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ISOMORPHIC FACTORIZATION OF REGULAR GRAPHS OF EVEN DEGREE

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Abstract

A graph G is divisible by t if its edge set can be partitioned into t subsets, such that the subgraphs (called factors) induced by the subsets are all isomorphic. Such an edge partition is an isomorphic factorization. It is proved that a $2k$-regular graph with an even number of vertices is divisible by $2k$ provided it contains either no 3-cycles or no 5-cycles. It is also shown that any 4-regular graph with an even number of vertices is divisible by 4. In both cases the components of the factors found are paths of length 1 and 2, and the factorizations can be constructed in polynomial time.

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1. Introduction

In this paper the problem of decomposing an r-regular graph into r isomorphic subgraphs is examined. A *graph* to us is a finite simple graph; if we wish to allow multiple edges (but not loops) the term *multigraph* will be used. The number of vertices and edges of a graph G will be denoted $v(G)$ and $e(G)$ respectively. A graph G is *divisible by* t if its edge set can be partitioned into t subsets, such that the subgraphs (called *factors*) of G induced by the subsets are all isomorphic. Such a partition is an *isomorphic factorization of* G *into* t *parts.* The obvious necessary condition $t|e(G)$ is called the *divisibility condition for* G *and* t. G is t*-rational* if G is divisible by t or if the divisibility for G and t is not satisfied.

In previous work on isomorphic factorizations of regular graphs, Wormald [8] has shown that for $2 \leq t < r/2$, "almost all" labelled r-regular graphs are not divisible by t. More positively, it is easy to establish that every r-regular graph is t-rational for all $t \geq r+1$ (a simple proof appears in [3]). Therefore, probably the most tractable unresolved questions is the folowing conjecture.

CONJECTURE 1.1. *All r-regular graphs are r-rational.*

So far this problem has not yielded to any unified approach. Wormald and the author [3] have proved that all 3-regular multigraphs are divisible by 3. Using a similar method, the author [2] has shown that for large values of r all r-regular graphs are r-rational. In this paper a different technique is used to attack the problem for regular graphs of even degree.

Since an isomorphic factorization into t parts is just an edge partition, it can be thought of as an edge-colouring of a graph G. Given an edge partition $\{E_1, E_2, \ldots, E_t\}$ we assign a distinct colour c_i to all edges in E_i. Conversely, any edge-colouring of G yields an edge partition by collecting together all edges of the same colour; if this is an isomorphic factorization the colouring will be called *isofactorial.*

2. Regular graphs of even degree

In this section we show how to find an isofactorial edge-colouring of a $2k$-regular graph with $2k$ colours, provided that the graph has an even number of vertices (this is necessary for the divisibility condition to hold), and is either free of 3-cycles or free of 5-cycles. The strategy used is a form of "divide and conquer": the graph is first divided into *2-factors* (2-regular spanning subgraphs), and each 2-factor is then suitably coloured with exactly 2 colours. In order to do this the following lemma, due to Petersen, is required.

LEMMA 2.1 [7]. *Every $2k$-regular graph, $k \geq 0$, is the union of k edge-disjoint 2-factors.*

Such a decomposition into 2-factors is known an a *2-factorization.*

An *r-edge-colouring* of a graph G is a function $\gamma : E(G) \to \Gamma$, where Γ is a set of *colours*, and $|\Gamma| = r$. An edge-colouring of a graph will be called *allowable* if the components of each subgraph induced by all edges of one colour are paths of length 1 and 2 (*1-paths* and *2-paths*). For graphs (but not multigraphs) this is equivalent to each being incident with at most one other edge of the same colour. In all cases so far where Conjecture 1.1 has been verified, it has been by finding isofactorial allowable colourings. The following conjecture, stronger than Conjecture 1.1, therefore seems to be true.

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CONJECTURE 2.2. *Every r-regular graph with an even number of vertices has an isofactorial allowable r-edge-colouring. (Equivalently, every such graph has an isomorphic factorization into r parts, the components of each factor being paths of length* 1 *and* 2.)

Let $\gamma: E(G) \to \Gamma$ be an edge-colouring of a graph G with colours from a set Γ. For any colour $c \in \Gamma$, $N(c,\gamma)$ is the subgraph of G induced by the edges of colour c, and $n(c,\gamma) = e(N(c,\gamma))$ is the number of edges coloured c. If γ is allowable, let $n_i(c,\gamma)$, $i = 1$ or 2, be the number of components which are i-paths in $N(c,\gamma)$. Also, let $n_i(\gamma) = \sum_{c\in\Gamma} n_i(c,\gamma)$. When a particular colouring γ is understood these symbols will be abbreviated to $N(c)$, $n(c)$, $n_i(c)$ and n_i. No confusion between $n_i(\gamma)$ and $n_i(c)$ should result, since edge-colourings will always be represented by Greek letters and colours by Roman letters.

In our strategy each 2-factor will receive a special sort of allowable colouring. A *bisection* of a graph G is an allowable colouring with two colours, say a and b, such that

(a) $n(a) - n(b) = 1, 0$ or -1, and

(b) $n_2(a) - n_2(b) = n(a) - n(b)$.

The first condition requires that the number of edges of each colour be the same if $e(G)$ is even, or differ by 1 if $e(G)$ is odd. The second condition requires that the number of 2-paths of each colour do the same, in such a way that if there are more a than b edges, then there are more a than b 2-paths, and vice versa. Note that $n_2 \equiv e(G) \pmod 2$ for any bisection of G.

We wish to construct bisections of the 2-factors of a graph. The components of each 2-factor are just cycles, and the bisections of the 2-factors will be constructed from bisections of cycles, using the following results.

LEMMA 2.3. *Let C be a cycle of length l. Then for any integer m such that $0 \le m \le l/2$ and $m \equiv l \pmod 2$, there is a bisection β of C with $n_2(\beta) = m$.*

Proof. Suppose that C has edges $e_1, e_2, \dots, e_l$ (in order). Let

$$\beta(e_i) = \begin{cases} a & \text{if } \le i \le 2m, i \equiv 1 \text{ or } 2 \pmod 4, \\ b & \text{if } 1 \le i \le 2m, i \equiv 3 \text{ or } 0 \pmod 4, \\ a & \text{if } 2m+1 \le i \le l, l-i \equiv 1 \pmod 2, \\ b & \text{if } 2m+1 \le i \le l, l-i \equiv 0 \pmod 2. \end{cases}$$

Then as we traverse C from e_1 to e_l we encounter m 2-paths, alternately coloured a and b, followed by $l-2m$ 1-paths, also alternately coloured a and b. The colouring β will be allowable if $\beta(e_1) \ne \beta(e_l)$ and, if $0 < l/2$, provided $\beta(e_{2m}) \ne \beta(e_{2m+1})$. The former is always true since $\beta(e_1) = a$ and $\beta(e_l) = b$. If l is odd then m is also odd, and $\beta(e_{2m}) = a \neq b = \beta(e_{2m+1})$, while if l is even then m is even and $\beta(e_{2m}) = b \neq a = \beta(e_{2m+1})$. Therefore β is allowable.

