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A NEW CHARACTERIZATION OF WIMAN SURFACES

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Abstract

Let $\mathcal{M}$ be a regular map of genus $g > 1$ and X be the underlying Riemann surface. A reflection of $\mathcal{M}$ fixes some simple closed curves on X, which we call *mirrors*. Each mirror passes through at least two of the geometric points (vertices, face-centers and edge-centers) of $\mathcal{M}$. In this paper we study the surfaces which contain mirrors passing through just two geometric points, and show that only Wiman surfaces have this property.

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1. Introduction

A compact Riemann surface X of genus $g > 1$ is called *symmetric* if it admits an anti-conformal involution $\sigma : X \to X$, which is called a *symmetry* of X. The fixed-point set of σ consists of k disjoint simple closed curves on X, and these curves are called the *mirrors* of σ. Here k is an integer and by a classical theorem of Harnack $0 \le k \le g + 1$. Let $\mathcal{M}$ be a regular map on X. (Maps and regular maps are described in the next section.) A *reflection* of $\mathcal{M}$ is a symmetry of X that leaves $\mathcal{M}$ invariant and fixes some mirrors. It follows from [8] that each mirror of a reflection of $\mathcal{M}$ passes through at least two of the geometric points of $\mathcal{M}$. By geometric points we mean the vertices, face-centers and edge-centers of $\mathcal{M}$. In this paper we study the surfaces which contain mirrors passing through just two geometric points, and show that only Wiman surfaces have this property. (We briefly describe these surfaces in the next section.) This result is, in fact, a new characterization of these surfaces.

2. Preliminaries

2.1. NEC groups A discrete subgroup of PSL(2, $\mathbb{R}$), the group of conformal isometries of the hyperbolic plane $\mathbb{H}$, is called a *Fuchsian group*. A discrete subgroup of PGL(2, $\mathbb{R}$), the group of isometries of $\mathbb{H}$, with compact quotient space is called a *non-Euclidean crystallographic (NEC) group*.

Let T be a hyperbolic triangle with angles π/l, π/m and π/n, where each of l, m and n is a positive integer greater than one and $1/l + 1/m + 1/n < 1$. Such a triangle is called a *(l, m, n)-triangle*. The group Γ^* generated by the reflections in the sides of T is a NEC group and is called the *NEC triangle group* $\Gamma^*(l, m, n)$, which has a presentation

$$\langle p, q, r \mid p^2 = q^2 = r^2 = (pq)^l = (qr)^m = (rp)^n = 1\rangle.$$

The subgroup Γ of Γ^* consisting of conformal isometries is called the *Fuchsian triangle group* $\Gamma[l, m, n]$, and it has a presentation

$$\langle x, y, z \mid x^l = y^m = z^n = xyz = 1\rangle.$$

See [7, 10] for details.

2.2. Automorphisms of Riemann surfaces A compact Riemann surface X of genus $g > 1$ can be expressed in the form $\mathbb{H}/\Omega$, where Ω is a torsion-free Fuchsian group. An *automorphism* of X is a conformal or anti-conformal homeomorphism $f : X \to X$. All automorphisms of X form a group under composition of maps and we denote it by $\mathrm{Aut}^{\pm} X$ and the subgroup consisting of conformal automorphisms by $\mathrm{Aut}^{+} X$. The groups $\mathrm{Aut}^{+} X$ and $\mathrm{Aut}^{\pm} X$ are isomorphic to $N^{+}(\Omega)/\Omega$ and $N^{\pm}(\Omega)/\Omega$, respectively. Here $N^{+}(\Omega)$ and $N^{\pm}(\Omega)$ denote the normalizers of Ω in PSL(2, $\mathbb{R}$) and PGL(2, $\mathbb{R}$), respectively. Any group G of automorphisms of X may be lifted to an NEC group Δ acting on $\mathbb{H}$. If G does not contain sense-reversing automorphisms, then Δ is Fuchsian. In either case there is an epimorphism from Δ to G with kernel Ω. Such an epimorphism is called *smooth*.

