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SUBSETS OF VERTICES GIVE MORITA EQUIVALENCES OF LEAVITT PATH ALGEBRAS

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Abstract

We show that every subset of vertices of a directed graph E gives a Morita equivalence between a subalgebra and an ideal of the associated Leavitt path algebra. We use this observation to prove an algebraic version of a theorem of Crisp and Gow: certain subgraphs of E can be contracted to a new graph G such that the Leavitt path algebras of E and G are Morita equivalent. We provide examples to illustrate how desingularising a graph, and in- or out-delaying of a graph, all fit into this setting.

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1. Introduction

Given a directed graph E, Crisp and Gow identified in [11, Theorem 3.1] a type of subgraph which can be 'contracted' to give a new graph G whose C^*-algebra $C^*(G)$ is Morita equivalent to $C^*(E)$. Crisp and Gow's construction is widely applicable, as they point out in [11, Section 4]. It includes, for example, Morita equivalences of the C^*-algebras of graphs that are elementary-strong-shift-equivalent [5, 12] or are in- or out-delays of each other [6]. Two of the basic moves discussed in [17] are special cases of the Crisp–Gow construction.

The C^*-algebra of a directed graph E is the universal C^*-algebra generated by mutually orthogonal projections p_v and partial isometries s_e associated to the vertices v and edges e of E, respectively, subject to relations. In particular, the relations capture the connectivity of the graph. For any subset V of vertices, $\sum_{v\in V} p_v$ converges to a projection p in the multiplier algebra of $C^*(E)$. (If V is finite, then p is in $C^*(E)$.) Then the module $pC^*(E)$ implements a Morita equivalence between the corner $pC^*(E)p$ of $C^*(E)$ and the ideal $C^*(E)pC^*(E)$ of $C^*(E)$. The difficult part is to identify $pC^*(E)p$ and $C^*(E)pC^*(E)$ with known algebras. The corner $pC^*(E)p$ may not be another graph algebra, but sometimes it is (see, for example, [10] and [4]). The projection p is called full when $C^*(E)pC^*(E) = C^*(E)$.

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Now let R be a commutative ring with identity. A purely algebraic analogue of the graph C^*-algebra is the Leavitt path algebra $L_R(E)$ over R. This paper is based on the very simple observation that every subset V of the vertices of a directed graph E gives an algebraic version of the Morita equivalence between $pC^*(E)p$ and $C^*(E)pC^*(E)$ for Leavitt path algebras (see Theorem 3.1). We show that this observation is widely applicable by proving an algebraic version of Crisp and Gow's theorem (see Theorem 4.1). A special case of this result has been very successfully used in both [2, Section 3] and [14].

If V is infinite, we cannot make sense of the projection p in $L_R(E)$, but we can make sense of the algebraic analogues of the sets $pC^*(E)$, $pC^*(E)p$ and $C^*(E)pC^*(E)$. For example,

$$pC^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \text{ are paths in } E \text{ and } \mu \text{ has range in } V\}$$

has analogue

$$M = \operatorname{span}_R\{s_\mu s_{\nu^*} : \mu, \nu \text{ are paths in } E \text{ and } \mu \text{ has range in } V\},$$

where we also use s_e and p_v for universal generators of $L_R(E)$. Theorem 3.1 below gives a surjective Morita context (M, M^*, MM^*, M^*M) between the R-subalgebra MM^* and the ideal M^*M of $L_R(E)$. The set V is full, in the sense that $M^*M = L_R(E)$, if and only if the saturated hereditary closure of V is the whole vertex set of E (see Lemma 3.2).

Recently, the first author and Sims proved in [9, Theorem 5.1] that equivalent groupoids have Morita equivalent Steinberg R-algebras. They then proved that the graph groupoids of the graphs G and E appearing in Crisp and Gow's theorem are equivalent groupoids [9, Proposition 6.2]. Since the Steinberg algebra of a graph groupoid is canonically isomorphic to the Leavitt path algebra of the graph, they deduced that the Leavitt path algebras of $L_R(G)$ and $L_R(E)$ are Morita equivalent.

In particular, we obtain a direct proof of [9, Proposition 6.2] using only elementary methods. There are two advantages to our elementary approach: it illustrates on the one hand where we have had to use different techniques from the C^*-algebraic analogue, and on the other hand where we can just use the C^*-algebraic results already established.