Now if l and m are odd, then $n_2(a) = (m+1)/2$, $n_2(b) = (m-1)/2$, $n(a) = (l+1)/2$ and $n(b) = (l-1)/2$, while if l and m are even, then $n_2(a) = n_2(b) = m/2$ and $n(a) = n(b) = l/2$. Hence β is a bisection, with $n_2(\beta) = m$ as desired.

LEMMA 2.4. *Suppose that G_1 and G_2 are disjoint graphs with bisections β_1 and β_2 respectively. Then $G_1 \cup G_2$ has a bisection β such that $n_2(\beta) = n_2(\beta_1) + n_2(\beta_2)$.*

PROOF. By recolouring if necessary we may assume that both β_1 and β_2 use the colours a and b; furthermore we may suppose that $n(a,\beta_1) \geq n(b,\beta_1)$ and $n(a,\beta_2) \leq n(b,\beta_2)$. Define β to be β_1 on G_1 and β_2 on G_2; then it is not difficult to verify that β is the required bisection.

COROLLARY 2.5. *If G_1, $G_2 \dots G_k$ are disjoint graphs, each G_i having a bisection β_i, then $G_1 \cup G_2 \cup \cdots \cup G_k$ has a bisection β with $n_2(\beta) = n_2(\beta_1) + n_2(\beta_2) + \cdots + n_2(\beta_k)$.*

PROOF. This follows from Lemma 2.4 by induction on k.

We now wish to use Lemma 2.3 and Corollary 2.5 to construct bisections of 2-regular graphs. However, the bounds on m (0 and $l/2$) in Lemma 2.3 are not tight because of the restriction that m be an integer congruent to l modulo 2. Therefore, for each $l \geq 3$ let $p(l)$ and $q(l)$ be respectively the smallest and largest integers congruent to l modulo 2 between 0 and $l/2$.

LEMMA 2.6. *Let G be a 2-regular graph, with c_l cycles of length l for each $l \geq 3$. Then*

(a) $v(G) = 3c_3 + 4c_4 + 5c_5 + 6c_6 + \cdots$.

(b) *For all m such that $m \equiv v(G) \pmod 2$ and $P \leq m \leq Q$, G has a bisection β such that $n_2(\beta) = m$, where*

$$P = \sum_{l=3}^{\infty} p(l)c_l = c_3 + c_5 + c_7 + c_9 + \cdots,$$

$$Q = \sum_{l=3}^{\infty} q(l)c_l = c_3 + 2c_4 + c_5 + 2c_6 + 3c_7 + 4c_8 + 3c_9 + \cdots.$$

(*Note that P, $Q \equiv v(G) \pmod 2$.*)

PROOF. (a) Count the vertices by cycles.

(b) Let $Z_1, Z_2 \dots, Z_k$ be the cycles which compose G, and let Z_i be of length l_i. Then $P = \sum_{l=3}^{\infty} p(l)c_l = \sum_{i=1}^{k} p(l_i)$ and $Q = \sum_{l=3}^{\infty} q(l)c_l = \sum_{i=1}^{k} q(l_i)$.

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Therefore, since $P \leq m \leq Q$ and $m \equiv v(G) = l_1 + l_2 + \cdots + l_k \pmod 2$, for each i we can choose $m_i \equiv l_i \pmod 2$ with $p(l_i) \leq m_i \leq q(l_i)$, such that $m = m_1 + m_2 + \cdots + m_k$. By Lemma 2.3 each Z_i has a bisection β_i with $n_2(\beta_i) = m_i$, and thus by Corollary 2.5 G has a bisection β with $n_2(\beta) = m$.

LEMMA 2.7. *Let G be a 2-regular graph having c_l cycles of length l for each $l \geq 3$. Suppose that $v = v(G)$ is even.*

(a) *If $c_3 = 0$ then G has a bisection such that $n_2(a) = n_2(b) = \lfloor v/10 \rfloor$.*

(b) *If $c_5 = 0$ then G has a bisection such that $n_2(a) = n_2(b) = \lfloor v/6 \rfloor$.*

PROOF. (a) Since $c_3 = 0$, by Lemma 2.6 we have

$$v(G) = 4c_4 + 5c_5 + 6c_6 + 7c_7 + \cdots$$

and G has a bisection β with $n_2(\beta) = m$ for any every m, $P \leq m \leq Q$, where

$$P = c_5 + c_7 + c_9 + c_{11} + \cdots$$
$$\therefore 5P = 5c_5 + 5c_7 + 5c_9 + 5c_{11} + \cdots \leq v$$

and

$$Q = 2c_4 + c_5 + 2c_6 + 3c_7 + 4c_8 + 3c_9 + 4c_{10} + \cdots$$
$$\therefore 5Q = 10c_4 + 5c_5 + 10c_6 + 15c_7 + 20c_8 + 15c_9 + 20c_{10} + \cdots \geq v.$$

Also P and Q are both even (since $v(G)$ is even) and thus $P \leq 2\lfloor v/10 \rfloor$, $Q \geq 2\lceil v/10 \rceil$. Hence $n_2(\beta)$ may be chosen to be $2\lfloor v/10 \rfloor$, and then it follows from the definition of a bisection that $n_2(a, \beta) = n_2(b, \beta) = \lfloor v/10 \rfloor$.

(b) Since $c_5 = 0$, from Lemma 2.6 we obtain

$$v(G) = 3c_3 + 4c_4 + 6c_6 + 7c_7 + \cdots$$

and G has a bisection β with $n_2(\beta) = m$ for any even m, $P \leq m \leq Q$, where

$$P = c_3 + c_7 + c_9 + c_{11} + \cdots$$
$$\therefore 3P = 3c_3 + 3c_7 + 3c_9 + 3c_{11} + \cdots \leq v$$

and

$$Q = c_3 + 2c_4 + 2c_6 + 3c_7 + 4c_8 + 3c_9 + 4c_{10} + \cdots$$
$$\therefore 3Q = 3c_3 + 6c_4 + 6c_6 + 9c_7 + 12c_8 + 9c_9 + 12c_{10} + \cdots \geq v.$$

Since P and Q are both even, $P \leq 2\lfloor v/6 \rfloor$ and $Q \geq 2\lceil v/6 \rceil$. Hence $n_2(\beta)$ may be chosen to be $2\lfloor v/6 \rfloor$, and therefore $n_2(a, \beta) = n_2(b, \beta) = \lfloor v/6 \rfloor$.