2.3. Maps and regular maps A *map* $\mathcal{M}$ on a Riemann surface X is an embedding of a finite connected graph $\mathcal{G}$ into X such that the components of $X - \mathcal{G}$, which are called the *faces* of $\mathcal{M}$, are each homeomorphic to an open disc. In our maps we require X to be orientable, compact, connected and without boundary. The *genus* of $\mathcal{M}$ is defined to be the genus of the underlying surface X. If $\mathcal{M}$ has genus g and consists of F faces, E edges and V vertices, then we have

$$V + F - E = 2 - 2g,$$

which is known as the *Euler–Poincaré formula*. A *dart* of $\mathcal{M}$ is a pair, consisting of a vertex v and an edge directed towards v. In our case an edge will be homeomorphic to either a closed interval or a circle. In the latter case it will be called a *loop*. In either case an edge will give two darts. However, we sometimes require an edge to have just one vertex and one dart. Such an edge is called a *free edge*. In Figure 1, (a) is an edge with two darts, (b) is a loop and (c) is a free edge. Here $\mathcal{M}$ is said to be of *type* (m, n) if every face of $\mathcal{M}$ has n sides and m darts meet at every vertex. An *automorphism* of $\mathcal{M}$ is an automorphism of X that leaves $\mathcal{M}$ invariant and preserves incidence. If $\mathcal{M}$

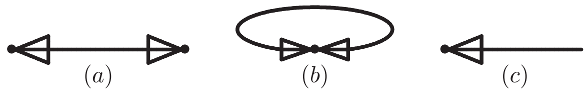

FIGURE 1. (a) An edge with two darts. (b) A loop. (c) A free edge.

admits an involution σ that fixes the midpoint of an edge and interchanges the two darts without interchanging the two neighboring faces, then $\mathcal{M}$ is called *reflexible* and σ is called a *reflection* of $\mathcal{M}$. All automorphisms of $\mathcal{M}$ form a group under composition. We denote this group and its subgroup consisting of conformal automorphisms by $\mathrm{Aut}^{\pm}\,\mathcal{M}$ and $\mathrm{Aut}^{+}\,\mathcal{M}$, respectively. Here $\mathcal{M}$ is said to be *regular* if $\mathrm{Aut}^{+}\,\mathcal{M}$ is transitive on the darts. It is clear that if $\mathcal{M}$ is regular, then the number of the darts is equal to $|\mathrm{Aut}^{+}\,\mathcal{M}|$. So, if $\mathcal{M}$ is of type (m, n) and regular, then $\mathcal{M}$ has $|\mathrm{Aut}^{+}\,\mathcal{M}|/2$ edges, $|\mathrm{Aut}^{+}\,\mathcal{M}|/m$ vertices and $|\mathrm{Aut}^{+}\,\mathcal{M}|/n$ faces.

In [4], it was shown that if $\mathcal{M}$ is a regular map of type (m, n) on a Riemann surface $X = \mathbb{H}/\Omega$, then Ω is normal in the Fuchsian triangle group $\Gamma[2, m, n]$. If Γ is the normalizer of Ω in $\mathrm{PSL}(2, \mathbb{R})$, then $\mathrm{Aut}^{+}\,\mathcal{M}$ is isomorphic to $\mathrm{Aut}^{+}\,X$, otherwise $\mathrm{Aut}^{+}\,\mathcal{M}$ is a proper subgroup of $\mathrm{Aut}^{+}\,X$.

A Fuchsian group is called *maximal* if it is not contained in any other Fuchsian group. A regular map $\mathcal{M}$ of type (m, n) is called *maximal* if the Fuchsian triangle group $\Gamma[2, m, n]$ is maximal. A complete list of nonmaximal Fuchsian groups is given in [9].

A map $\mathcal{M}^*$ on the sphere is called an *m-star map* if it consists of a single vertex v and m free edges incident to v. It is clear that $\mathcal{M}^*$ has a single face which can be regarded as an m-gon. We assume that $\mathrm{Aut}^{+}\,\mathcal{M}^*$ is isomorphic to C_m whose generators fix the unique vertex and the face-center, and cyclically permute the free edges, where C_m denotes the cyclic group of order m.

For more details on maps and regular maps, the reader might consult [4].

2.4. Wiman surfaces According to a classical theorem of Wiman [13], the largest possible order of an automorphism of a Riemann surface of genus $g > 1$ is $4g + 2$ and the second largest possible order is $4g$. (Also, see [3].) It is known that the corresponding surfaces are obtained as kernels of smooth homomorphisms of the Fuchsian triangle groups $\Gamma[2, 2g + 1, 4g + 2]$ and $\Gamma[2, 4g, 4g]$ onto C_{4g+2} and C_{4g}, respectively. Following Kulkarni [5, 6] we call these surfaces *Wiman surfaces of types I and II*, respectively. Thus, these surfaces underly regular maps of types $(2g + 1, 4g + 2)$ and $(4g, 4g)$, respectively. The Wiman surface of type II of genus g also underlies a regular map of type $(4, 4g)$. This follows from the inclusion relationship $\Gamma[2, m, m] < \Gamma[2, 4, m]$ given in [9].