2. Preliminaries

A directed graph $E = (E^0, E^1, r, s)$ consists of countable sets E^0 and E^1, and range and source maps $r, s : E^1 \to E^0$. We think of E^0 as the set of vertices, and of E^1 as the set of edges directed by r and s. A vertex v is called an *infinite receiver* if $|r^{-1}(v)| = \infty$ and is called a *source* if $|r^{-1}(v)| = 0$. Sources and infinite receivers are called *singular* vertices.

We use the convention established in [16] that a path is a sequence of edges $\mu = \mu_1\mu_2\cdots$ such that $s(\mu_i) = r(\mu_{i+1})$. We denote the ith edge in a path μ by μ_i. We say that a path μ is finite if the sequence is finite and denote its length by $|\mu|$. Vertices are

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regarded as paths of length 0. We denote the set of finite paths by E^* and the set of infinite paths by E^∞. We usually use the letters x, y for infinite paths. We extend the range map r to $\mu \in E^* \cup E^\infty$ by $r(\mu) = r(\mu_1)$; for $\mu \in E^*$, we also extend the source map s by $s(\mu) = s(\mu_{|\mu|})$.

Let $(E^1)^* := \{e^* : e \in E^1\}$ be a set of formal symbols called *ghost edges*. If $\mu \in E^*$, then we write μ^* for $\mu^*_{|\mu|} \cdots \mu^*_2 \mu^*_1$ and call it a *ghost path*. We extend r and s to the ghost paths by $r(\mu^*) = s(\mu)$ and $s(\mu^*) = r(\mu)$.

Let R be a commutative ring with identity and let A be an R-algebra. A *Leavitt E-family in A* is a set $\{P_v, S_e, S_{e^*} : v \in E^0, e \in E^1\} \subset A$, where $\{P_v : v \in E^0\}$ is a set of mutually orthogonal idempotents, and:

(L1) $P_{r(e)}S_e = S_e = S_e P_{s(e)}$ and $P_{s(e)}S_{e^*} = S_{e^*} = S_{e^*}P_{r(e)}$ for $e \in E^1$;
(L2) $S_{e^*}S_f = \delta_{e,f}P_{s(e)}$ for $e, f \in E^1$; and
(L3) for all nonsingular $v \in E^0$, $P_v = \sum_{r(e)=v} S_e S_{e^*}$.

For a path $\mu \in E^*$, we set $S_\mu := S_{\mu_1} \cdots S_{\mu_{|\mu|}}$. The *Leavitt path algebra $L_R(E)$* is the universal R-algebra generated by a universal Leavitt E-family $\{p_v, s_e, s_{e^*}\}$: that is, if A is an R-algebra and $\{P_v, S_e, S_{e^*}\}$ is a Leavitt E-family in A, then there exists a unique R-algebra homomorphism $\pi : L_R(E) \to A$ such that $\pi(p_v) = P_v$ and $\pi(s_e) = S_e$ [18, Section 3]. It follows from (L2) that

$$L_R(E) = \operatorname{span}_R\{s_\mu s_{\nu^*} : \mu, \nu \in E^*\}.$$

Remark 2.1. Our definition of the Leavitt path algebra $L_R(E)$ as a universal algebra comes from [18]. Often $L_R(E)$ is presented concretely as a quotient of the free algebra generated by the edges and vertices subject to the relations (L1)–(L3) above. The relations (L1)–(L3) are formulated to fit our path convention.

3. Subsets of vertices of a directed graph give Morita equivalences

Theorem 3.1. *Let E be a directed graph, let R be a commutative ring with identity and let $\{p_v, s_e, s_{e^*}\}$ be a universal generating Leavitt E-family in $L_R(E)$. Let $V \subset E^0$ and*

$$M := \operatorname{span}_R\{s_\mu s_{\nu^*} : \mu, \nu \in E^*, r(\mu) \in V\} \text{ and } M^* := \operatorname{span}_R\{s_\mu s_{\nu^*} : \mu, \nu \in E^*, r(\nu) \in V\}.$$

Then:

(1) *MM^* is an R-subalgebra of $L_R(E)$;*
(2) *$MM^* = \operatorname{span}\{s_\mu s_{\nu^*} : r(\mu), r(\nu) \in V\}$ and M^*M is an ideal of $L_R(E)$ containing MM^*;*
(3) *with actions given by multiplication in $L_R(E)$, M is an MM^*–M^*M-bimodule and M^* is an M^*M–MM^*-bimodule;*
(4) *there are bimodule homomorphisms*

$$\Psi\colon M \otimes_{M^*M} M^* \to MM^* \quad \text{and} \quad \Phi\colon M^* \otimes_{MM^*} M \to M^*M$$

such that $(MM^*, M^*M, M, M^*, \Psi, \Phi)$ is a surjective Morita context.

