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CONFORMALLY FLAT HYPERSURFACES OF SYMMETRIC SPACES

YOSHIO MATSUYAMA

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Abstract

In this paper we consider how much we can say about an irreducible symmetric space M which admits a single hypersurface with at most two distinct principal curvatures. Then we prove that if N is conformally flat, then N is quasiumbilical and M must be a sphere, a real projective space or the noncompact dual of a sphere or a real projective space.

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Recently, the following problem was proposed by B. Y. Chen and L. Verstraelen [3]: if we assume that an irreducible symmetric space M admits a single submanifold with a particular property, how much can we say about the ambient space? With respect to this problem, the author showed in [4] the following: (1) If M admits a (connected) locally symmetric hypersurface N ($\dim N \geqslant 3$) with at most two distinct principal curvatures, then M must be a sphere, a real projective space, or the noncompact dual of a sphere or a real projective space. (2) If an irreducible symmetric space M admits an Einstein hypersurface N ($\dim N \geqslant 3$) with at most two distinct principal curvatures, then M must be of rank 1.

The purpose of this paper is to prove the following:

THEOREM. *If an irreducible symmetric space M admits a conformally flat hypersurfaced N* ($\dim N \geqslant 4$) *with at most two distinct principal curvatures, then M must be a sphere, a real projective space, or the noncompact dual of a sphere or a real projective space.*

It is well-known that an n-dimensional ($n \geqslant 4$) hypersurface N in a sphere, a real projective space, or the noncompact dual of a sphere or a real projective space is conformally flat if and only if it is quasiumbilical (see [1] for instance). Hence, we know that: *A conformally flat hypersurface* N (dim $N \geqslant 4$) *with at most two distinct principal curvatures in an irreducible symmetric space is quasiumbilical* (*see Theorem* 8.1 *of* [3]).

1. Symmetric spaces and basic formulas

Let M be a connected Riemannian symmetric space. As usual if G denotes the closure of the group of isometries generated by an involutive isometry for each point of M, then G acts transitively on M; hence the isotropy subgroup H, say at 0, is compact and $M = G/H$. Let $\mathfrak{G}$, $\mathfrak{H}$ denote the Lie algebras corresponding to G, H, respectively. Then we call

$$\mathfrak{G} = \mathfrak{H} + \mathfrak{M}, \quad \text{and} \quad \mathfrak{H} = [\mathfrak{M}, \mathfrak{M}]$$

by the Cartan decomposition. It is well-known the space $\mathfrak{M}$ consists of the Killing vector field X whose covariant derivative vanishes at 0; in particular, the evaluation map at 0 gives a linear isomorphism of $\mathfrak{M}$ onto T_0M: $X \mapsto X(0)$. Hence we have

LEMMA 1.1. *For the curvature tensor R at* 0

$$R(X, Y)Z = -[[X, Y], Z] \quad \text{for } X, Y, Z \in \mathfrak{M}.$$

LEMMA 1.2. *A linear subspace L of the tangent space T_0M to a symmetric space M is the tangent space to some totally geodesic submanifold N of M if and only if L satisfies the condition* $[[\mathfrak{N}, \mathfrak{N}], \mathfrak{N}] \subset \mathfrak{N}$, *where*

$$\mathfrak{N} = \{X \in \mathfrak{M}; X(0) \in L\}.$$

Next, let N be a hypersurface of an $(n + 1)$-dimensional Riemannian manifold M. And let ∇ and ∇' be the covariant differentiations on M and N, respectively. Then the second fundamental form A of the immersion is given by

$$\nabla_X Y = \nabla'_X Y + g(AX, Y)\xi, \tag{1.1}$$

$$\nabla_X \xi = -AX, \tag{1.2}$$

for vector fields X, Y tangent to N and a unit vector field ξ normal to N, where g is the metric tensor of N induced by the immersion from the metric tensor g of M. The equations of Gauss and Codazzi are then given respectively

$$R'(X, Y; Z, W) = R(X, Y; Z, W) + g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W), \tag{1.3}$$

(1.4) $$R(X, Y; Z, \xi) = g((\nabla'_X A)Y, Z) - g((\nabla'_Y A)X, Z),$$

for vector fields X, Y, Z, W tangent to N and ξ normal to N, where R and R' are the curvature tensors of M and N, respectively, and $R(X, Y; Z, W) = g(R(X, Y)Z, W)$.

