Developing new picture proofs that the sums of the first *n* odd integers are squares

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Introduction

What is it that makes us judge two proofs of the same theorem to be the same or different? This is not an idle question: one central aspect of judging mathematics is the *novelty* of the mathematics presented. This is important everywhere, from the peer-review system, to assigning international prestige, to funding agencies' grant decisions. It even matters to some extent in examinations, to avoid accusations of collusion. Surprisingly, philosophers of mathematics have not paid the question of novelty much attention. In this Article, we will consider the appealing conjecture that the main ideas that make up the proof, the essence of a proof, can indeed be identified and that very different styles of proofs can share common main ideas. Further, that a particular theorem can be proved using quite different, independent main ideas. As a means of exploring whether this is plausible, we will present a number of novel proofs of the following theorem.

Theorem 1: The sum of the first n odd integers, starting from 1, is n^2.

Expressed in algebraic notation, Theorem 1 becomes

$$1 + 3 + 5 + 7 + \ldots + (2n - 1) = \sum_{k=1}^{n} (2k - 1) = n^2. \qquad (1)$$

In particular we will record the genesis of three novel ‘proofs without words’, and look at how they relate to several algebraic proofs. This will let us compare the main ideas of a proof to how it is presented or, in other words, compare the content and form of a proof.

Mathematicians often do not record the genesis of an idea. Gauss is often attributed with the sentiment that no architect leaves the scaffolding in place after completing the building. Some writers, notably [1], balance Gauss's austere view, but students often see mathematics as a static, ‘finished’, product, see e.g. [2].

Without wanting to create an artificial product/process dichotomy, we think it is important both to record mathematics in a formal traditional way and to encourage intuitive, creative thinking. Educators, in particular, have discussed the different roles *proof plays*. These roles include proving *that* a theorem is true juxtaposed against proving why a theorem is true, [3]. Other authors have supported the proposition that collecting proofs is valuable. For example [4] suggested a variety of proofs are valuable for at least four reasons: (a) variety helps us understand *why* a theorem is true; (b) there is potential for generalising in different directions; (c) they provide an opportunity to discuss aesthetics in mathematics; (d) they can have historical interest, see e.g. [5].

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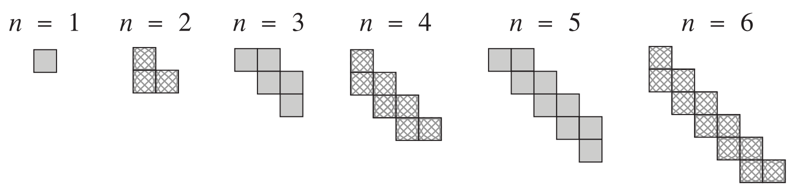
This work started as a reaction to [6] who collected together different proofs of the same theorem as a vehicle through which to discuss mathematical style. Mathematical style is also discussed by [7] who collected together proofs that $\sqrt{2}$ is irrational, and the famous work of [8] provides many proofs of the Pythagorean proposition.

Since this paper is about style of proof, and the *essence of a mathematical proof*, we thought hard and discussed at length how to present the genesis of our novel proofs without words. In the end we chose this direct style, hoping to narrate accurately the process of discovery, and record honestly the philosophical and educational issues which occurred to us through our exploration of these proofs.

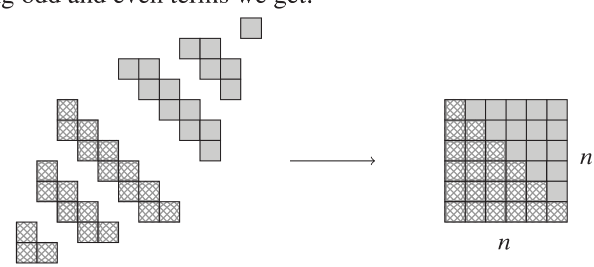
Genesis of new proofs

The story started in December 2020 as the first author (CJS) wanted to create a collection of proofs. The goals for the collection of proofs were (i) personal satisfaction, (ii) possible future use as teaching materials, and (iii) with potential as materials for research on students' conceptions of rigour and insight. The chosen Theorem 1 and an initial collection of proofs, with commentary, has been published as [9], and the research with undergraduate students is now also submitted. This collection of proofs was sent to the second author (FST), who has a professional interest in the nature of proof and the philosophy of mathematical arguments. In correspondence he proposed the following diagrammatic proof. (The naming convention used here, e.g. "Pictorial III", is to retain compatibility with names used in [9] for future disambiguation.)

