. P. Lotz, 'Extensions and liftings of positivep linear mappings', *Trans. Amer. Math. Soc.* **211** (1975$^{XI}$), 85–100.

[15] W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces* I, (North-Holland Publishing Company, Amsterdam, London, 1971).

[16] W. A. J. Luxemburg and A. R. Schep, 'An extension theorem for Riesz homomorphisms', *Indag. Math.* **41** (1979), 145–154.

[17] B. de Pagter, *f-algebras and orthomorphisms*, Thesis, 1981, University of Leiden, Holland.

[18] A. L. Peressini, *Ordered topological vector spaces*, (Harper & Row, New York, 1967).

[19] John Riedl, 'Partially ordered locally convex vector spaces and extensions of positive continuous linear mappings', *Math. Ann.* **157** (1964/65), 95–124.

[20] A. I. Veksler and V. A. Geiler, 'Order and disjoint completeness of linear partially ordered spaces', *Siberian Math. J.* **13** (1972), 30–35.

[21] Anthony W. Wickstead, 'Extensions of orthomorphisms', *J. Austral. Math. Soc. Ser. A* **29** (1980), 87–98.

[22] Stephen Willard, *General topology*, (Addison-Wesley Publishing Company, Reading, Massachussetts, Menlo Park, California, London, Don Mills, Ontario, 1970).

[23] A. C. Zaanen, 'The universal completion of an Archimedean Riesz space', *Proc. Netherl. Acad. Sci. A* (4) **86** (1983), 435–441.

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