108.30 Nearly isosceles right-angled triangles and square triangular numbers

In Vol. 106 No. 566 (July 2022) of *The Mathematical Gazette*, Geoffrey Strickland invited students to solve the following problem 2022.4. "A *nearly isosceles* right-angled triangle with integer side-lengths is defined as one in which the two sides adjacent to the right-angle differ in length by just 1 unit. A triangle with side lengths 20, 21 and 29 is an example. Find, with proof, a method of generating a sequence of such triangles." The problem is indeed a good one for a student and can be solved using known results from Number Theory. But a more detailed analysis leads to an unexpected connection with another problem, that of triangular numbers which are also square numbers. As is shown below, every square triangular number generates two nearly isosceles right-angled triangles and furthermore all nearly isosceles right-angled triangles can be generated in this way.

So consider the equation $a^2 + b^2 = c^2$, where a, b and c are integers and $b = a \pm 1$. Perhaps the simplest method of solving the problem is to introduce the obvious parameterisation $a = \frac{1}{2}(x - 1)$, $b = \frac{1}{2}(x + 1)$, $c = y$ to give the equation $x^2 - 2y^2 = -1$. This is the negative Pell's equation and can be solved using continued fractions, see Theorem 11.10 of [1]. Extracting the relevant information from the theorem, we have the following. Let d be a positive integer that is not a perfect square. Let p_k / q_k be the k th convergent of the simple continued fraction of $\sqrt{d}$, $k = 1, 2, \ldots$ and let n be the length of the continued fraction. If n is odd, the solutions of the equation $x^2 - dy^2 = -1$ are $x = p_{(2j-1)n-1}$, $y = q_{(2j-1)n-1}$, $j = 1, 2, \ldots$. Now the continued fraction of $\sqrt{2}$ is $[1 : 2, 2, \ldots]$ so $n = 1$. The first four solutions are given in the following Table.

j	p_{2j}	q_{2j}	(a, b, c)
1	7	5	(3, 4, 5)
2	41	29	(20, 21, 29)
3	239	169	(119, 120, 169)
4	1393	985	(696, 697, 985)

A recurrence relation can be deduced from the continued fraction and is given by $p_{2j+2} = 3p_{2j} + 4q_{2j}$, $q_{2j+2} = 2p_{2j} + 3q_{2j}$, $j = 1, 2, \ldots$, $p_0 = q_0 = 1$.

A more direct solution can be obtained from Theorem 244 of the classic text by Hardy and Wright [2]. From that theorem, the solutions of the equation $x^2 - 2y^2 = -1$ are given by $x + y\sqrt{2} = \pm(1 + \sqrt{2})^{2n+1}$, (and the solutions of the equation $x^2 - 2y^2 = 1$, which is used later, by $x + y\sqrt{2} = \pm(1 + \sqrt{2})^{2n}$), $n = 1, 2, \ldots$.

This second method immediately above is probably the simplest way of finding all nearly isosceles right-angled triangles. In order to exhibit the relationship with square triangular numbers consider a slightly different approach. A solution of the equation $a^2 + b^2 = c^2$ in positive integers is

https://doi.org/10.1017/mag.2024.81 Published online by Cambridge University Press

said to be *primitive* if $\gcd(a, b, c) = 1$. Euclid's formula, see Theorem 11.1 of [1], states that the parameterization $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$ where m and n are co-prime with one of them odd and the other even, $m > n$, generates all primitive solutions. Clearly any solution for nearly isosceles right-angled triangles is primitive. There are two cases to consider.

Suppose first that $(m^2 - n^2) - 2mn = -1$. Then $n^2 + 2mn - (m^2 + 1) = 0$. So $n = \frac{1}{2}\left(-2m \pm \sqrt{4m^2 + 4(m^2 + 1)}\right)$, i.e. $n = \sqrt{2m^2 + 1} - m$. Now $2m^2 + 1$ is odd and so to have an integer square root must be of the form $(2t + 1)^2 = 4t^2 + 4t + 1$. Therefore $m^2 = 2t(t + 1)$, i.e. $\left(\frac{m}{2}\right)^2 = \frac{1}{2}t(t + 1)$. The second case where $(m^2 - n^2) - 2mn = 1$ leads similarly to the equations $m = \sqrt{2n^2 + 1} + n$ and $\left(\frac{n}{2}\right)^2 = \frac{1}{2}t(t + 1)$.

As is well-known, a *triangular number* is an integer of the form $\frac{1}{2}t(t + 1)$, $t = 1, 2, \ldots$. So for a triangular number to be a square we must have $\frac{1}{2}t(t + 1) = s^2$ leading to $(2t + 1)^2 - 2(2s)^2 = 1$ which is Pell's equation $x^2 - 2y^2 = 1$. As is easily verified, all solutions have x odd and y even so every solution of Pell's equation gives a square triangular number. Solutions to finding nearly isosceles right-angled triangles are found by putting either $m = 2s$ and then $n = \sqrt{2m^2 + 1} - m$ or $n = 2s$ and then $m = \sqrt{2n^2 + 1} + n$ as indicated in the following Table.