THEOREM 2.8. *Let G be a $2k$-regular graph, $k \geq 0$, with $v = v(G)$ even. Suppose that G has either no 3-cycles or no 5-cycles. Then G has an isofactorial* *allowable $2k$-edge-colouring. (In other words, G has an isomorphic factorization into $2k$-parts, the components of each factor being paths of length 1 and 2.)*

PROOF. By Lemma 2.1 G has a 2-factorization, consisting of 2-factors $F_1, F_2, \ldots, F_k$. If G has no 3-cycles, then no F_i has any 3-cycles, and so by Lemma 2.7 (a) each F_i has a bisection β_i, using colours a_i and b_i, such that $n_2(a_i, \beta_i) = n_2(b_i, \beta_i) = \lfloor v/10 \rfloor$. Also, $n(a_i, \beta_i) = n(b_i, \beta_i) = e(F_i)/2 = v/2$. Therefore the colouring γ of G defined by $\gamma(e) = \beta_i(e)$ for all $e \in E(F_i)$ is an allowable colouring with each $N(c, \gamma)$ isomorphic to $\lfloor v/10 \rfloor P_3 \cup (v/2 - 2\lfloor v/10 \rfloor) P_2$ (where P_k represents a path of length $k - 1$). Thus γ is isofactorial, as required.

Similar reasoning, using Lemma 2.7(b), applies if G has no 5-cycles.

The method developed in this section unfortunately does not work for a $2k$-regular graph which contains both 3-cycles and 5-cycles. Consider for example, a 6-regular graph with 30 vertices and a 2-factorization containing three 2-factors, one consisting of ten 3-cycles and the other two of six 5-cycles each. A bisection of the first 2-factor necessarily has $n_2 = 10$, while bisections of the other two must have $n_2 = 6$. Is it therefore impossible to match bisections of these three 2-factors to form an isofactorial colouring of the whole graph. In the next section we shall see how to surmount this difficulty in the particular case of 4-regular graphs.

3. 4-regular graphs

In this section we prove that every 4-regular graph G with $v(G)$ even has an isofactorial allowable 4-edge-colouring ($e(G)$ is divisible by 4 if and only if $v(G)$ is even). In order to find such a colouring G will be decomposed into two subgraphs, which will be generalizations of the 2-factors used in the previous section, and the colouring of G will be assembled from bisections of these subgraphs.

A *near 2-factor* of a graph is a spanning subgraph in which all vertices have degree 2, except for one vertex of degree 1 and one vertex of degree 3. An (r, s)-*lollipop*, or $L_{r,s}$, consists of a cycle of length r and a path of length s which intersect in exactly one vertex, one of the ends of the path. The vertices of a lollipop are therefore *cycle vertices* or *path vertices*; there is one *common vertex* which is both. The other end of the path from the common vertex is the *extreme vertex*. The components of a near-2-factor must all be cycles, except for one which is a lollipop.

As in Section 2, bisections of the subgraphs of interest will be constructed from bisections of their components, which here will be cycles and lollipops. In order to describe the range of n_2 values achievable by this method, the following terminology will be employed. An ordered pair of integers (f, g) is a *bisection limit pair* for a graph G if

(a) $f + g \leq v(G)$,

(b) $f,\ v(G) - g \equiv 2v(G) \pmod 4$, and

(c) for any $m \equiv v(G) \pmod 2$ such that $f \leq 2m \leq v(G) - g$, there is a bisection β of G with $n_2(\beta) = m$.

In other words, G has bisections with $n_2 = f/2$ and $n_2 = (v(G) - g)/2$, and with $n_2 = m$ for any m of the correct parity between these two values.

COROLLARY 3.1. *If $G_1, G_2, \ldots G_k$ are disjoint graphs, and each G_i has a bisection limit pair (f_i, g_i), then $G_1 \cup G_2 \cup \cdots \cup G_k$ has the bisection limit pair $(f_1 + f_2 + \cdots + f_k\ g_1 + g_2 + \cdots + g_k)$.*

PROOF. This follows easily from Corollary 2.5.

Henceforth, let $\bar{x}$ denote the congruence class of an integer x modulo 4.

COROLLARY 3.2. *For each $l \geq 3$ let*

$$f(C_l) = 2p(l) = \begin{cases} 0 & \textit{if } \bar{l} = \bar{0} \textit{ or } \bar{2}, \textit{ that is, } l \textit{ is even}, \\ 2 & \textit{if } \bar{l} = \bar{1} \textit{ or } \bar{3}, \textit{ that is, } l \textit{ is odd}, \end{cases}$$

and

$$g(C_l) = l - 2q(l) = \begin{cases} 0 & \textit{if } \bar{l} = \bar{0}, \\ 3 & \textit{if } \bar{l} = \bar{1}, \\ 2 & \textit{if } \bar{l} = \bar{2}, \\ 1 & \textit{if } \bar{l} = \bar{3}. \end{cases}$$

Then $(f(C_l), g(C_l))$ is a bisection limit pair for the l-cycle C_l.

PROOF. By Lemma 2.3 and the definitions of $p(l)$ and $q(l)$, C_l has bisections with $n_2 = m$ for any $m \equiv v(G) \pmod 2$ between (and including) $p(l) = f(C_l)/2$ and $q(l) = (v(C_l) - g(C_l))/2$, as required. It follows from their definitions that $p(l)$ and $q(l)$ depend only on $\bar{l}$, giving $f(C_l)$ and $g(C_l)$ the values listed.

Notice that $g(C_l)$ depends only on $\bar{l}$. This is why the second element of a bisection limit pair was taken to be g, rather than the perhaps more natural $v(G) - g$; for cycles and lollipops g will take only finitely many values.

LEMMA 3.3. *Suppose that $r \geq 3$ and $s \geq 1$. Let*

$$f(L_{r,s}) = \begin{cases} 4 & \textit{if } \bar{r} + \bar{s} = \bar{0} \textit{ or } \bar{2}, \textit{ that is, } r + s \textit{ is even}, \\ 2 & \textit{if } \bar{r} + \bar{s} = \bar{1} \textit{ or } \bar{3}, \textit{ that is, } r + s \textit{ is odd}, \end{cases}$$

and

$$g(L_{r,s}) = \begin{cases} 0 & \text{if } \bar{r}+\bar{s}=\bar{0}, \bar{r}=\bar{0} \text{ or } \bar{3}, \\ 4 & \text{if } \bar{r}+\bar{s}=\bar{0}, \bar{r}=\bar{1} \text{ or } \bar{2}, \\ 3 & \text{if } \bar{r}+\bar{s}=\bar{1}, \\ 2 & \text{if } \bar{r}+\bar{s}=\bar{2}, \\ 1 & \text{if } \bar{r}+\bar{s}=\bar{3}, \bar{r}=\bar{0}, \bar{2} \text{ or } \bar{3} \\ 5 & \text{if } \bar{r}+\bar{s}=\bar{3}, \bar{r}=\bar{1}. \end{cases}$$

Then $(f(L_{r,s}), g(L_{r,s}))$ *is a bisection limit pair for the* (r,s)*-lollipop* $L_{r,s}$.