Pictorial III Consider the following sequence of arranging the odd numbers into 'zigzags'.



Combining odd and even terms we get:



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The idea for this version of the proof was found through the classic method of scribbling on notepaper, looking for new patterns of odd numbers that could combine into a square.

This proof then triggered the idea of considering other ways of proving the result diagrammatically, and the ensuing discussion included the suggestion that the 'Rearranging II' proof from [9] could be turned into a diagrammatic proof.

Rearranging II We use the standard result $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and rearrange:

$$\underbrace{\sum_{k=1}^{n}(2k-1)}_{\text{odd}} = \underbrace{(1+2+3+\ \dots\ +2n)}_{\text{all}} - \underbrace{(2+4+6+\ \dots\ +2n)}_{\text{even}}$$

$$= (1+2+3+\ \dots\ +2n) - 2(1+2+3+\ \dots\ +n).$$

Hence

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{2n} k - 2\sum_{k=1}^{n} k = \frac{2n(2n+1)}{2} - 2\frac{n(n+1)}{2} = n^2.$$

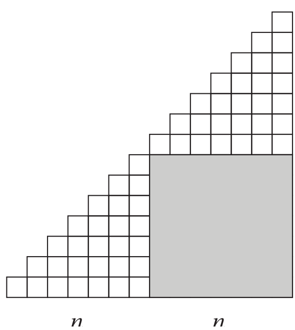
To produce a diagrammatic version of this proof, we were inspired by [10, p. 70] to incorporate the above into 'Rearranging II', producing this argument.

Pictorial IV

To sum the odd numbers we notice

$$\underbrace{\sum_{k=1}^{n}(2k-1)}_{\text{odd}} = \underbrace{(1+2+3+\ \dots\ +2n)}_{\text{all}} - \underbrace{(2+4+6+\ \dots\ +2n)}_{\text{even}}$$

$$= (1+2+3+\ \dots\ +2n) - 2(1+2+3+\ \dots\ +n).$$



On subtracting the triangular areas the grey square remains: n^2.

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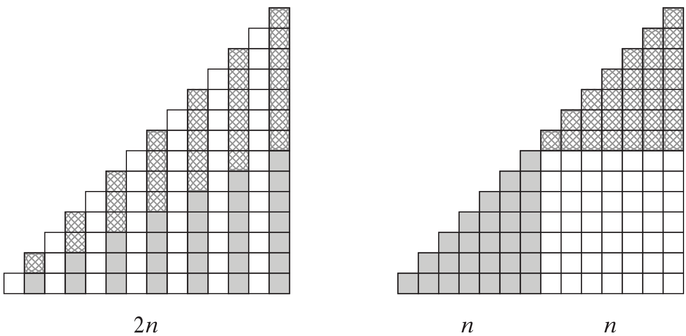
One worry remained: this argument is not a pure proof without words. The essential algebraic observation in 'Rearranging II' was not replaced by a diagram, but was retained in algebraic form. As a proof without words, this is deeply unsatisfactory, and so we sought a purely diagrammatic version of this mathematical observation. The key stumbling block, for some reason, was how to represent the trivial factoring of

$$2 + 4 + 6 + \ldots + 2n = 2(1 + 2 + 3 + \ldots + n)$$

in diagrammatic form. How to represent this factoring was eventually resolved in the following argument:

Pictorial V



Both Pictorial III and Pictorial V are 'proofs without words', i.e. a diagrammatic demonstration of a result, without any accompanying explanatory text. Three classic collections of proofs without words are by Robert B. Nelsen, which are published as [10], [11] and [12]. Three proofs without words of Theorem 1 are included in [10], and a further two in [12]. Recently [13] suggested some criteria for accepting a proof without words. In particular he suggested the following three criteria.

1. It should illuminate the result so that it is not just a technical exercise.
2. It should illustrate the result pictorially.
3. It should not require any words or equations.

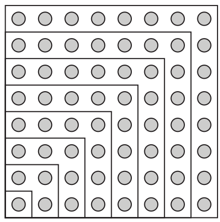
We believe both Pictorial III and Pictorial V satisfy these three criteria. In addition, to merit publication, [13] suggested it should also be original! We could not find Pictorial III and Pictorial IV proofs here or elsewhere, and following private communication with Professor Nelsen, we think they may be novel.