3. Patterns and rotary automorphisms

Let $\mathcal{M}$ be a reflexible regular map of type (m, n) of genus $g > 1$ and $X = \mathbb{H}/\Omega$ be the underlying Riemann surface. By [11], Ω is normal in the NEC triangle group

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$\Gamma^*(2, m, n)$ and we have a triangulation of X with $|\mathrm{Aut}^{\pm} \mathcal{M}|$ $(2, m, n)$-triangles. If $\mathrm{Aut}^{\pm} \mathcal{M}$ is isomorphic to $\mathrm{Aut}^{\pm} X$, then the union of the sides of these triangles gives us all of the mirrors of the symmetries of X. If $\mathrm{Aut}^{\pm} \mathcal{M}$ is a proper subgroup of $\mathrm{Aut}^{\pm} X$, then $\mathcal{M}$ is not maximal and the sides of the triangles give us only those mirrors fixed by the reflections of $\mathcal{M}$.

Following Coxeter [2], let us label the vertices, edge-centers and face-centers of $\mathcal{M}$ with **0**, **1** and **2**, respectively. Then we see that each corner of a $(2, m, n)$-triangle T on X is either **0**, **1** or **2**. It follows that each side of T corresponds to one of the pairs **01**, **02** and **12**. We call the side of T joining the corners **0** and **1** the **01**-*side*. We call the other sides of T the **02**-*side* and the **12**-*side* in the same manner. Let P, Q and R denote the reflections in the sides of T, and let them satisfy

$$P^2 = Q^2 = R^2 = (PQ)^2 = (QR)^m = (RP)^n = 1. \tag{3.1}$$

The reflections P, Q and R generate $\mathrm{Aut}^{\pm} \mathcal{M}$. Since $1/m + 1/n < 1/2$, Equation (3.1) is not a presentation for $\mathrm{Aut}^{\pm} \mathcal{M}$ and hence to obtain a presentation we need at least one more relation.

A mirror M on X passes through some geometric points of $\mathcal{M}$ and these geometric points form a periodic sequence which we call the *pattern* of M. For example, each mirror on the sphere fixed by a reflection of the map (3, 5) has pattern **010212010212** which we abbreviate to $(\mathbf{010212})^2$ (see [2]). Each repeated part of a pattern is called a *link*, and the number of links is called the *order* of the pattern. In the above example, **010212** is a link and the pattern has order two.

Now let M be a mirror on X and let the order of the pattern of M be greater than one. As shown in [8], there exist two conformal automorphisms of $\mathcal{M}$, each of which fixes M setwise and cyclically permutes the links of the pattern of M. These automorphisms are inverses of each other and they rotate M in opposite directions. They are called the *rotary automorphisms* of M. If the order of the pattern of M is one, that is, the pattern of M consists of one link, then M has only one rotary automorphism, which is the identity.

If a mirror passes through only two geometric points, then we call it a mirror with a *short pattern*. Clearly, a short pattern is either **01**, **02** or **12** (or in reverse order).

4. The main result

In this section, we work out the mirrors on surfaces with short patterns. First, we begin with those lying on the sphere and tori.

Let $\mathcal{M}^*$ be an m-star map on the sphere. It can easily be seen that if m is odd, then each reflection of $\mathcal{M}^*$ fixes a mirror with pattern **021**. If m is even, then the pattern of the mirror of each reflection of $\mathcal{M}^*$ is either **02**, which is a short pattern, or **1012**. The other regular maps on the sphere are well known and it is not difficult to see that they do not admit reflections fixing mirrors with short patterns.

It is known that there are two regular maps of genus one (up to duality), which are of types (4, 4) and (3, 6). The underlying surfaces are known as the *square* and the

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rhombic tori and they can be obtained by identifying the opposite sides of a square and a regular hexagon in the Euclidean plane, respectively. Clearly, the square torus underlies a regular map $\mathcal{M}_1$, which is of type (4, 4) and has one face, one vertex and two edges. Since $\mathcal{M}_1$ has a single vertex, its edges are loops. By examining the picture of $\mathcal{M}_1$ it can be seen that $\mathcal{M}_1$ admits six reflections whose mirrors have patterns **01**, **02** and **12** such that each pattern is shared by two mirrors. Similarly, the rhombic torus underlies a regular map $\mathcal{M}_2$ of type (3, 6) with one face, two vertices and three edges. The map $\mathcal{M}_2$ admits six reflections whose mirrors have patterns **12** and **0102**. In this case each pattern is shared by three mirrors.