PROOF. The proof is by induction on $r+s$. It can be verified that bisections with the required n_2 values exist for all lollipops with $r+s \leq 7$, or in fact for all sixteen lollipops with $r = 3, 4, 5$ or 6 and $s = 1, 2, 3$ or 4 (see Figures 3.1 and 3.2, where the two colours are indicated by solid and dashed lines). Assume therefore that $r+s \geq 8$, and that the result holds for all lollipops $L_{t,u}$ with $t+u < r+s$. Moreover we may suppose that $r \geq 7$ or $s \geq 5$.

Consider any $m \equiv v(L_{r,s}) \pmod 2$ such that $f(L_{r,s}) \leq 2m \leq v(L_{r,s}) - g(L_{r,s})$. Since either $r \geq 7$ or $s \geq 5$, one of the cycle or the path which constitute $L_{r,s}$ must contain a path of length 5, say between vertices w and x, which can be replaced by a single edge wx to form a lollipop $L_{t,u}$ (where either $t = r-4$, $u = s$ or $t = r$, $u = s-4$). Thus $f(L_{t,u}) = f(L_{r,s})$, $g(L_{t,u}) = g(L_{r,s})$, and $v(L_{t,u}) = v(L_{r,s}) - 4$. Hence $m \equiv v(L_{t,u}) \pmod 2$ and

$$f(L_{t,u}) \leq 2m \leq v(L_{t,u}) - g(L_{t,u}) + 4.$$

Suppose first that $2m \leq v(L_{t,u}) - g(L_{t,u})$. Then by the induction hypothesis $L_{t,u}$ has a bisection β with $n_2(\beta) = m$ using, say, colours a and b. Without loss of generality $\beta(wx) = a$. If wx is replaced by a path of length 5 coloured as in Figure 3.3 (a), a bisection β' of $L_{r,s}$ results with $n_s(\beta') = m$.

Now if $2m > v(L_{t,u}) - g(L_{t,u})$ then since $m \equiv v(L_{t,u}) \pmod 2$, $2m = v(L_{t,u}) - g(L_{t,u}) + 4$. By the induction hypothesis $L_{t,u}$ has a bisection β with $n_2(\beta) = m - 2 = (v(L_{t,u}) - g(L_{t,u}))/2$. Assume that $\beta(wx) = a$. There is at most one other edge coloured a incident with wx; therefore without loss of generality it may be suppose that all other edges incident with w are coloured b. Then, by replacing wx by a path of length 5 coloured as in Figure 3.3(b), a bisection β' of $L_{r,s}$ is obtained for which $n_2(\beta') = m$, as required.

Now that bisection limit pairs for cycles and lollipops have been found, it is possible to derive them for any graph of all whose components are cylces are lollipops. Such a graph will be called a *cluster*. Given a cluster G with components $G_1, G_2, \ldots, G_k$, let $f(G) = f(G_1) + f(G_2) + \cdots + f(G_k)$ and $g(G) = g(G_1) + g(G_2) + \cdots + g(G_k)$. Then by Corollary 3.1 $(f(G), g(G))$ is a bisection limit pair for G.

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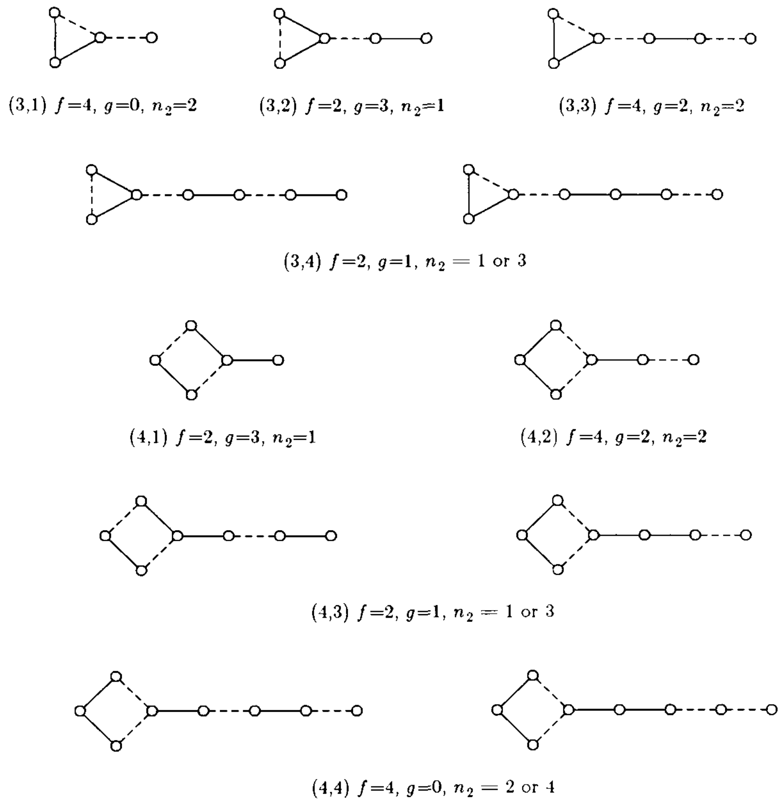
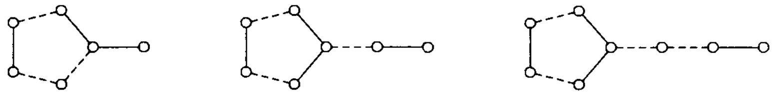


FIGURE 3.1

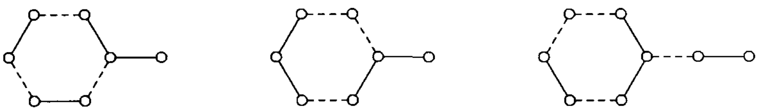
Henceforth G will be a 4-regular graph with $v = v(G)$ even. Each component of G has a closed eulerian trail. By concatenating the edge sequences of one closed eulerian trail from each component of G (call these the *component trails*) an *eulerian sequence* $e_1, e_2, \ldots, e_{2v}$ (note that $e(G) = 2v$) in G is obtained, which contains every edge exactly once. If the edges of this sequence are coloured alternately a and b, then each component trail is coloured alternately, and thus each vertex must be incident with two a edges and two b edges. This is true even for the *terminal vertex* (the one incident with the first and last edges) of each component trail, because each component, and hence each trail, has an even number of edges. Therefore the edges of both colours induce 2-factors of G. (Closed eulerian trails have been used previously to generate k-factors of $2k$-regular graphs in a paper by A. J. W. Hilton on proper edge-colourings.)

