Canad. Math. Bull. Vol. **29** (3), 1986

ON RANK ONE COMMUTATORS AND TRIANGULAR REPRESENTATIONS

BY

Tsoy-Wo Ma

ABSTRACT. Starting with the extension of Lomonosov's Lemma by Tychonoff fixed point theorem, a result of Daughtry and Kim–Pearcy–Shields on rank-one commutators is extended to the context of locally convex spaces. Non-zero diagonal coefficients, eigenvalues and simultaneous triangular representations of compact operators on locally convex spaces are studied.

1. **Introduction**. This paper extends certain results for compact operators on Banach spaces to the context of locally convex spaces. Lomonosov's Lemma [8] is extended by Tychonoff's fixed point theorem [15]. The original proof is slightly simplified by our special way to define the related non-linear map. Following Daughtry [2] and Kim–Pearcy–Shields [5], a result on rank-one commutators and hyperinvariant subspaces is obtained. Our proof does not distinguish the commuting and non-commuting cases separately. A result of Lindenstrauss [6; p. 231] becomes a simple consequence. No special generalization of fixed point theorem [6; p. 230 (5)] is needed. It in turn establishes the existence of simultaneous triangular representations of commuting families of compact operators as in Ringrose [11]. Since the spectrum of a compact operator is a null sequence, the Riesz Decomposition Theorem [12; p. 31] can be derived directly from the Riesz theory [13] without functional calculus in our proof of (3.2). This allows us to identify the non-zero diagonal coefficients and eigenvalues. Following Laurie–Nordgren–Radjavi–Rosenthal [7], common triangular representations of non-commuting families of compact operators are studied. Thanks to Professor P. Rosenthal for sending us their preprint [7].

2. **Rank-one commutators**. Let E be a complex separated locally convex space of infinite dimension. Let $L(E)$ denote the algebra of all (continuous linear) operators on E. A closed vector subspace M of E is an invariant subspace of an operator T if $TM \subset M$; a hyperinvariant subspace of T if $SM \subset M$ for every operator S commuting

Received by the editors September 14, 1983, and, in final revised form, February 20, 1985.

AMS Subject Classification 47A10, 47A67.

Key Words: Lomonosov technique, rank one commutators, invariant subspaces, compact non-self-adjoint, diagonal coefficients, eigenvalues, triangular representations, Riesz theory.

with T, i.e. $ST = TS$. A closed vector subspace M is an invariant subspace of an algebra $\mathcal{A}$ of operators if M is invariant under every operator of $\mathcal{A}$. A vector subspace is trivial if it is either $\{0\}$ or E. An algebra $\mathcal{A}$ of operators is transitive if $\mathcal{A}$ has no non-trivial invariant subspace. An operator A is non-scalar if $A \neq \lambda I$ for every complex number λ where I is the identity operator on E. The following lemma is also true for TK but it requires slightly more work.

(2.1) LEMMA. *Let $\mathcal{A}$ be a transitive algebra of operators. Then for every non-zero compact operator K there exists an operator T in $\mathcal{A}$ such that* 1 *is an eigenvalue of the compact operator KT.*

PROOF. Let $Ka \neq 0$ for some $a \in E$. There exists an open convex balanced 0-neighbourhood U such that $Ka \notin 2U$. For the compact operator K there is an open convex 0-neighbourhood V_1 such that $K(V_1)$ is relatively compact. For the bounded set $K(V_1)$ we have $\lambda K(V_1) \subset U$ for some $\lambda > 0$. Let $V = \lambda V_1$. Then V is an open convex 0-neighbourhood such that $K(V)$ is relatively compact in E and it is a subset of U. Then $X = c\ell K(a + V)$ is a non-empty compact convex set. We claim

$$K(a + V) \subset E\backslash U. \tag{A}$$

In fact, suppose to the contrary that $K(a + v) = u$ for some $v \in V$ and $u \in U$. Then $Ka = u - Kv \in U + U = 2U$ contradicts to the choice of U. Thus (A) is valid. Since U is open we have $X \subset E\backslash U$ and in particular every $x \in X$ is non-zero. Next we claim