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Proof. We have

$$MM^* = \operatorname{span}_R\{p_v s_\mu s_{\nu^*} s_\alpha s_\beta^* p_w : v, w \in V, \alpha, \beta, \mu, \nu \in E^*\}.$$

Products of the form $s_\mu s_{\nu^*} s_\alpha s_\beta^*$ are either zero or of the form $s_\mu s_\gamma s_{\delta^*} s_{\nu^*} = s_{\mu\gamma} s_{(\nu\delta)^*}$ for some $\gamma, \delta \in E^*$. Thus, it is easy to see that MM^* is a subalgebra of $L_R(E)$ and

$$MM^* = \operatorname{span}_R\{p_v s_\mu s_{\nu^*} p_w : v, w \in V, \mu, \nu \in E^*\} = \operatorname{span}\{s_\mu s_{\nu^*} : \mu, \nu \in E^*, r(\mu), r(\nu) \in V\}.$$

Similarly, M^*M is an ideal.

To see that $MM^* \subset M^*M$, take a spanning element $s_\mu s_{\nu^*}$ of MM^*. Then $r(\mu) \in V$, $s_\mu s_{\nu^*} \in M$ and $s_\mu s_{\nu^*} = p_{r(\mu)} p_{r(\mu)^*} s_\mu s_{\nu^*} \in M^*M$. Thus, $MM^* \subset M^*M$.

Since the module actions are given by multiplication in $L_R(E)$, it is easy to verify that M is an MM^*–M^*M-bimodule and M^* is an M^*M–MM^*-bimodule. The function $f\colon M \times M^* \to MM^*$ defined by $f(m, n) = mn$ is bilinear and $f(md, n) = f(m, dn)$ for all $d \in M^*M$. By the universal property of the balanced tensor product, there is a bimodule homomorphism $\Psi\colon M \otimes_{M^*M} M^* \to MM^*$ with $\Psi(m \otimes n) = f(m, n) = mn$. Similarly, there is a bimodule homomorphism $\Phi\colon M^* \otimes_{MM^*} M \to M^*M$ such that $\Phi(n, m) = nm$. Both Ψ and Φ are surjective. Since multiplication in $L_R(E)$ is associative, for $m, m' \in M, n, n' \in M^*$,

$$m\Phi(n \otimes m') = mnm' = \Psi(m \otimes n)m' \quad \text{and} \quad n\Psi(m \otimes n') = nmn' = \Phi(n \otimes m)n'.$$

Thus, $(MM^*, M^*M, M, M^*, \Psi, \Phi)$ is a surjective Morita context. □

In the situation of Theorem 3.1, we say that a subset V of E^0 is *full* if the ideal M^*M is all of $L_R(E)$. We want a graph-theoretic characterisation of fullness, so we want the algebraic version of [6, Lemma 2.2]. We need some definitions.

For $v, w \in E^0$, we write $v \le w$ if there is a path $\mu \in E^*$ such that $s(\mu) = w$ and $r(\mu) = v$. We say that a subset H of E^0 is *hereditary* if $v \in H$ and $v \le w$ implies $w \in H$. A hereditary subset H of E^0 is *saturated* if

$$v \in E^0, \quad 0 < |r^{-1}(v)| < \infty \quad \text{and} \quad s(r^{-1}(v)) \subset H \quad \Longrightarrow \quad v \in H.$$

We denote by $\Sigma H(V)$ the smallest saturated hereditary subset of E^0 containing V. For a saturated hereditary subset H of E^0, we write I_H for the ideal of $L_R(E)$ generated by $\{p_v : v \in H\}$.

Lemma 3.2. *Let E be a directed graph and let $V \subset E^0$. Then V is full if and only if $\Sigma H(V) = E^0$.*

Proof. Let R be a commutative ring with identity and $\{p_v, s_e, s_{e^*}\}$ a universal generating Leavitt E-family in $L_R(E)$. As in Theorem 3.1, let $M = \operatorname{span}_R\{s_\mu s_{\nu^*} : r(\mu) \in V\}$.