Consider now a fixed eulerian sequence $e_1, e_2, \ldots, e_{2v}$ in G. If the colours a and b are interchanged along some initial segment, of even length, of this



(5,1) $f=4$, $g=2$, $n_2=2$ (5,2) $f=2$, $g=5$, $n_2=1$ (5,3) $f=4$, $g=4$, $n_2=2$

(5,4) $f=2$, $g=3$, $n_2 = 1$ or 3



(6,1) $f=2$, $g=1$, $n_2 = 1$ or 3 (6,2) $f=4$, $g=4$, $n_2=2$

(6,3) $f=2$, $g=3$, $n_2 = 1$ or 3

(6,4) $f=4$, $g=2$, $n_2 = 2$ or 4

FIGURE 3.2

(a)

(b)

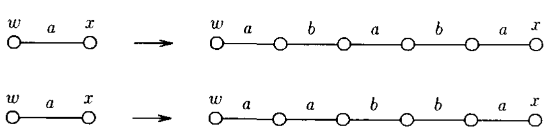


FIGURE 3.3

sequence, a new edge-colouring γ_j results for each j, $0 \le j \le v$, where

$$\gamma_j(e_i) = \begin{cases} b & 1 \le i \le 2j, i \text{ odd}, \\ a & 1 \le i \le 2j, i \text{ even}, \\ a & 2j+1 \le i \le 2v, i \text{ odd}, \\ b & 2j+1 \le i \le 2v, i \text{ even}. \end{cases}$$

Thus γ_0 is the colouring discussed in the preceding paragraph. Let A_j and B_j be the subgraphs induced by the edges coloured a and b respectively under γ_j.

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If $j = 0$ or v, or if e_{2j} and e_{2j+1} are in different components of G, then A_j and B_j are 2-factors of G, obtained from A_0 and B_0 by interchanging the cycles in certain components of G. Otherwise, e_{2j} and e_{2j+1} lie in the same component H of G. Let w be the vertex common to e_{2j} and e_{2j+1}, and let z be the terminal vertex of the component trail in H. If $w = z$ then A_j and B_j once again form 2-factors. If $w \neq z$ then w will have degree 3 in A_j and degree 1 in B_j, while z has degree 1 in A_j and degree 3 in B_j; A_j and B_j will therefore be near-2-factors.

For any j, $v(A_j) = v$ is even. Also, A_j is a cluster, and therefore has a bisection limit pair $(f(A_j), g(A_j))$. That is, for any $m \equiv v \pmod 2$ such that $f(A_j) \leq 2m \leq v - g(A_j)$, A_j has a bisection for which $n_2 = m$. Let $[i,j]$ denote the interval in the integers between i and j; then this may be rephrased by saying that for any $i \in I_j = [f(A_j)/4, (v - g(A_j))/4]$, A_j has a bisection, using colours a' and a'', say, for which $n_2(a') = n_2(a'') = i$. Each B_j has a similar interval J_j. It will be shown that for some j, $I_j \cap J_j \neq \varnothing$. Suitable bisections of A_j and B_j can then be combined to obtain an isofactorial colouring of G.

In order to show that $I_j \cap J_j \neq \varnothing$ for some j, a "continuity" argument will be used. It is therefore necessary to show that, for each j, I_{j+1} and J_{j+1} do not differ too much from I_j and J_j respectively. Thus we shall investigate how A_{j+1} and B_{j+1} are obtained from A_j and B_j.

The colouring γ_{j+1} differs from γ_j in that e_{2j+1} and e_{2j+2} exchange colours. Now e_{2j+1} and e_{2j+2} must be consecutive edges in some component trail, since each component trail has an even number of edges. Therefore suppose that $e_{2j+1} = wx$ and $e_{2j+2} = xy$, and let z be the terminal vertex of the component trail containing these edges. Now $A_{j+1} = A_j \cup \{xy\} - \{wx\}$. Form A_j^* from A_j by deleting the component or components containing w, x and y; then A_j^* is a subgraph of A_{j+1} also. Thus $A_j = A_j^* \cup A'_j$ and $A_{j+1} = A_j^* \cup A''_j$ for some clusters A'_j, A''_j. Similarly $B_j = B_j^* \cup B'_j$ and $B_{j+1} = B_j^* \cup B''_j$, where B_j^* is obtained by deleting from B_j the components containing w, x and y.

LEMMA 3.4. *The transformation $a'_j \to A''_j$ has one of the following forms, where r, $t \geq 3$ and s, $u \geq 1$.*

(a) $L_{r,s} \to L_{t,u}$ (*where* $r + s = t + u$).
(b) $L_{r,s} \cup C_t \to L_{t,r+s}$.
(c) $L_{t,r+s} \to L_{r,s} \cup C_t$.
(d) $L_{r,s} \cup C_t \to C_r \cup L_{t,s}$.
(e) $C_{r+s} \to L_{r,s}$.
(f) $L_{r,s} \to C_{r+s}$.
(g) $C_r \cup C_s \to L_{r,s}$.
(h) $L_{r,s} \to C_r \cup C_s$.

PROOF. Let w, x, y and z be as in the preceding discussion. Either w or y may be equal to z, but not both (or G would contain a 2-cycle).