$$X \subset \cup\{T^{-1}(a + V) : T \in \mathcal{A}\}. \tag{B}$$

Suppose to the contrary there exists $y \in X$ such that $y \notin T^{-1}(a + V)$ for every $T \in \mathcal{A}$. Since $\mathcal{A}$ is an algebra of operators the set $M = c\ell\,\{Ty : T \in \mathcal{A}\}$ is an invariant subspace of $\mathcal{A}$. If $Ty = 0$ for every T in $\mathcal{A}$ then the subspace generated by $y \neq 0$ is a non-trivial invariant subspace of $\mathcal{A}$ contrary to the transitivity of $\mathcal{A}$. Hence $M \neq \{0\}$. Due to transitivity of $\mathcal{A}$ again, we have $M = E$. On the other hand, $Ty \in E\backslash(a + V)$ for all $T \in \mathcal{A}$ by the choice of y. Since V is open we have $M \subset E\backslash(a + V)$, i.e. $M \neq E$. This contradiction establishes the open cover of the compact set X as we claimed in (B). So there are $T_1, T_2, \ldots, T_n$ in $\mathcal{A}$ such that $X \subset \cup_{i=1}^{n} T_i^{-1}(a + V)$. Let $\{\alpha_i\}$ be a partition of unity on X subordinated to this finite subcover. Define a continuous map $f: X \to E$ by $f(x) = \Sigma_{i=1}^{n} \alpha_i(x)\, KT_i(x)$, $\forall x \in X$. If $\alpha_i(x) \neq 0$ then $x \in T_i^{-1}(a + V)$, i.e. $KT_i(x) \in K(a + V) \subset X$. Since X is convex we have $f(X) \subset X$. The Tychonoff theorem [15] gives a fixed point $b \in X$ for the continuous map F. Define $T = \Sigma_{i=1}^{n} \alpha_i(b) T_i$. Then T is an operator in the algebra $\mathcal{A}$. Since X contains only non-zero vectors we have $b \neq 0$. Finally

$$KT(b) = K\left[\sum_{i=1}^{n} \alpha_i(b) T_i(b)\right] = \sum_{i=1}^{n} \alpha_i(b)\, KT_i(b) = f(b) = b.$$

Consequently 1 is an eigenvalue of KT.

https://doi.org/10.4153/CMB-1986-041-1 Published online by Cambridge University Press

(2.2) THEOREM. *Let A be a non-scalar operator. If there exists a non-zero compact operator K such that rank* $(AK - KA) \leqq 1$, *then A has a non-trivial hyperinvariant subspace.*

PROOF. Suppose to the contrary that A has no non-trivial hyperinvariant subspace. Then the algebra of all operators commuting with A is transitive. For the non-zero compact operator K there is an operator T such that $AT = TA$ and 1 is an eigenvalue of KT. Let $x \neq 0$ in E such that $KTx = x$. Following [5] let V be the closed vector subspace generated by $\{A^n x : n \geqq 0\}$. Clearly $V \neq \{0\}$ and $A(V) \subset V$. Suppose V is finite dimensional. Then $A \mid V$ has an eigenvalue, say λ. Now the subspace $H \equiv \ker(A - \lambda I)$ is non-trivial since A is non-scalar. Clearly it is a hyperinvariant subspace of A and the proof is complete for this case. Secondly assume V to be infinite dimensional. Let $S = KT - I$ and $R = AS - SA$. Then $R = (AK - KA)T$ has rank $\leqq 1$. Since $\ker S$ is finite dimensional we have $V \not\subset \ker S$. Let n be the smallest integer such that $A^n x \notin \ker S$. Since $x \in \ker S$ we have $n \geqq 1$. Thus $SA^{n-1}x = 0$ and $SA^n x \neq 0$. It follows $RA^{n-1}x = -SA^n x \neq 0$. Since R has rank at most one we have $R(E) \subset S(E)$. Let $N = S(E)$. Since $(AS - SA)(E) \subset S(E)$ we have $A(N) \subset N$. Since KT is compact, N is a closed vector subspace of E. Its polar N^0 in E' is an invariant subspace of the transpose A'. Since 1 is an eigenvalue of KT, the operator $S = KT - I$ is not surjective, e.g. [13; Cor 2, p. 172], i.e. $N \neq E$. Let p be the index of the eigenvalue 1 of KT. Then $N = S(E) \supset S^p(E)$. Hence $1 \leqq \dim E/N \leqq \dim E/S^p(E) = \dim \ker S^p < \infty$. Since $(E/N)'$ and N^0 are isomorphic, N^0 is non-zero finite dimensional. Therefore $A' \mid N^0$ has an eigenvalue, say λ. Let J be the identity map on E'. Then $A' - \lambda J$ is not injective. Hence range$(A - \lambda I)$ is not $\sigma(E, E')$-dense and consequently it is not E-dense. Therefore the subspace $H \equiv c\ell(A - \lambda I)(E)$ is $\neq E$. Since A is non-scalar, $H \neq \{0\}$. It is routine to verify that H is a hyperinvariant subspace of A. This completes the proof.