$(2t + 1) + 2s\sqrt{2}$	m	n	a	b	c
$(1 + \sqrt{2})^2$	2	1	3	4	5
$= 3 + 2\sqrt{2}$	5	2	21	20	29
$(3 + 2\sqrt{2})(3 + 2\sqrt{2})$	12	5	119	120	1695
$= 17 + 12\sqrt{2}$	29	12	697	696	985
$(17 + 12\sqrt{2})(3 + 2\sqrt{2})$	70	29	4059	4060	5741
$= (99 + 70\sqrt{2})$	169	70	23661	23660	33461
$(99 + 70\sqrt{2})(3 + 2\sqrt{2})$	408	169	137903	137904	195025
$= 577 + 408\sqrt{2}$	985	408	803761	803760	1136689

From the above calculations, given a square triangular number, two isosceles right-angled triangles can be obtained directly. So let $N = s^2 = \frac{1}{2}t(t + 1)$ be a square triangular number. The two nearly isosceles right-angled triangles that it generates are obtained by putting either $m = 2s$ and $n = \sqrt{2m^2 + 1} - m = 2t - 2s + 1$ or $n = 2s$ and $m = \sqrt{2n^2 + 1} + n = 2t + 2s + 1$ in the above Euclid's formula parameterization.

The preceding theory can easily be extended to finding solutions of the equation $a^2 + b^2 = c^2$ where $b = a \pm 2$. Here the obvious parameterization $a = x - 1$, $b = x + 1$, $c = y$ gives the equation $2x^2 + 2 = y^2$. Hence $y = 2z$ is even, again leading to the negative Pell equation $x^2 - 2z^2 = -1$.

https://doi.org/10.1017/mag.2024.81 Published online by Cambridge University Press

Further, x must be odd, but then all of $x - 1$, $x + 1$ and y are even meaning that solutions of the equation $a^2 + (a + 2)^2 = c^2$ are obtained from solutions of $a^2 + (a + 1)^2 = c^2$ by multiplication by 2. There are no other solutions.

References

1. K. H. Rosen, *Elementary Number Theory and its Applications*, Addison-Wesley (1992).
2. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press (1938).

10.1017/mag.2024.81
Published by Cambridge University Press on behalf of The Mathematical Association

TERRY S. GRIGGS
School of Mathematics and Statistics,
The Open University,
Milton Keynes MK7 6AA
e-mail: *terry.griggs@open.ac.uk*

108.31 Generalised Thales intercept theorem

According to T. Heath [1, p. 124], [2, p. 128] and C. R. Fletcher [3, p. 268], Thales (about 624-547 B.C.) is a central figure in the evolution of geometry, as he was the first scientist to introduce proofs alongside empirical methods. One of the main results attributed to Thales is the so-called "intercept theorem", which the Greek scientist used to measure the heights of pyramids and distances of ships at sea [1, p. 124]. In [4, p. 9], John Stillwell underlines the importance of this theorem saying that it "is the key to using algebra in geometry". E. Moise, in [5, pp. 136-141], provides the following simple statement of Thales intercept theorem:

Parallel projections are one-to-one correspondences that preserve betweenness, congruence and ratio;

moreover, the author shows that the result on ratios and his converse can be deduced from the fact that parallel projections preserve the midpoint of a segment. Therefore, we focus our attention on the following statement:

Take two parallel segments A_1A_1' and A_2A_2' and find the midpoints M and M' of the segments A_1A_2 and $A_1'A_2'$. Then, the segment MM' is parallel to A_1A_1' and A_2A_2'.

Using the technique shown by N. Lord in [6], we can generalise the previous statement as follows:

Take n parallel segments A_1A_1', A_2A_2', $\dots$, A_nA_n' and find the centres of gravity M and M' of the sets $\{A_1, A_2, \dots, A_n\}$ and $\{A_1', A_2', \dots, A_n'\}$. Then, the segment MM' is parallel to the segments A_1A_1', A_2A_2', $\dots$, A_nA_n'.

Indeed, let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ and $\mathbf{a}_1', \mathbf{a}_2', \dots, \mathbf{a}_n'$ be the position vectors of consecutive vertices $A_1, A_2, \dots, A_n$ and $A_1', A_2', \dots, A_n'$. Then, the position vectors of the centres of gravity M and M' are given by $\mathbf{m} = \frac{1}{n}(\mathbf{a}_1 + \mathbf{a}_2 + \dots + \mathbf{a}_n)$ and $\mathbf{m}' = \frac{1}{n}(\mathbf{a}_1' + \mathbf{a}_2' + \dots + \mathbf{a}_n')$. It follows that