On the square torus we can find infinitely many regular maps of types (4, 4) which do not admit reflections fixing mirrors with short patterns. This is because in the Euclidean plane there exist similar polygons of different sizes. As is well known, this situation does not occur in the sphere and the hyperbolic plane. A similar discussion applies to the rhombic torus.

THEOREM 4.1. *For every $g > 1$, only the Wiman surfaces of genus g contain mirrors with short patterns.*

PROOF. Let $\mathcal{M}$ be a regular map of type (m, n) of genus $g > 1$ and let $X = \mathbb{H}/\Omega$ be the underlying Riemann surface. Suppose that M is a mirror on X with a short pattern. Then the pattern of M is either **01**, **02** or **12**.

CASE 1 (M has pattern **01**). It follows from [8] that m is even. Let T be a $(2, m, n)$-triangle on X whose **01**-side lies on M. Let P, Q and R be the reflections of $\mathcal{M}$ in the sides of T, and let them satisfy (3.1). They generate $\mathrm{Aut}^{\pm}\,\mathcal{M}$ and by [8] M has a rotary automorphism $(RQ)^{(m/2-1)}RP$. Since the order of the pattern of M is one, $(RQ)^{(m/2-1)}RP$ is the identity. From this we have $(RQ)^{(m/2-1)}R = P$ and this means that P is redundant. Thus, $\mathrm{Aut}^{\pm}\,\mathcal{M}$ is generated by Q and R, and is isomorphic to D_m, the dihedral group of order $2m$. Obviously, QR generates the group $\mathrm{Aut}^{+}\,\mathcal{M}$, which is isomorphic to C_m. Since $|\mathrm{Aut}^{+}\,\mathcal{M}| = m$, $\mathcal{M}$ has one vertex, m/n faces and $m/2$ edges. It follows from the Euler–Poincaré formula that m cannot be greater than $2n$, otherwise we would have $m > 4g + 2$. However, this is not possible as the order of an automorphism of a compact Riemann surface of genus $g > 1$ cannot exceed $4g + 2$ (see [3, 13]). Now if $m = 2n$, then by the Euler–Poincaré formula we find that $m = 4g + 2$ and $n = 2g + 1$. This implies that $\mathcal{M}$ is of type $(4g + 2, 2g + 1)$ and X is the Wiman surface of type I. Similarly, if $m = n$, then by the Euler–Poincaré formula we find that $m = n = 4g$. Clearly, $\mathcal{M}$ is of type $(4g, 4g)$ and X is the Wiman surface of type II.

CASE 2 (M has pattern **12**). According to [8], n is even and hence every face of $\mathcal{M}$ is an even-sided polygon. We may suppose that M is a geodesic arc joining the midpoints of a pair of opposite edges of a face, say F, of $\mathcal{M}$. It is clear that every pair of opposite edges of F are congruent under Ω. Thus, $\mathcal{M}$ has a single face and so $|\mathrm{Aut}^{+}\,\mathcal{M}| = n$. It follows from the Euler–Poincaré formula that n cannot be greater than $2m$, otherwise we would have $n > 4g + 2$ and this is not possible as we pointed out above. Now

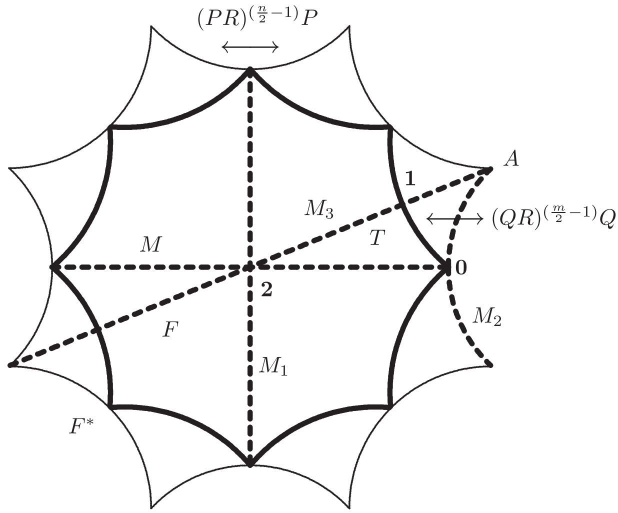

FIGURE 2. Illustration showing mirror M_2 orthogonal to M at the corner **0** of T and fixed by the reflection $(QR)^{(m/2-1)}Q$.

following the same discussion as in Case 1, we can see that X is the Wiman surface of type I if $n = 2m$ and the Wiman surface of type II if $n = m$.