(2.3) THEOREM. [6; p. 231] *If a non-scalar operator A commutes with a non-zero compact operator K, then A has a non-trivial hyperinvariant subspace.*

(2.4) COROLLARY. *Every commuting family of compact operators has a non-trivial invariant subspace provided the underlying space is of dimension at least two.*

3. **Triangular representations**. Triangular representations are now introduced in the context of locally convex spaces. The set of all vector subspaces of E is ordered by inclusion. A totally ordered set of closed vector subspaces is simply called a chain. Let $\mathscr{C}$ be a chain. For every M in $\mathscr{C}$ let M_- denote the closure of union of subspaces L in $\mathscr{C}$ such that $L \subset M$ and $L \neq M$. If there is no such L then define $M_- = \{0\}$. Clearly M_- is a closed vector subspace of M. A chain $\mathscr{C}$ is complete if both $\{0\}$ and E are in $\mathscr{C}$ and for every subfamily $\mathscr{D}$ of $\mathscr{C}$ both subspaces $\cap \mathscr{D}$ and $c\ell \cup \mathscr{D}$ are in $\mathscr{C}$. A complete chain $\mathscr{C}$ is simple if for each M in $\mathscr{C}$ we have $\dim(M/M_-) \leqq 1$. A chain $\mathscr{C}$ is invariant under an operator T if every M in $\mathscr{C}$ is an invariant subspace of T. A simple chain which is invariant under T is called a triangular representation of T. A chain is a simultaneous triangular representation of a family $\mathscr{A}$ of operators if it is a triangular representation

of every operator in $\mathscr{A}$. It is routine application of Zorn's Lemma that every commuting family $\mathscr{A}$ of compact operators has a chain which is maximal among all invariant chains. As a result of maximality it is always complete. Consequently for triangular representation it suffices to prove $\dim(M/M_-) \leqq 1$ for every M in a given maximal chain.

(3.1) THEOREM. *Every commuting family $\mathscr{A}$ of compact operators has a simultaneous triangular representation.*

PROOF. Let $\mathscr{C}$ be a maximal invariant chain of $\mathscr{A}$. Suppose to the contrary that there is some M in $\mathscr{C}$ such that $\dim(M/M_-) \geqq 2$. Let $\varphi: M \rightarrow M/M_-$ denote the quotient map. For each T in $\mathscr{A}$ let T_M be the restriction of T to M. Since both M and M_- are invariant under T there exists a unique operator Q_T on M/M_- such that $Q_T\varphi = \varphi T_M$. Let V be a 0-neighbourhood in E such that $T(V)$ is relatively compact in E. Then $V \cap M$ is a 0-neighbourhood in M such that $T_M(V \cap M)$ is relatively compact in the closed subspace M. Because φ is open, $\varphi(V \cap M)$ is a 0-neighbourhood in M/M_- such that $Q_T[\varphi(V \cap M)] = \varphi[T_M(V \cap M)]$ is relatively compact in M/M_-. For every S, T in $\mathscr{C}$ we have $(Q_SQ_T)\varphi = \varphi(S_MT_M) = \varphi(ST|M) = \varphi(TS|M) = (Q_TQ_S)\varphi$, i.e. $Q_SQ_T = Q_TQ_S$. Therefore $\{Q_T: T \in \mathscr{A}\}$ is a commuting family of compact operators on M/M_-. Let H be a non-trivial invariant subspace. Then $N = \varphi^{-1}(H)$ is an invariant subspace of $\mathscr{A}$. Since $M_- \subset N \subset M$ and $M_- \neq N \neq M$ is contradicts the maximality of $\mathscr{C}$. This completes the proof.