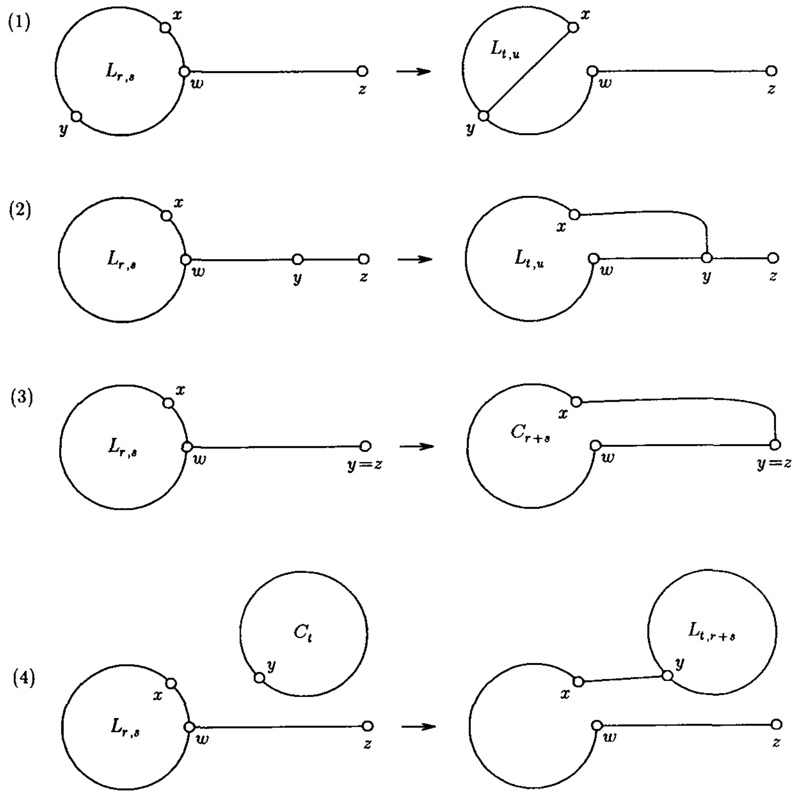

FIGURE 3.4

Suppose that $w \neq z$. Then w is the common vertex of the unique lollipop L in A_j, x is adjacent to w in L, and z is the extreme vertex of L; y may be any vertex of A_j except w or x. Let C, P and R denote respectively the sets of cycle vertices of L, of path vertices of L, and of all the vertices in the other, cyclic, components of A_j. There are eight cases according to whether $x \in C$ or P and $y \in C$, $P - \{z\}$, $\{z\}$ or R (see Figures 3.4 and 3.5). (Note that in cases (5) and (8) x and z may be equal; this is impossible in all other cases.)

(1) $x \in C$, $y \in C$ gives transformation (a).

(2) $x \in C$, $y \in P - \{z\}$ gives transformation (a).

(3) $x \in C$, $y = z$ gives transformation (f).

(4) $x \in C$, $y \in R$ gives transformation (b).

(5) $x \in P$, $y \in C$ gives transformation (a) (here $t = r$ and $u = s$).

(6) $x \in P$, $y \in P - \{z\}$ gives transformation (c).

(7) $x \in P$, $y = z$ gives transformation (h).

(8) $x \in P$, $y \in R$ gives transformation (d).

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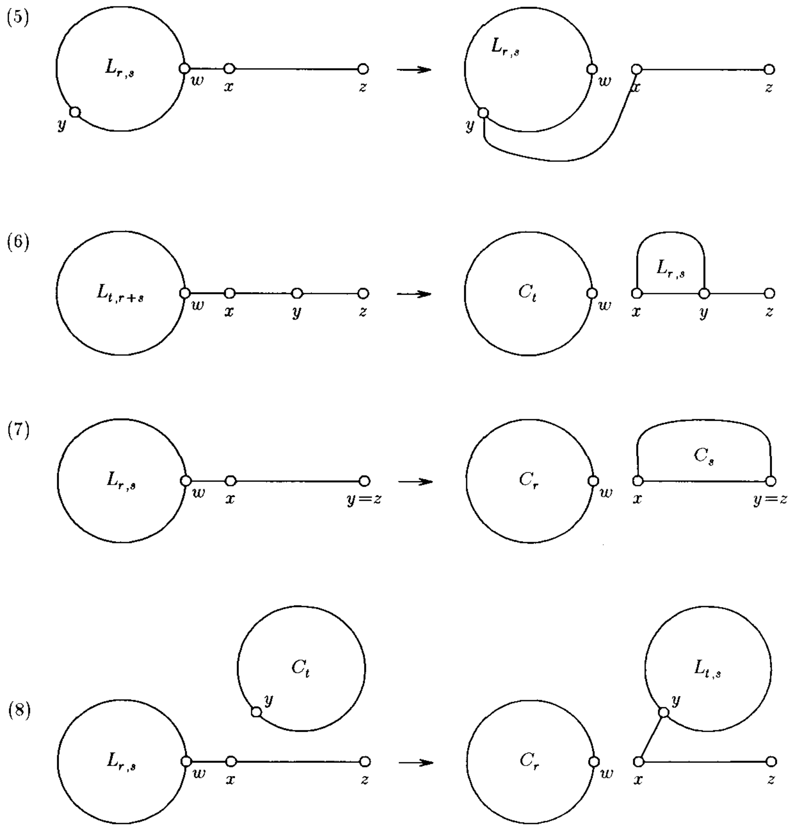

FIGURE 3.5

Now suppose that $w = z$ (and hence $x, y \neq z$). Then A_j is a 2-factor of G because the terminal vertex $z = w$ appears in $e_{2j+1} = wx$, indicating that it is the first edge of a new component trail (the last edge of a component trail must have an even subscript), and thus e_{2j} was in a different component. Hence all components of A_j are cycles, including one, H, containing the edge wx. There are now two cases (see Figure 3.6).

(9) $y \in V(H)$ gives transformation (e).

(10) $y \notin V(H)$ gives transformation (g).

COROLLARY 3.5. *The forms of transformation listed in Lemma* 3.4 *also apply to* $B'_j \to B''_j$.

PROOF. Another eulerian sequence for G is $e_{2v}, e_{2v-1}, \ldots, e_1$. Applying Lemma 3.4 to this sequence, each transformation $B''_j \to B'_j$ must be one of types (a) to (h). However, since the inverse of each form of transformation is also a form of transformation, these eight types also apply to $B'_j \to B''_j$.

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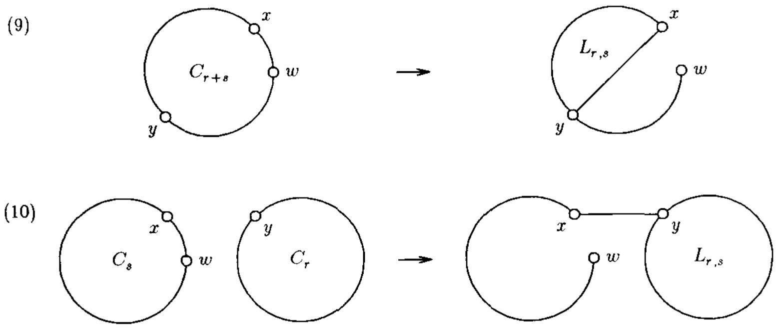

FIGURE 3.6

For each j, $0 \leq j \leq v$, write $I_j = [a_j, c_j]$ and $J_j = [b_j, d_j]$.

LEMMA 3.6. *For each* j, $0 \leq j < v$, $|a_{j+1} - a_j|$, $|b_{j+1} - b_j|$, $|c_{j+1} - c_j|$ *and* $|d_{j+1} - d_j|$ *are at most* 1.