CASE 3 (M has pattern **02**). It follows from [8] that m and n are both even. We may suppose that M is a diagonal of a face F of $\mathcal{M}$ joining two opposite vertices. Let T be a $(2, m, n)$-triangle whose **02**-side lies on M. Let P, Q and R be the reflections of $\mathcal{M}$ in the sides of T, and let them satisfy (3.1). As we pointed out in Section 3, these reflections generate $\mathrm{Aut}^{\pm}\,\mathcal{M}$. Since n is even, there is a mirror M_1 that intersects M orthogonally at the corner **2** of T and is fixed by the reflection $(PR)^{(n/2-1)}P$. Similarly, there is a mirror M_2 that is orthogonal to M at the corner **0** of T and is fixed by the reflection $(QR)^{(m/2-1)}Q$ (see Figure 2). By [8], the automorphism $(PR)^{(n/2-1)}P(QR)^{(m/2-1)}Q$ is a rotary automorphism for M. Clearly, it is a product of the reflections $(PR)^{(n/2-1)}P$ and $(QR)^{(m/2-1)}Q$, which fix M_1 and M_2, respectively. Since the order of the pattern of M is one, $(PR)^{(n/2-1)}P(QR)^{(m/2-1)}Q$ is the identity and, hence, $(RP)^{(n/2-1)} = P(QR)^{(m/2-1)}Q$. Obviously, $(RP)^{(n/2-1)}$ is a rotation and so is $P(QR)^{(m/2-1)}Q$. Let M_3 be the mirror containing the **12**-side of T, which is fixed by P. As M_2 is fixed by $(QR)^{(m/2-1)}Q$, and $P(QR)^{(m/2-1)}Q$ is a rotation, M_2 and M_3 intersect at a point A. Since M_3 has pattern $(\mathbf{12})^k$, the point A is either **1** (edge-center) or **2** (face-center), where k is a positive integer. It is clear that the internal angle of a $(2, m, n)$-triangle at an edge-center is $\pi/2$ and so A cannot be an edge-center. Thus, A is a face-center and by elementary hyperbolic geometry we can show that $m = 4$.

If we repeat this process, we obtain a fundamental polygon F^* for Ω. It is a regular hyperbolic n-gon and has twice the area of F. It is clear that the opposite sides of F^*

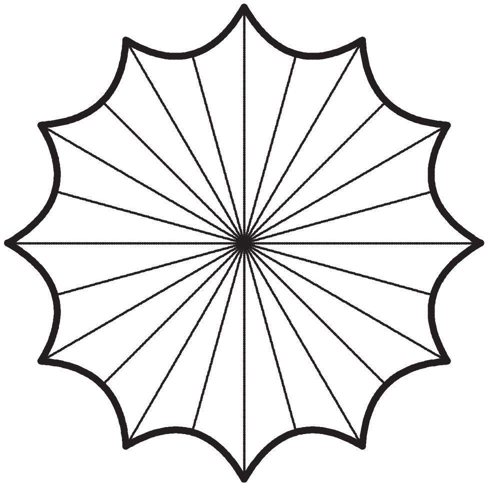

FIGURE 3. The face of $\mathcal{M}_1$ divided into $8g$ $(2, 4g, 4g)$-triangles.

are congruent under Ω. So $\mathcal{M}$ has n edges, $n/2$ vertices and two faces. By the Euler–Poincaré formula we obtain $n = 4g$ and, hence, $\mathcal{M}$ has type $(4, 4g)$. This means that X is the Wiman surface of type II. □