The essence of a proof

Returning to the question of the essence of a proof, to what extent are 'Rearranging II', 'Pictorial IV' and 'Pictorial V' the same proof? At some level, the two arguments rely on the same observations and main idea, so at their core they share the same *essence*. By essence we do not mean the *type* or *style* of the argument. Types of proofs include 'proof by induction', 'proofs by contradiction' and 'exhaustive cases'. The proof 'Rearranging II' is an algebraic calculation whereas 'Pictorial V' is a proof without words. We consider the essence of a proof to be the fundamental mathematical idea or pattern which enables us to establish the theorem. To investigate this, let us consider some of the other proofs without words of Theorem 1 and their main ideas.

The original "proof by picture" is attributed to Nicomachus of Gerasa, circa 100CE, which is included as [10, p. 71].

Pictorial I



The *essence* of this argument appears to us to be that successively adding the odd numbers we can build up squares step by step, the difference between two adjacent squares is an odd number, in modern algebraic notation,

$$(n+1)^2 - n^2 = 2n + 1.$$

Many proofs rely on this observation. The induction step in a typical proof by induction, including the 'Induction' proofs of [9], makes use of this observation directly to move from case n to $n + 1$. So does the following 'Telescoping' argument.

Telescoping

Notice that $2k - 1 = k^2 - (k-1)^2$, so that adding up we have

$$\sum_{k=1}^{n} (2k-1) = \sum_{k=1}^{n} \left(k^2 - (k-1)^2\right).$$

However, in

$$\sum_{k=1}^{n} \left(k^2 - (k-1)^2\right) = \left(1^2 - 0^2\right) + \left(2^2 - 1^2\right) + \ \dots \ + \left(n^2 - (n-1)^2\right)$$

all terms cancel except two, one from the first term and one from the last, i.e. $-0^2 + n^2$, leaving n^2.

The Telescoping argument is self-contained and appears very specific to this situation. The proof entitled 'Backwards reasoning' in [9] makes use of the Fundamental Theorem of Finite Differences.

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Fundamental theorem of finite differences

$S_n = \sum_{k=1}^{n} a_k$ if, and only if, (i) $a_{n+1} = S_{n+1} - S_n$ and (ii) $S_1 = a_1$.

A rigorous proof of the Fundamental Theorem of Finite Differences requires mathematical induction, indeed induction is inherent in many of the ideas in this paper. Notice that a base case is needed to accommodate the discrete counterpart of a constant of integration. The Fundamental Theorem of Finite Differences is a generalisation of the algebraic observation explicit in the Telescoping argument. Rather than using a specific example the theorem abstracts the criteria needed for the essence of the argument to work. The Fundamental Theorem of Finite Differences takes a specific argument and makes it into a more abstract general argument.

Translating essential ideas

Without the explicit narrative given here (or something explicit but more succinct), would people make the connection that two proofs share an essential idea? As an informal test to see whether the main ideas of Pictorial III could be reliably translated into an algebraic argument, the authors independently tried to develop an algebraic counterpart of Pictorial III, and our results are below.

FST version: The essence of this proof is to split the sum of odd numbers into two sequences: the 'odd-odds' of the form $4n - 3$ and the 'even-odds' of the form $4n - 1$. Now

$$\sum_{k=1}^{m} (4k - 3) = \frac{4m(m + 1)}{2} - 3m = \frac{(2m - 1)2m}{2}$$

and

$$\sum_{k=1}^{m} (4k - 1) = \frac{4m(m + 1)}{2} - m = \frac{2m(2m + 1)}{2}.$$

If we are summing an even number of terms we can combine these as follows:

$$\sum_{k=1}^{2m} (2k - 1) = \sum_{k=1}^{m} (4k - 3) + \sum_{k=1}^{m} (4k - 1) = \frac{(2m - 1)2m}{2} + \frac{2m(2m + 1)}{2}$$

$$= \frac{2m(2m + 1 + 2m - 1)}{2} = 4m^2 = (2m)^2.$$

If we are summing an odd number of terms we can combine these as follows:

$$\sum_{k=1}^{2m-1} (2k - 1) = \sum_{k=1}^{m} (4k - 3) + \sum_{k=1}^{m-1} (4k - 1)$$

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$$= \frac{(2m-1)2m}{2} + \frac{2(m-1)(2(m-1)+1)}{2}$$

$$= \frac{(2m-1)(4m-2)}{2} = (2m-1)^2.$$

Notice how this proof repeatedly uses $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, just as does the argument Rearranging I:

Rearranging I We use the standard results $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} 1 = n$ and rearrange:

$$\sum_{k=1}^{n} (2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2\frac{n(n+1)}{2} - n = n^2.$$

Indeed, a general approach to finding sums of the form $\sum_{k=1}^{p} p(k)$, where p is a polynomial, is to expand and use standard results for the sum of each power $\sum_{k=1}^{n} k^m$, where $m \in \mathbb{N}$. The essential idea is to represent a complex problem, i.e. $\sum p(k)$, with a basis of terms. We now have many simpler problems for which the results are known.