PROOF. By definition of I_j, $|a_{j+1} - a_j| = |f(A_j) - f(A_{j+1}|/4$. Therefore, to prove that $|a_{j+1} - a_j| \leq 1$ it suffices to show that $|f(A_j) - f(A_{j+1})| \leq 7$, since $f(A_j) \equiv 2v \equiv f(A_{j+1}) \pmod 4$. For each of the transformations $A'_j \to A''_j$ in Lemma 3.4, A'_j and A''_j both consist of a cycle, a lollipop, two cycles, or a cycle and a lollipop. Therefore, by Corollary 3.2 and Lemma 3.3, $f(A'_j)$ and $f(A''_j)$ are both between 0 and 6. Thus

$$|f(A_{j+1}) - f(A_j)| = |f(A_j^* \cup A''_j) - f(A_j^* \cup A'_j)| = |f(A''_j) - f(A'_j)| \leq 6 \leq 7,$$

as required.

Similarly, to prove that $|c_{j+1} - c_j| \leq 1$, it is enough to show that $|g(A_{j+1}) - g(A_j)| = |g(A''_j) - g(A'_j)|$ is at most 7. By Corollary 3.2 and Lemma 3.3, $g(A'_j)$ and $g(A''_j)$ are both between 0 and 8, and thus the only way in which this can fail to hold is if one of $g(A'_j)$ or $g(A''_j)$ is 0 and the other is 8. Also, $g(A'_j)$ or $g(A''_j)$ can equal 8 only if the graph is of the form $L_{r,s} \cup C_t$ with $\bar{r} + \bar{s} = \bar{3}$, $\bar{r} = \bar{1}$ (so that $\bar{s} = \bar{2}$), and $\bar{t} = \bar{1}$. Therefore the transformation $A'_j \to A''_j$ must be of the types (b), (c), or (d) listed in Lemma 3.4.

If the transformation is of type (b) then $g(A'_j) = 8$ and $A'_j = L_{r,s} \cup C_t$ as above. But then $A''_j = L_{t,r+s}$ with $\bar{t} = \bar{1}$ and $\bar{r} + \bar{s} = \bar{3}$, and thus by Lemma 3.3 $g(A''_j) = 4 \neq 0$. The same argument, interchanging A'_j and A''_j, holds for type (c). Suppose therefore that the transformation is of type (d), namely $L_{r,s} \cup C_t \to C_r \cup L_{t,s}$. Without loss of generality assume that $g(A'_j) = 8$. Then, since $\bar{r} = \bar{1}$, $\bar{s} = \bar{2}$ and $\bar{t} = \bar{1}$, by Corollary 3.2 and Lemma 3.3 $g(A''_j) = 3 + 5 = 8 \neq 0$. Therefore $|c_{j+1} - c_{j+1}| \leq 1$, as required.

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By Corollary 3.5 the transformations $B'_j \to B''_j$ also have the eight possible forms listed in Lemma 3.4, and thus the same arguments show that $|b_{j+1} - b_j|$ and $|d_{j+1} - d_j|$ are ≤ 1.

A cluster C will be said to be *inflexible* if $f(C)+g(C) = v(C)$. If C is inflexible then there is only one n_2 value (namely $f(C)/2$) for which a bisection of C can be constructed by our methods. Every component or union of components of an inflexible cluster is also inflexible. The final stage of our proof will show that if no $I_j \cap J_j$ is nonempty, then some A_j and A_{j+1} are both flexible, but with different n_2 values. The next two lemmas prove that this is impossible.

LEMMA 3.7. *The only inflexible cycles are C_3 and C_5. The only inflexible lollipops are $L_{3,1}, L_{3,2}, L_{3,3}, L_{4,1}, L_{4,2}, L_{5,1}, L_{5,2}, L_{5,3}$ and $L_{6,2}$.*

PROOF. By Corollary 3.2 any inflexible cycle must have at most 5 vertices, and by Lemma 3.3 any inflexible lollipop must have at most 9 vertices. Check all cycles with 5 or fewer vertices and all lollipops with 9 or fewer vertices.

LEMMA 3.8. *If A_j and A_{j+1} are both inflexible then $a_j = c_j = a_{j+1} = c_{j+1}$.*

PROOF. Since A_j and A_{j+1} are inflexible, $a_j = c_j = f(A_j)/4$ and $a_{j+1} = c_{j+1} = f(A_{j+1})/4$. Therefore it suffices to prove that $f(A_j) = f(A_{j+1})$, or, equivalently, that $f(A'_j) = f(A''_j)$. Note that since A'_j and A''_j are unions of components of A_j and A_{j+1} respectively, they are inflexible. There are eight possible situations, as listed in Lemma 3.4.

(a) $A'_j = L_{r,s}$ and $A''_j = L_{t,u}$, $r+s = t+u$. Then $\bar{r} + \bar{s} = \bar{t} + \bar{u}$, and hence $f(A'_j) = f(A''_j)$ by Lemma 3.3.

(b) $A'_j = L_{r,s} \cup C_t$ and $A''_j = L_{t,r+s}$. Since $r \geq 3$ and $s \geq 1$, $r+s \geq 4$. But then A''_j cannot be inflexible, by Lemma 3.7.

(d) $A'_j = L_{r,s} \cup C_t$ and $A''_j = C_r \cup L_{t,s}$. Then C_t and C_r must be inflexible, and thus by Lemma 3.7 r and t must both be odd (3 or 5, in fact). Therefore $f(L_{r,s}) = f(L_{t,s})$ by Lemma 3.3, and $f(C_t) = f(C_r)$ by Corollary 3.2, giving $f(A'_j) = f(A''_j)$ as required.

(e) $A'_j = C_{r+s}$ and $A''_j = L_{r,s}$. Since $r+s \geq 4$ and C_{r+s} is inflexible, $r+s = 5$. Therefore $f(C_{r+s}) = 2$ by Corollary 3.2 and $f(L_{r,s}) = 2$ by Lemma 3.3, whence $f(A'_j) = f(A''_j)$.

(g) $A_j = C_r \cup C_s$ and $A''_j = L_{r,s}$. Since C_r and C_s are inflexible, r and s are both odd (3 or 5) and thus $r+s$ is even. Hence $f(C_r \cup C_s) = 2+2 = 4$ by Corollary 3.2 and $f(L_{r,s}) = 4$ by Lemma 3.3. Therefore $f(A'_j) = f(A''_j)$.

Now transformations of types (c), (f) and (h) are the inverses of types (b), (e) and (g) respectively, and therefore the proof in these cases can be obtained by interchanging A'_j and A''_j in the appropriate argument above.

THEOREM 3.9. *Let G be a 4-regular graph with an even number of vertices. Then G has an isofactorial allowable 4-edge-colouring. (In other words, G has an isomorphic factorization into 4 parts, the components of each factor being paths of length 1 and 2.)*

PROOF. Suppose that, for some j, there exists $i \in I_j \cap J_j$. Then A_j has a bisection, using say colours a' and a'', such that $n_2(a') = n_2(a'') = i$, and B_j has a bisection using b' and b'' such that $n_2(b') = n_2(b'') = i$. Together they form an allowable 4-edge-colouring γ of G. For each $c \in \{a', a'', b', b''\}$, $N(c, \gamma)$ is isomorphic to $iP_3 \cup (v/2 - 2i)\ P_2$ (where P_k is the path of length $k - 1$), and thus γ is isofactorial.