To introduce diagonal coefficients it suffices to work with a particular triangular representation $\mathscr{C}$ of a given compact operator T. Take any M in $\mathscr{C}$. Suppose $M \neq M_-$. There is z_M in $M \backslash M_-$. Since $\mathscr{C}$ is simple we have $Tz_M = y_M + \alpha_M z_M$ for some y_M in M_- and some number α_M. It can be proved that α_M is independent of the choice of z_M. For $M = M_-$ define $\alpha_M = 0$. The number α_M is called the diagonal coefficient of T and M relative to $\mathscr{C}$. The diagonal multiplicity of a given complex number λ is the cardinal number of all M in $\mathscr{C}$ satisfying $\alpha_M = \lambda$.

(3.2) THEOREM. *A non-zero number λ is a diagonal coefficient iff it is an eigenvalue.*

PROOF. Suppose α_M is a non-zero diagonal coefficient. Then $(T - \alpha_M I)M \subset M_-$ and $M \neq M_-$. Thus for the compact operator $T|M$, the map $(T|M) - \alpha_M(I|M)$ is not subjective on M. Therefore α_M is an eigenvalue of $T|M$, e.g. [13; Cor. 2, p. 172]. Consequently α_M is also an eigenvalue of T. Conversely let $\lambda \neq 0$ be an eigenvalue of T. Then the dimension of $F \equiv \ker(T - \lambda I)$ is finite. Let B be the boundary of any compact convex balanced 0-neighbourhood V in F. Then every vector in B is non-zero because $\dim F \geqq 1$. Parallel to [12] define $M = \cap\{N \in \mathscr{C}: B \cap N \neq \phi\}$. There is at least one such N, e.g. E. Since $\mathscr{C}$ is complete we have $M \in \mathscr{C}$. Since B is compact and $\mathscr{C}$ is totally ordered we have $B \cap M \neq \phi$. Suppose we can prove $M \neq M_-$. Then $B \cap M_- = \phi$ and there is $z_M \in B \cap (M \backslash M_-)$. So $z_M \neq 0$ and $Tz_M = \lambda z_M$. Since $(T - \alpha_M I)z_M \in M_-$ we have $(\lambda - \alpha_M)z_M \in M_-$, i.e. $\lambda = \alpha_M$. Consequently λ is a diagonal coefficient and the proof is complete. Now suppose to the contrary that $M = M_-$. Then

M must be infinite dimensional. Let p be the index of the compact operator $T|M$. Let $Q = \ker[(T|M) - \lambda(I|M)]^p$. Then Q is an invariant subspace of $T|M$ and λ is the only eigenvalue of $T|Q$. Take any $L \in \mathscr{C}$ such that $L \subset M$ and $L \neq M$. Suppose to the contrary that $\dim(L \cap Q) \geqq 1$. Then $T|L \cap Q$ has exactly one eigenvalue λ. Let $x \neq 0$ in $L \cap Q$ satisfy $Tx = \lambda x$. Let f be the gauge of V in F and let $y = x/f(x)$. Since $x \in F$ we have $y \in B \cap L$. It follows from the choice of M we have $M \subset L$. This again contradicts the choice of L. Therefore $M \neq M_-$ as required. The proof is now complete.

(3.3) THEOREM. *Let λ be a non-zero complex number. Then its diagonal multiplicity and algebraic multiplicity are equal.*

A compact operator T is quasinilpotent if zero is the only possible eigenvalue. As immediate consequence of our (3.2) and (3.3), T is quasinilpotent if $T(M) \subset M_-$ for every M in any triangular representation $\mathscr{C}$. Also if $M = M_-$ for every M in $\mathscr{C}$ then T is quasinilpotent. The proof of our (3.3) is identical with Ringrose [11] and hence omitted.