First suppose that V is full, that is, that $M^*M = L_R(E)$. To see that $\Sigma H(V) = E^0$, fix $v \in E^0$. Then $p_v \in M^*M$, and we can write p_v as a linear combination

$$p_v = \sum_{(\alpha,\beta)\in F_1,\, (\mu,\nu)\in F_2} r_{\alpha,\beta,\mu,\nu} s_\alpha s_{\beta^*} s_\mu s_{\nu^*},$$

where F_1, F_2 are finite subsets of $E^* \times E^*$, each $r_{\alpha,\beta,\mu,\nu} \in R$ and $r(\beta) = r(\mu) \in V$.

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Since $\Sigma H(V)$ is a hereditary subset containing V, we have $s(\beta), s(\mu) \in \Sigma H(V)$ and hence $p_{s(\beta)}, p_{s(\mu)} \in I_{\Sigma H(V)}$. Thus, each summand

$$s_\alpha s_{\beta^*} s_\mu s_{\nu^*} = s_\alpha p_{s(\alpha)} s_{\beta^*} s_\mu p_{s(\mu)} s_{\nu^*} \in I_{\Sigma H(V)}.$$

It follows that $p_v \in I_{\Sigma H(V)}$. Thus, $v \in \Sigma H(V)$ and hence $E^0 \subset \Sigma H(V)$. The reverse set inclusion is trivial. Thus, $\Sigma H(V) = E^0$.

Conversely, suppose that $\Sigma H(V) = E^0$. To see that V is full, we need to show that the ideal M^*M is all of $L_R(E)$. For this, suppose that I is an ideal of $L_R(E)$ containing MM^*. It suffices to show that $L_R(E) = I$. By Theorem 3.1, M^*M is an ideal of $L_R(E)$ containing MM^* and, taking $I = M^*M$, gives $L_R(E) = M^*M$, as needed.

By [18, Lemma 7.6], the subset $H_I := \{v \in E^0 : p_v \in I\}$ of E^0 is a saturated hereditary subset of E^0. Since I contains MM^*, we have $p_v \in I$ for all $v \in V$. Thus, $V \subset H_I$ and, since H_I is a saturated hereditary subset, we get $\Sigma H(V) \subset H_I$. By assumption, $\Sigma H(V) = E^0$ and now $L_R(E) = I_{E^0} = I_{\Sigma H(V)} \subset I_{H_I} \subset I \subset L_R(E)$. So, $L_R(E) = I$ for any ideal I containing MM^*. Thus, V is full. □

4. Contractible subgraphs of directed graphs

We start by stating the algebraic version of the result of Crisp and Gow [11, Theorem 3.1]. For this we need a few more definitions. Our path convention differs from that used in [11] and we make the appropriate adjustment.

Let E be a directed graph. A finite path $\alpha = \alpha_1\alpha_2 \cdots \alpha_{|\alpha|}$ in E with $|\alpha| \geq 1$ is a *cycle* if $s(\alpha) = r(\alpha)$ and $s(\alpha_i) \neq s(\alpha_j)$ when $i \neq j$. Then E (respectively, a subgraph) is *acyclic* if it contains no cycles. An acyclic infinite path $x = x_1x_2 \cdots$ in E is a *head* if each $r(x_i)$ receives only x_i and each $s(x_i)$ emits only x_i.

If E has a head, we can get a new graph F by collapsing the head down to a source. This is an example of a desingularisation and hence $L_R(F)$ and $L_R(E)$ are Morita equivalent by [1, Proposition 5.2]. Thus, the 'no-heads' hypothesis in Theorem 4.1 below is not restrictive.

We thank the referee for pointing us to [15]. Our Theorem 4.1 generalises [15, Theorem 3.1] to graphs with infinitely many vertices and to commutative rings instead of fields.

THEOREM 4.1. *Let R be a commutative ring with identity, let E be a directed graph with no heads and let $\{p_v, s_e, s_{e^*}\}$ be a universal generating Leavitt E-family in $L_R(E)$. Suppose that $G^0 \subset E^0$ contains the singular vertices of E. Suppose also that the subgraph T of E defined by*

$$T^0 := E^0 \backslash G^0 \quad \textit{and} \quad T^1 := \{e \in E^1 : s(e), r(e) \in T^0\}$$

is acyclic. Suppose that:

(T1) *each vertex in G^0 is the range of at most one infinite path $x \in E^\infty$ such that $s(x_i) \in T^0$ for all $i \geq 1$.*

Also, suppose that for each $y \in T^\infty$:

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(T2) *there is a path from* $r(y)$ *to a vertex in* G^0;
(T3) $|s^{-1}(r(y_i))| = 1$ *for all* i; *and*
(T4) $e \in E^1,\ s(e) = r(y) \implies |r^{-1}(r(e))| < \infty$.