The other attempt took quite a different approach to creating an algebraic interpretation of the diagrams in Pictorial III.

CJS version: Consider the first n odd numbers from 1 to $2n-1$:

$$1 + 3 + 5 + 7 + \ldots + (2n-3) + (2n-1).$$

Split this into two groups.

$$1 + 5 + 9 + \ldots + (2n-3) = 1 + (2+3) + (4+5) + \ldots + \big((n-2) + (n-1)\big)$$

$$3 + 7 + \ldots + (2n-1) = (1+2) + (3+4) + (5+6) + \ldots + \big((n-1) + n\big).$$

This shows that the odd numbers from 1 to $(2n-1)$ can be split and rearranged into two groups, which are the two consecutive triangular numbers

$$1 + 2 + 3 + \ldots + (n-1) \text{ and } 1 + 2 + 3 + \ldots + n.$$

Reversing the first list and adding we have (essentially the 'Reversed' argument)

$$\begin{array}{ccccccccccc} n-1 & + & n-2 & + & n-3 & +\ldots+ & 1 & + & 0 \\ 1 & + & 2 & + & 3 & +\ldots+ & n-1 & + & n \end{array}$$

where each column totals to n and there are n columns so the total is n^2.

Curiously, the two algebraic proofs we came up with separately appear substantially different, despite both being attempts to find an algebraic presentation with the essence of the pictorial proof. One proof concentrated

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on the odd and even parts of the sequence and used the formula $\frac{1}{2}n(n + 1)$, while the other focused on the reversing argument. This suggests that form and content may not be so easily separated: the presentation of the argument has an effect on the way we understand the key ideas. What appear to be straightforward details in one setting may need careful scrutiny in another.

A reader helpfully pointed out that the CJS proof only applies when n is even. A slight variation is required when n is odd:

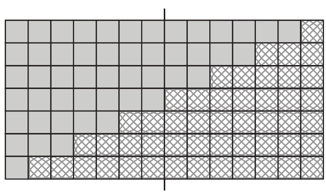
> The changes are simply that $2n - 3$ and $2n - 1$ have to be swapped and also $(n - 2) + (n - 1)$ with $(n - 1) + n$. After that the proof is the same for odd or even n.

While experts might just say 'similarly when n is odd', the argument as written by CJS certainly has an omission. With this omission restored both proofs treat n odd/even separately.

Notice how the FST version used the formula $\frac{1}{2}n(n + 1)$ whereas the CJS proof contains the 'Reversed' argument. The reversed argument is attributed to Gauss (but see [15]) to sum the first n integers. There is a clear pictorial counterpart to 'Reversed' proofs in general, and it is not difficult to create a diagrammatic counterpart to show

$$2(1 + 3 + 5 + 7 + \dots + (2n - 1)) = 2n^2.$$

Pictorial VI



For this 'Reversed' argument, it does seem that the essence of a proof has both an algebraic and pictorial representation. As with the Fundamental Theorem of Finite Differences, this essence is also manifested in a general theorem: the sum of an arithmetic progression is the average of the first and last terms, multiplied by the number of terms. The horizontal visual matching in this picture helps to justify why considering only the first and last terms is sufficient. This is another example where a specific argument has been generalised into a more general theorem.

Our next example is to transform *Rearranging I* into a picture, giving another novel proof without words. In an abbreviated algebraic form Rearranging I is

$$\sum_{k=1}^{n}(2k - 1) = 2\sum_{k=1}^{n}k - \sum_{k=1}^{n}1 = 2\left(\frac{n(n + 1)}{2}\right) - n = n^2. \qquad (2)$$

Our first attempt to create a picture was to reuse the algebraic factoring idea developed for Pictorial IV as a guide for grouping squares.