Therefore suppose that $I_j \cap J_j = \varnothing$ for all j, $0 \le j \le v$. Then $I_0 \cap J_0 = [a_0, c_0] \cap [b_0, d_0] = \varnothing$, and without loss of generality it can be assumed that $c_0 < b_0$. Then $c_v = d_0 \ge b_0 > c_0 \ge a_0 = b_v$, and hence $j^* = \min\{j : c_{j+1} \ge b_{j+1}\}$ exists. Let $j = j^*$.

Then $b_j \ge c_j + 1$, but $c_{j+1} \ge b_{j+1}$. Also, since $I_{j+1} \cap J_{j+1} = [a_{j+1}, c_{j+1}] \cap [b_{j+1}, d_{j=1}] = \varnothing$, it must be true that $a_{j+1} \ge d_{j+1} + 1$. Hence, from these inequalities and from Lemma 3.6 it follows that

$$b_{j+1} + 1 \ge b_j \ge c_j + 1 \ge a_j + 1 \ge a_{j+1} \ge d_{j+1} + 1 \ge b_{j+1} + 1,$$

and

$$b_{j+1} + 1 \ge b_j \ge c_j + 1 \ge c_{j+1} \ge a_{j+1} \ge d_{j+1} + 1 \ge b_{j+1} + 1.$$

Both of the above chains of inequalities must hold with equality at every stage, and therefore $a_j + 1 = c_j + 1 = a_{j+1} = c_{j+1}$. Thus A_j is inflexible because $a_j = c_j$, and A_{j+1} is inflexible because $a_{j+1} = c_{j+1}$. But $a_j \ne a_{j+1}$, which contradicts Lemma 3.8. It is therefore impossible to have $I_j \cap J_j = \varnothing$ for all j, and hence G has the required isofactorial colouring.

4. Algorithms

Sections 2 and 3 provide not only proofs that isofactorial edge-colourings exist, but also polynomial time algorithms for finding them. In order to give asymptotic bounds on the number of operations each algorithm requires, the following three results are needed.

LEMMA 4.1. *If G is a graph all of whose vertices are of even degree, then a collection of closed eulerian trails, one for each component of G, can be found in $O(e(G))$ operations.*

PROOF. The maze-searching algorithm of Edmonds and Johnson [1, 5.3, page 114], slightly modified to allow for disconnected graphs, accomplishes this.

LEMMA 4.2. *A 1-factor (1-regular spanning subgraph) of a regular bipartite graph G can be found in $O(v(G)^{1/2}e(G))$ operations.*

PROOF. The algorithm of Hopcroft and Karp [5] produces a maximum matching (1-regular subgraph) of a bipartite graph G in $O(v(G)^{1/2}e(G))$ operations; if G is regular then by a theorem of König [6] this matching is a 1-factor.

LEMMA 4.3. *A 2-factorization of a $2k$-regular graph G, $k \geq 1$, can be found in $O(k^2v(G)^{3/2})$ operations.*

PROOF. As explained by Fleischner [4, page 30], this can be done as follows. First, find closed eulerian trails in each component of G, and orient each edge in the direction of the trail containing it. Split each vertex of G into two, one incident with the inwardly oriented edges and the other with the outwardly oriented edges, to form a k-regular bipartite graph G^*, where $v(G^*) = 2v(G)$ and $e(G^*) = 2e(G)$. By k applications of Lemma 4.2, a decomposition of G^* into 1-factors can be found in $O(kv(G)^{1/2}e(G)) = O(k^2v(G)^{3/2})$ (since $e(G) = kv(G)$) operations, and this corresponds to a 2-factorization of G. The bound on the number of operations is determined by this last step.

These results may now be used to give an upper bound on the number of operations required to implement the methods of Section 2.

THEOREM 4.4. *Let G be a $2k$-regular graph, $k \geq 1$, with $v(G)$ even. Then the following algorithm, which attempts to find an isofactorial allowable $2k$-edge-colouring of G, can be implemented so as to take $O(k^2v(G)^{3/2})$ operations. It can fail only if G contains both 3-cycles and 5-cycles.*

First, find a 2-factorization of G into 2-factors $F_1, F_2, \dots, F_k$. For each F_i calculate the length of each component cycle of F_i, and thence the numbers $P(F_i)$ and $Q(F_i)$ as in Lemma 2.6 (b). Let $P^* = \max\{P(F_i): i = 1, 2, \dots k\}$ and $Q^* = \min\{Q!(F_i): i = 1, 2, \dots k\}$. If $P^* > Q^*$ stop: G contains both 3-cycles and 5-cycles. Otherwise, construct for each F_i a bisection with $n_2 = P^*$, giving the required colouring of G.

PROOF. Each step of the algorithm can be implemented in $O(e(G)) = O(kv(G))$ operations, except for finding a 2-factorization of G, which by Lemma 4.3 can be done in $O(k^2v(G)^{3/2})$ operations. Hence is obtained the bound on the number of operations required.

If G has either no 3-cycles or no 5-cycles then it follows from Lemma 2.7 that $P^* \leq Q^*$, and therefore the algorithm will succeed.

The algorithm obtained from Section 3 is asymptotically a little less efficient than that of Section 2, when applied to 4-regular graphs, but of course it works for *all* such graphs with an even number of vertices.

THEOREM 4.5. *Let G be a 4-regular graph, with $v(G)$ even. Then an isofactorial allowable 4-edge-colouring of G can be found in $O(v(G)^2)$ operations by an implementation of the following algorithm.*

First, find a eulerian sequence in G. For each j, $0 \leq j \leq v(G)$, construct the graphs A_j and B_j as in Section 3, identify their components, and calculate the intervals I_j and J_j. If there exists $i \in I_j \cap J_j$, construct bisections of A_j and B_j with $n_2 = i$, and hence obtain the required colouring of G.

PROOF. In Theorem 3.9 it was proved that $I_j \cap J_j$ is nonempty for some j, and therefore the algorithm will find the desired colouring. Construction of an eulerian sequence can be done in $O(e(G))$ operations, by Lemma 4.1, and the final step of constructing the bisections for a particular A_j and B_j also can be done in $O(e(G))$ operations. Constructing each A_j and B_j and identifying their components also takes $O(e(G))$ operations, but this may have to be done $v(G)+1$ times. Therefore the algorithm overall can be implemented in $O(v(G)e(G)) = O(v(G)^2)$ operations.

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