Let G *be the graph with vertex set* G^0 *and one edge* e_β *for each* $\beta \in E^*\backslash E^0$ *with* $s(\beta), r(\beta) \in G^0$ *and* $s(\beta_i) \in T^0$ *for* $1 \leq i < |\beta|$, *such that* $s(e_\beta) = s(\beta)$ *and* $r(e_\beta) = r(\beta)$. *Then* $L_R(G)$ *is Morita equivalent to* $L_R(E)$.

In words, the new graph G of Theorem 4.1 is obtained by replacing each path $\beta \in E^*$ with $s(\beta), r(\beta)$ in G^0 of length at least 1 which passes through T by a single edge e_β, which has the same source and range as β. Note that the edges e in E with $r(e)$ and $s(e)$ in G^0 remain unchanged.

Let $v \in E^0$. As in [11], define

$$B_v = \{\beta \in E^*\backslash E^0 : r(\beta) = v, s(\beta) \in G^0 \text{ and } s(\beta_i) \in T^0 \text{ for } 1 \leq i < |\beta|\}.$$

Then $\bigcup_{w \in G^0} B_w$ of E^* corresponds to the set of edges G^1 in G.

To prove Theorem 4.1, we apply Theorem 3.1 with $V = G^0$, so that

$$M = \operatorname{span}_R\{s_\mu s_{\nu^*} : r(\mu) \in G^0\}.$$

Then M^*M is an ideal of $L_R(E)$ containing the subalgebra MM^*, and M^*M and MM^* are Morita equivalent. We need to show that $M^*M = L_R(E)$ and that MM^* is isomorphic to $L_R(G)$. Our proof uses quite a few of the arguments from Crisp and Gow's proof of [11, Theorem 3.1]. In particular, Lemma 3.6 of [11] gives a Cuntz–Krieger G-family in $C^*(E)$ and, since the proof is purely algebraic, it also gives a Leavitt G-family in $L_R(E)$. The universal property of $L_R(G)$ then gives a unique homomorphism $\phi : L_R(G) \to L_R(E)$. Crisp and Gow used the gauge-invariant uniqueness theorem to show that their C^*-homomorphism is one-to-one. The analogue here would be the graded uniqueness theorem; however, ϕ is not graded. Instead, to show that ϕ is one-to-one, we adapt some clever arguments from the proof of [1, Proposition 5.1] in Lemma 4.3 below which uses a reduction theorem.

Theorem 4.2 (Reduction theorem). *Let* R *be a commutative ring with identity, let* E *be a directed graph and let* $\{p_v, s_e, s_{e^*}\}$ *be a universal Leavitt* E*-family in* $L_R(E)$. *Suppose that* $0 \neq x \in L_R(E)$. *There exist* $\mu, \nu \in E^*$ *such that either:*

(1) *for some* $v \in E^0$ *and* $0 \neq r \in R$, *we have* $0 \neq s_{\mu^*} x s_\nu = r p_v$; *or*
(2) *there exist* $m, n \in \mathbb{Z}$ *with* $m \leq n$, $r_i \in R$ *and a nontrivial cycle* $\alpha \in E^*$ *such that* $0 \neq s_{\mu^*} x s_\nu = \sum_{i=m}^n r_i s_\alpha^i$. *(If* i *is negative, then* $s_\alpha^i := s_{\alpha^*}^{|i|}$.*)*

Proof. For Leavitt path algebras over a field, this is proved in [3, Proposition 3.1]. We checked carefully that the same proof works over a commutative ring R with identity. □

Lemma 4.3. *Let* R *be a commutative ring with identity. Let* E *and* G *be directed graphs and let* $\phi : L_R(G) \to L_R(E)$ *be an* R*-algebra* $*$*-homomorphism. Denote by* $\{p_v, s_e, s_{e^*}\}$ *and* $\{q_v, t_e, t_{e^*}\}$ *universal Leavitt* E*- and* G*-families in* $L_R(E)$ *and* $L_R(G)$, *respectively. Suppose that:*

(1) *for all* $v \in G^0$, $\phi(q_v) = p_{v'}$ *for some* $v' \in E^0$*; and*
(2) *for all* $e \in G^1$, $\phi(t_e) = s_\beta$ *for some* $\beta \in E^*$ *with* $|\beta| \geq 1$.