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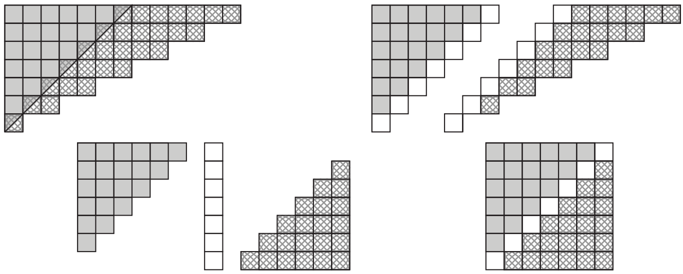
Pictorial VII



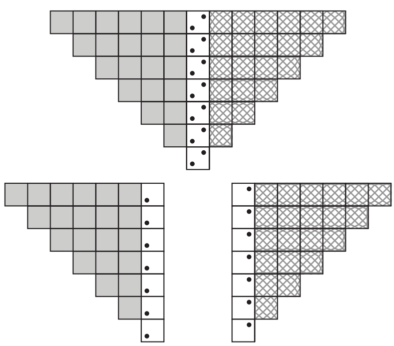
The algebraic proof (2) makes use of two copies of the sum $1 + \ldots + n$, i.e. with n terms. Hence we need the white squares in Pictorial VII to overlap, so that one copy needs to be subtracted algebraically giving the $-n$ term. This subtraction is far from clear, indeed Pictorial VII went through many iterations and is still not entirely satisfactory.

In an e-mail response to a draft of this paper Donald Smith also expressed dissatisfaction with Pictorial VII and suggested the following remedy:

> What if you were to start with a left-right symmetrical diagram by centring all the rows? You might even make the central column white, with two dots in each square to clarify the next stage. Split the diagram down the axis, but repeat the centre column in both halves, with white squares but only one dot in each one. Turn the left-hand half upside down. Merge the two halves by overlapping the white squares, now putting two dots in each one to indicate a full square again. Finally, justify the rows to make the $n \times n$ square.

This idea generates the following diagrams as the pictorial counterpart of Rearranging I.

Pictorial VIII



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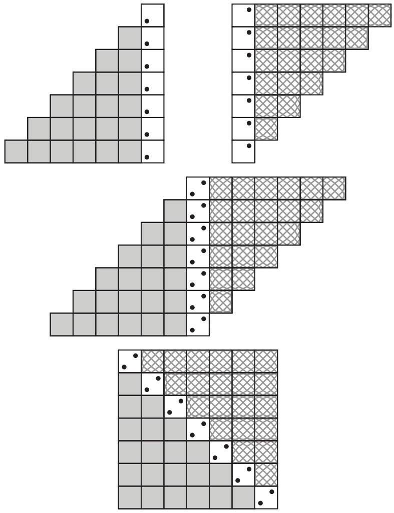


Look again at the final diagrams in Pictorial VII/VIII. Perhaps we could interpret the final diagram as two copies of the numbers $1 + 2 + \ldots + (n - 1)$ together with one copy of n. Can that observation lead to another algebraic proof? Indeed, it can.

Rearranging III

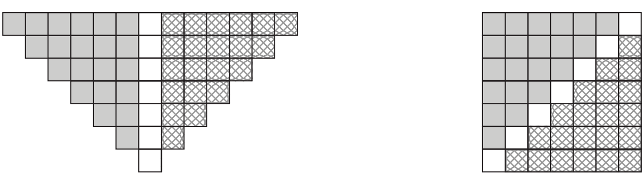
$$\begin{aligned} & 1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) \\ &= (1 + 0) + (1 + 2) + (1 + 4) + \ldots + \big(1 + (2n - 2)\big) \\ &= n + \big(2 + 4 + \ldots + 2(n - 1)\big) \\ &= n + 2\frac{(n - 1)n}{2} = n^2. \end{aligned}$$

Put another way,

$$\sum_{k=1}^{n} (2k - 1) = \sum_{k=0}^{n-1} (1 + 2k) = \sum_{k=0}^{n-1} 1 + 2\sum_{k=0}^{n-1} k = n + 2\frac{(n-1)n}{2} = n^2.$$

We have therefore used the picture in Pictorial VII to create yet another algebraic proof. This leads to one final pictorial proof, which is a much more direct pictorial interpretation of Rearranging III.

Pictorial IX



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Discussion

In summary we have identified three essential ideas.

1. Balance reversed lists.
2. The difference of adjacent squares is odd: $(n + 1) - n^2 = 2n + 1$.
3. Use a basis of known basic results.