Let $\mathscr{C}$ be a simultaneous triangular representation of compact operators S, T. As a simple chain it is also a triangular representation of $S + T$, αT and ST. Let $d_M(T)$ denote the diagonal coefficient of T at M. Clearly $d_M(S + T) = d_M(S) + d_M(T)$, $d_m(\alpha T) = \alpha d_M(T)$ and $d_M(ST) = d_M(S)d_M(T)$. Since invertibility of an operator is an algebraic property, the standard argument is applicable to the following spectral theorem which is needed in the proofs of the subsequent theorems.

(3.4) THEOREM. *For every polynomial f we have $\sigma[f(T)] = f[\sigma(T)]$ where σ denotes the spectrum of relevant operator.*

The following results are modified from [7]. The only difference is due to the fact that the compact operator is in the front in our first lemma which has to be used in the following omitted proofs. A non-commutative polynomial in two variables is a formal linear combination of words in variables.

(3.5) THEOREM. *The following statements are equivalent for all compact operators S, T.*

(*a*) *S, T have a simultaneous triangular representation.*

(*b*) *$(ST - TS)f(S, T)$ is quasinilpotent for every non-commutative polynomial f.*

(*c*) *$\sigma[f(S, T)] \subset f[\sigma(S), \sigma(T)]$ for every non-commutative polynomial f.*

(3.6) THEOREM. *If $ST - TS$ commutes with both S and T, then there is a simultaneous triangular representation.*

(3.7) THEOREM. *Let $\mathscr{A}$ be an algebra of compact operators. If every pair of operators in $\mathscr{A}$ has a simultaneous triangular representation, then so does the algebra $\mathscr{A}$.*

REFERENCES

1. M. S. Brodskii, *Triangular and Jordan Representations of Linear Operators*, Amer. Math. Soc. Transl., **32** (1971).

2. J. Daughtry, *An Invariant Subspace Theorem*, Proc. Amer. Math. Soc. **49** (1975), pp. 267–268.

3. K. Fan, *Invariant Subspaces of Certain Linear Operators*, Bull. Amer. Math. Soc. **69** (1963), pp. 773–777.

4. I. C. Gohberg and M. G. Krein, *Theory and Applications of Volterra Operators in Hilbert Space*, Amer. Math. Soc. Transl., **24** (1970).

5. H. W. Kim, C. Pearcy and A. L. Shields, *Rank One Commutators and Hyperinvariant Subspaces*, Mich. Math. J. **22** (1975), pp. 193–194.

6. G. Köthe, *Topological Vector Space II*, Springer, 1979.

7. C. Laurie, E. Nordgren, H. Radjavi and P. Rosenthal, *On Triangularization of Algebras of Operators*, J. Reine Agnew. Math. **327** (1981), pp. 143–155.

8. V. I. Lomonosov, *Invariant Subspaces for the Family of Operators Commuting with a Completely Continuous Operator*, Funct. Anal. and Appl. **7** (1973), pp. 213–214.

9. N. H. McCoy, *On Quasi-Commutative Matrices*, Trans. Amer. Math. Soc. **36** (1934), pp. 327–340.

10. N. H. McCoy, *On Characteristic Roots of Matrix Polynomials*, Bull. Amer. Math. Soc., **42** (1936), pp. 592–600.

11. J. R. Ringrose, *Compact Non-Self-Adjoint Operators*, Von Nostrand, 1971.

12. H. Radjavi and P. Rosenthal, *Invariant Subspaces,* Springer, 1973.

13. A. P. Robertson and W. Robertson, *Topological Vector Spaces*, Cambridge, 1973.

14. H. H. Schaefer, *Topological Vector Spaces*, Macmillan, 1966.

15. A. Tychonoff, *Ein Fixpunktsatz*, Math. Ann. **111** (1935), pp. 767–776.

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF WESTERN AUSTRALIA,
NEDLANDS, W.A. 6009, AUSTRALIA.