Then ϕ *is injective.*

Proof. We follow an argument made in [1, Proposition 5.1]. Let $x \in \ker \phi$. Aiming for a contradiction, suppose that $x \neq 0$. By Theorem 4.2, there exist $\mu, \nu \in G^*$ such that either condition (1) or (2) of the theorem holds.

First, suppose that (1) holds, that is, there exist $v \in G^0$ and $0 \neq r \in R$ such that $0 \neq t_{\mu^*} x t_\nu = r q_v$. Using assumption (1), there exists $v' \in E^0$ such that $\phi(q_v) = p_{v'}$. Now

$$0 = \phi(t_{\mu^*} x t_\nu) = \phi(r q_v) = r\phi(q_v) = r p_{v'}.$$

But $r p_{v'} \neq 0$ since $r \neq 0$, giving a contradiction.

Second, suppose that (2) holds, that is, there exist $m, n \in \mathbb{Z}$ with $m \leq n$, $r_i \in R$ and a nontrivial cycle $\alpha \in E^*$ such that $0 \neq t_{\mu^*} x t_\nu = \sum_{i=m}^n r_i t_\alpha^i$. Since α is a nontrivial cycle, it has length at least 1. By assumption (2), $\phi(t_\alpha) = s_{\alpha'}$, where α' is a path in E such that $|\alpha'|_E \geq |\alpha|_G \geq 1$. Since ϕ is an R-algebra $*$-homomorphism,

$$0 = \phi(t_{\mu^*} x t_\nu) = \phi\Big(\sum_{i=m}^n r_i t_\alpha^i\Big) = \sum_{i=m}^n r_i \phi(t_\alpha)^i = \sum_{i=m}^n r_i s_{\alpha'}^i.$$

Since $|\alpha'| = k$ for some $k \geq 1$, $s_{\alpha'}$ has grading k and hence each $s_{\alpha'}^i$ has grading ik. Thus, each term in the sum $\sum_{i=m}^n r_i s_{\alpha'}^i$ is in a distinct graded component. But, since $s_{\alpha'} \neq 0$, we must have $r_i = 0$ for all i. Thus, $\sum_{i=m}^n r_i t_\alpha^i = 0$, which is a contradiction. In either case, we obtained a contradiction to the assumption that $x \neq 0$. Thus, $x = 0$ and ϕ is injective. □

Proof of Theorem 4.1. Let $\{p_v, s_e, s_{e^*}\}$ be a universal Leavitt E-family in $L_R(E)$. We apply Theorem 3.1 with $V = G^0$ to get a surjective Morita context between MM^* and M^*M.

Since M and $M^* \subset L_R(E)$, we have $M^*M \subset L_R(E)$. To see that $L_R(E) \subset M^*M$, let $s_\mu s_{\nu^*} \in L_R(E)$. We may assume that $s(\mu) = s(\nu)$, for otherwise $s_\mu s_{\nu^*} = 0$. If $s(\mu) \in G^0$, then the Leavitt E-family relations give $s_\mu s_{\nu^*} = s_\mu s_{s(\mu)^*} s_{s(\mu)} s_{\nu^*} \in M^*M$ and we are done. So, suppose that $s(\mu) \in T^0$. Then the graph-theoretic [11, Lemma 3.4(c)] implies that $B_{s(\mu)} \neq \emptyset$. Suppose first that $B_{s(\mu)}$ is finite. It then follows from the first part of [11, Lemma 3.6] that $s(\mu)$ is a nonsingular vertex. The second part of [11, Lemma 3.6] implies that for any Cuntz–Krieger E-family $\{P_v, S_e, S_{e^*}\}$ in $C^*(E)$,

$$P_{s(\mu)} = \sum_{\beta \in B_{s(\mu)}} S_\beta S_{\beta^*};$$

the proof is purely algebraic and works for any Leavitt E-family in $L_R(E)$. Thus,

$$s_\mu s_{\nu^*} = s_\mu p_{s(\mu)} s_{\nu^*} = \sum_{\beta \in B_{s(\mu)}} s_\mu s_\beta s_{\beta^*} s_{\nu^*} = \sum_{\beta \in B_{s(\mu)}} s_{\mu\beta} s_{s(\beta)^*} s_{s(\beta)} s_{(\nu\beta)^*} \in M^*M.$$