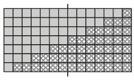
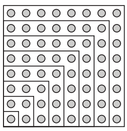
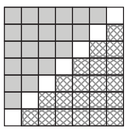
Essence	Picture	Specific algebraic	General theorem
Balance reversed lists		Reversed list	Sum of arithmetic progression
Difference of adjacent squares is odd		Telescoping	Fundamental Theorem of Finite Differences, more generally induction
Use basis of known results		Rearranging I, Rearranging III, and many others	Expand $\sum_{k=1}^{n} p(k)$ to use known results $\sum_{k=1}^{n} k^m$

TABLE 1: Three essential ideas and the corresponding proofs

For each of these ideas we have a proof without words, and one or more specific algebraic arguments. Furthermore, for each of these essential ideas we can derive a general theorem which can be applied to a wide range of similar problems. The balance reversed lists generalise to arithmetic progressions. The essence of the difference of adjacent terms becomes the Fundamental Theorem of Finite Differences, and even more generally the step-by-step approach becomes proof by induction. Lastly, using a basis of known basic results allows us to find the sum of any polynomial provided we have the sum of the basic terms $\sum_{k=1}^{n} k^m$ where $m \in \mathbb{N}$. These observations are summarised in Table 1.

It seems to us that Pictorial III and Pictorial V make use of very different essential ideas which are much more specific to Theorem 1. A different picture proof in which four copies of the sum of the odd numbers are arranged into a larger square was given by [10, p. 72].

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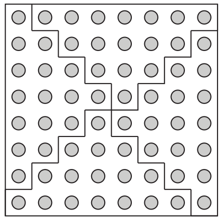
Pictorial II



The side length of the square is one more than the last term in the sum. The last term in the sum is $2n - 1$ and so the area of the square is $4n^2$. This gives the sum of the odd numbers as n^2 as before.

The essence of this argument does not rely on differences, but derives the total area directly. The difficult step seems to be noticing the last term is $2n - 1$. When using this proof with students, adding two sentences explaining (1) the stepped triangles are the sums of the odd numbers, and (2) four copies of this triangle can be fitted together to give a square, changed significantly how much the proof helped them to understand the theorem. The utility of the notion of essence is in recognising a mathematical idea shared between proofs. Just because, in some situations, we did not find something shared does not mean we cannot make sense of the concept, or use the concept fruitfully as a tool for discussing proofs. Rather, it is a warning that we cannot take for granted that an essence always exists. The essential ideas in Pictorial II, Pictorial III and Pictorial IV certainly have algebraic counterparts. At this stage we have not been able to identify corresponding general theorems. Perhaps each essential idea is specific to adding adjacent odd numbers. It would be slightly surprising if all these arguments did generalise in the same useful ways the essential ideas recorded in Table 1 have done. What is more surprising is the richness of Theorem 1 and the variety of proofs we have found and the relationships between these proofs.

Conclusions

In considering the different proofs of Theorem 1 we have identified the following:

- Algebraic arguments with corresponding purely pictorial arguments, and vice versa.
- Cases where an essence can be generalised to apply to a range of examples. Specifically, the essential observation in the proof can become a general theorem, and we then collect together the range of examples to which the observation applies via a formal definition (e.g. arithmetic progression). We have given three different examples of essential ideas, as summarised in Table 1.
- Cases where an algebraic proof can be translated into multiple pictorial proofs, and cases where a pictorial proof can be translated

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into multiple algebraic proofs. We have used pictures to create new algebraic arguments, and algebraic arguments to create new proofs without words.

The conclusion from this is that it is often possible to translate the key ideas from a proof in one kind of presentation, or style, to another style, but that this is not a promising route for finding what makes proofs the same or different. The representational form affects the essential ideas too. Translation is a creative act, and can lead to new ideas and generalisations. That said, different presentations also allow for different ways of seeing and understanding the proof, understanding the theorem and understanding how a particular theorem relates to other ideas.

Proofs without words can be fun and cryptic, but can also achieve a clarity that is helpful for understanding a proof better and for generating ideas that are also of algebraic significance. This leads to several questions of pedagogical importance: are collections of proofs helpful to students, and if so how can we use them productively? Can students identify common ideas when presented with a variety of proofs, perhaps as a 'matching task' with discussion? Can students transform proofs of a particular theorem from one style (algebraic) into another (picture)? How does such an activity change their conception of the particular proof, and of their broader understanding of what proving is?

We thus believe that proofs without words should be considered more than a mere curiosity for mathematical enthusiasts, and should be taken seriously as a pedagogical tool for helping students to understand proving strategies, develop mathematical creativity, and identify the main ideas used in a proof.

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