REMARK 4.2. Let $S = \mathbb{H}/\Omega$ be the Wiman surface of type II of genus $g > 1$. It is known that S can be obtained by identifying the opposite sides of a regular hyperbolic $4g$-gon whose angles equal to $\pi/2g$. All of the corners of the polygon become a single point on S under this identification. We can easily observe that S underlies a regular map $\mathcal{M}_1$ whose edges and vertices correspond to the sides and corners of the polygon, respectively. Clearly, $\mathcal{M}_1$ has one face, one vertex and $2g$ edges. The face of $\mathcal{M}_1$ can be divided into $8g$ $(2, 4g, 4g)$-triangles as shown in Figure 3, where $g = 3$. So, $\mathcal{M}_1$ has type $(4g, 4g)$. From Figure 3, it can be deduced that $\mathcal{M}_1$ admits $6g$ reflections whose mirrors have patterns **01**, **02** and **12** such that each pattern is shared by $2g$ mirrors. Here S also underlies a regular map $\mathcal{M}_2$ of type $(4, 4g)$ and this follows from the inclusion relationship $\Gamma[2, 4g, 4g] < \Gamma[2, 4, 4g]$ given in [9] (see Figure 4). Note that the edges of $\mathcal{M}_1$ and $\mathcal{M}_2$ are illustrated by thick line segments. It is not difficult to see that $\mathcal{M}_2$ has two faces, $2g$ vertices and $4g$ edges. As shown in Figure 4, each face of $\mathcal{M}_2$ can be divided into $8g$ $(2, 4, 4g)$-triangles and so S contains $16g$ such triangles. It can be deduced from Figure 4 that $\mathcal{M}_2$ admits $8g$ reflections whose mirrors have patterns **02**, $(\mathbf{01})^2$ and $(\mathbf{12})^2$ such that these patterns are shared by $4g$, $2g$ and $2g$ mirrors, respectively. It follows from [9] that in the cases where $g > 2$ the group $\Gamma[2, 4, 4g]$ is maximal and so is $\mathcal{M}_2$. For $g = 2$ the group $\Gamma[2, 4, 8]$ is contained in $\Gamma[2, 3, 8]$ with index three and Ω is normal in $\Gamma[2, 3, 8]$ (see [1] and [9]). Thus, S underlies a regular map $\mathcal{M}_3$ of type $(3, 8)$. According to [9], the group $\Gamma[2, 3, 8]$ is maximal and so is $\mathcal{M}_3$. It follows from [8] that the reflections of $\mathcal{M}_3$ fix mirrors with

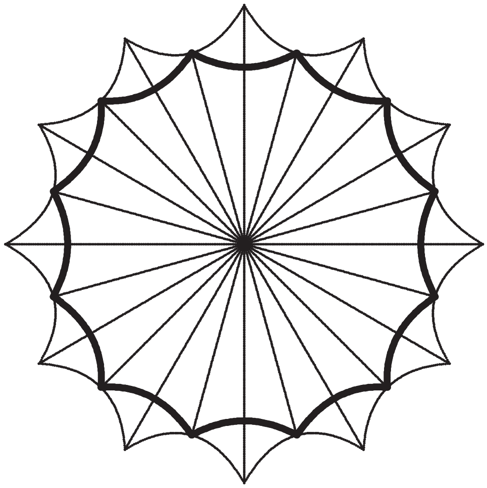

FIGURE 4. The inclusion relationship $\Gamma[2, 4g, 4g] < \Gamma[2, 4, 4g]$ given in [9].

patterns $(\mathbf{12})^a$ and $(\mathbf{0102})^b$, where a and b are positive integers. Here $a \neq 1$, otherwise $\mathcal{M}_3$ would have just one face. Clearly, $\mathcal{M}_3$ has more than one face and so the mirrors fixed by the reflections of $\mathcal{M}_3$ do not have short patterns.

The following results follow immediately from Theorem 4.1 and Remark 4.2. Corollary 4.4 is also given in [12]. See [12, Theorem 4.1].

COROLLARY 4.3. *Let $\mathcal{M}$ be a regular map of genus $g > 1$ on a Riemann surface X and let $\mathcal{M}$ has a loop. Then either $\mathcal{M}$ is of type $(4g + 2, 2g + 1)$ and X is the Wiman surface of type I, or $\mathcal{M}$ is of type $(4g, 4g)$ and X is the Wiman surface of type II.*

COROLLARY 4.4. *Let $\mathcal{M}$ be a regular map of genus $g > 1$ and X be the underlying Riemann surface. Suppose that $\mathcal{M}$ has a single face. Then:*

(i) *if $\mathcal{M}$ is maximal, then $\mathcal{M}$ is of type $(2g + 1, 4g + 2)$ and X is the Wiman surface of type I;*
(ii) *if $\mathcal{M}$ is nonmaximal, then $\mathcal{M}$ is of type $(4g, 4g)$ and X is the Wiman surface of type II.*

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