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Next suppose that $B_{s(\mu)}$ is infinite. Since $s(\mu) \in T^0$ and $B_{s(\mu)}$ is infinite, the graph-theoretic [11, Lemma 3.4(d)] implies that there exists $x \in T^\infty$ such that $s(\mu) = r(x)$. By assumption (T2), there is a path $\alpha \in E^*$ with $r(\alpha) \in G^0$ such that $s(\alpha) = r(x) = s(\mu)$. Now

$$s_\mu s_{\nu^*} = s_\mu p_{s(\mu)} s_{\nu^*} = s_\mu s_{\alpha^*} s_\alpha s_{\nu^*} \in M^* M.$$

Thus, $L_R(E) = M^*M$. (We could have used Lemma 3.2 to prove that $L_R(E) = M^*M$, as Crisp and Gow do, but this seemed easier.)

Next we show that $L_R(G)$ and M^*M are isomorphic. For $v \in G^0$ and $\beta \in \bigcup_{w \in G^0} B_w$, define

$$Q_v = p_v, \quad T_{e_\beta} = s_\beta \quad \text{and} \quad T_{e_\beta^*} = s_{\beta^*}.$$

Then $\{Q_v, T_e, T_{e^*}\}$ is a Leavitt G-family in $L_R(E)$; again this follows as in the proof of [11, Theorem 3.1]. To see what is involved, we briefly step through this. Relations (L1) follow immediately from the relations for $\{p_v, s_e, s_{e^*}\}$. To see that (L2) holds, let $\gamma, \beta \in \bigcup_{w \in G^0} B_w$. Then $T_{e_\beta^*} T_{e_\gamma} = s_{\beta^*} s_\gamma$. By the graph-theoretic [11, Lemma 3.4(a)], neither γ nor β can be a proper extension of the other. Thus,

$$T_{e_\beta^*} T_{e_\gamma} = s_{\beta^*} s_\gamma = \delta_{\beta,\gamma} p_{s(\beta)} = \delta_{e_\beta e_\gamma} Q_{s(e_\beta)}$$

and (L2) holds.

To see that (L3) holds, let $v \in G^0$ be a nonsingular vertex. Then B_v is finite and nonempty because it is equinumerous with $r_G^{-1}(v)$. Using the algebraic analogue of [11, Lemma 3.6] again,

$$Q_v = p_v = \sum_{\beta \in B_v} s_\beta s_{\beta^*} = \sum_{e_\beta \in r_G^{-1}(v)} T_{e_\beta} T_{e_\beta^*}.$$

Thus, (L3) holds and $\{Q_v, T_e, T_{e^*}\}$ is a Leavitt G-family in $L_R(E)$.

Now let $\{q_v, t_e, t_{e^*}\}$ be a universal Leavitt G-family in $L_R(G)$. The universal property of $L_R(G)$ gives a unique homomorphism $\phi\colon L_R(G) \to L_R(E)$ such that for $v \in G^0$, $\beta \in \bigcup_{w \in G^0} B_w$,

$$\phi(q_v) = Q_v = p_v, \quad \phi(t_{e_\beta}) = T_{e_\beta} = s_\beta \quad \text{and} \quad \phi(t_{e_\beta^*}) = T_{e_\beta^*} = s_{\beta^*}.$$

If $v \in G^0$, we have $p_v = s_v s_{v^*} \in MM^*$; if $\beta \in B_w$ for some $w \in G^0$, then $r(\beta)$ is in G^0 and $s_\beta = s_\beta s_{s(\beta)^*}$ and $s_{\beta^*} = s_{s(\beta)} s_{\beta^*} \in MM^*$. It follows that the range of ϕ is contained in MM^*. That ϕ is onto MM^* again follows from work of Crisp and Gow. They take a nonzero spanning element $s_\mu s_{\nu^*} \in MM^*$ and use the graph-theoretic [11, Lemma 3.4(b)], the algebraic [11, Lemma 3.6] and assumptions (T1)–(T4) to show that $s_\mu s_{\nu^*}$ is in the range of ϕ. Thus, ϕ is onto.

Finally, ϕ satisfies the hypotheses of Lemma 4.3 and hence is one-to-one. Thus, ϕ is an isomorphism of $L_R(G)$ onto MM^*. □

Remark 4.4. A version of Theorem 3.1 should hold for the Kumjian–Pask algebras associated to locally convex or finitely aligned k-graphs [7, 8]. But the challenge would be to formulate an appropriate notion of contractible subgraph in that setting.

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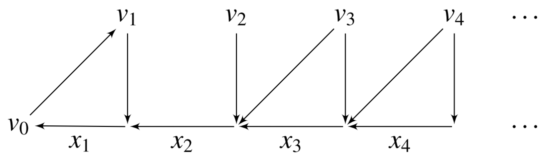

FIGURE 1. The graph E of Example 5.1.

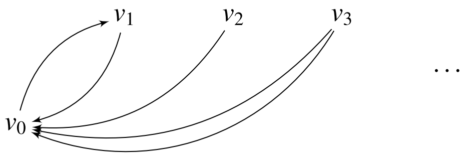

FIGURE 2. The collapsed graph F of Example 5.1.

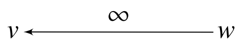

FIGURE 3. The graph F of Example 5.2.

5. Examples

As mentioned in the introduction, the setting of Theorem 4.1 includes many known examples. We found it helpful to see how some concrete examples fit.

EXAMPLE 5.1. An infinite path $x = x_1x_2\cdots$ in a directed graph is *collapsible* if x has no exits except at $r(x)$, the set $r^{-1}(r(x_i))$ of edges is finite for every i and $r^{-1}(r(x)) = \{x_1\}$ (see [16, Ch. 5]). Consider the row-finite directed graph E shown in Figure 1.

The infinite path $x = x_1x_2\cdots$ is collapsible. When we collapse x to the vertex v_0, as described in [16, Proposition 5.2], we get the graph F in Figure 2 with an infinite receiver at v_0.

This fits the setting of Theorem 4.1. Take $G^0 = \{v_i : i \geq 0\}$. Then T is the subgraph defined by $T^0 = \{s(x_i) : i \geq 1\}$ and $T^1 = \{x_i : i \geq 2\}$, and T contains none of the singular vertices $\{v_i : i \geq 2\}$ of E, is acyclic and satisfies the conditions (T1)–(T4). Thus, F is the graph G described in the theorem.

EXAMPLE 5.2. Consider the directed graph F in Figure 3 with source w and infinite receiver v.

An example of a Drinen–Tomforde desingularisation [13] of F is the row-finite graph E' with no sources on the left in Figure 4: a head has been added at the source w of F and each edge from w to v in F has been replaced with paths as shown. (This desingularisation is an example of an out-delay.) Since we are interested in Morita equivalence, we delete the head at w to get the graph E on the right in Figure 4.

Set $T^0 = E^0\backslash\{v, w\}$ and $T^1 = \{e \in E^1 : s(e), r(e) \in T^0\}$. Then the subgraph T contains none of the singularities of E, is acyclic and satisfies conditions (T1)–(T4) of Theorem 4.1. The graph F we started with is the graph G of Theorem 4.1.

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FIGURE 4. The desingularised graphs E' (left) and E (right) of Example 5.2.

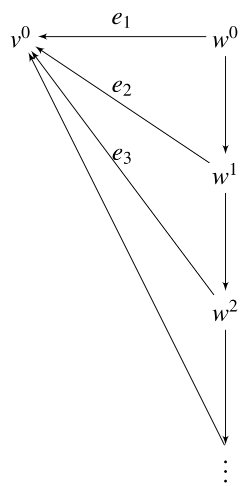

FIGURE 5. The in-delayed graph $d_s(E)$ of Example 5.3.

EXAMPLE 5.3. Consider again the graph F of Example 5.2. Label the infinitely many edges from w to v by e_i for $i \geq 1$. This time we will consider the in-delayed graph $d_s(E)$ given by the Drinen source-vector $d_s\colon E^0 \cup E^1 \to \mathbb{N} \cup \{\infty\}$ (see [6, Section 4]) to be the function defined by $d_s(e_i) = i - 1$ for $i \geq 1$, $d_s(v) = 0$ and $d_s(w) = \infty$. Then the in-delayed graph $d_s(E)$ given by d_s, as described in [6], is shown in Figure 5.

Now take $T^0 = d_s(E)^0 \backslash \{v^0, w^0\}$. Then T^0 contains none of the singular vertices of $d_s(E)$ and the corresponding subgraph T is acyclic. There are no infinite paths in $d_s(E)$ and hence conditions (T1)–(T4) of Theorem 4.1 hold trivially. The graph G of the theorem is again the graph F that we started out with.

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