The following result is basic:

LEMMA 1.3 (CHEN & NAGANO [2]). *If an irreducible symmetric space M admits a totally geodesic hypersurface, then M must be a sphere, a real projective space, or the noncompact dual of a sphere or a real projective space.*

2. Proof of Theorem

Let N be a hypersurface in M and $E_1, \dots, E_n$ be an orthonormal basis of T_xN, $x \in N$. Then the Ricci tensor S' of N satisfies

(2.1) $$\begin{aligned} S'(Y, Z) &= \sum_{i=1}^{n} R'(E_i, Y; Z, E_i) \\ &= S(Y, Z) - R(\xi, Y; Z, \xi) + \operatorname{trace} A g(AY, Z) - g(A^2Y, Z) \end{aligned}$$

for $Y, Z \in T_xN$, where S denotes the Ricci tensor of M.

We suppose that there is a point x_0 at which two principal curvatures α, β are exactly distinct. Then we can choose a neighborhood U of x_0 on which $\alpha \neq \beta$. We put $T_\alpha = \{X \in TU | AX = \alpha X\}$ and $T_\beta = \{X \in TU | AX = \beta X\}$. Then the equation (2.1) gives

(2.1)′ $$\begin{aligned} S'(Y, Z) = S(Y, Z) &- R(\xi, Y; Z, \xi) \\ &+ (p\alpha + (n-p)\beta) g(AY, Z) - g(A^2Y, Z), \end{aligned}$$

where p denotes the multiplicity of α. Thus the scalar curvatures ρ' and ρ of N and M satisfy

(2.2) $$\begin{aligned} \rho' &= \sum_{i=1}^{n} S'(E_i, E_i) \\ &= \rho - 2S(\xi, \xi) + (p\alpha + (n-p)\beta)^2 - (p\alpha^2 + (n-p)\beta^2) \\ &= \frac{n-1}{n+1}\rho + p(p-1)\alpha^2 + 2p(n-p)\alpha\beta + (n-p)(n-p-1)\beta^2, \end{aligned}$$

where the last equality holds since M is Einsteinian. Now, by the assumption that N is conformally flat, the Weyl conformal curvature tensor of N vanishes. Thus by (2.1)′ and (2.2), we see that the curvature tensor R of M satisfies

(2.3)

$$\begin{aligned}(n-2)\{&R(X,Y;Z,W)+g(AX,W)g(AY,Z)-g(AX,Z)g(AY,W)\}\\ &=g(Y,W)\{R(\xi,X;Z,\zeta)-(p\alpha+(n-p)\beta)g(AX,Z)+g(A^2X,Z)\}\\ &\quad-g(X,W)\{R(\xi,Y;Z,\xi)X-(p\alpha+(n-p)\beta)g(AY,Z)+g(A^2Y,Z)\}\\ &\quad+g(X,Z)\{R(\xi,Y;W,\xi)-(p\alpha+(n-p)\beta)g(AY,W)+g(A^2Y,W)\}\\ &\quad-g(Y,Z)\{R(\xi,X;W,\xi)-(p\alpha+(n-p)\beta)g(AX,W)+g(A^2X,W)\}\\ &\quad+\frac{\rho}{n+1}\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\}\\ &\quad-\frac{1}{n-1}(p(p-1)\alpha^2+2p(n-p)\alpha\beta+(n-p)(n-p-1)\beta^2)\\ &\quad\cdot\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\}\end{aligned}$$

for X, Y, Z, W tangent to N.

Let X, Y, Z, W and T be vector fields tangent to N. By differentiation of (2.3) with respect to T, we may obtain, after a straightforward computation, that

(2.4)

$$\begin{aligned}(n-2)\{&g(AT,X)R(W,Z;Y,\xi)+g(AT,Y)R(Z,W;X,\xi)\\ &+g(AT,Z)R(Y,X;W,\xi)+g(AT,W)R(X,Y;Z,\xi)\\ &+g((\nabla'_TA)g(AY,Z)-g((\nabla'_TA)X,Z)g(AY,W)\\ &\qquad+g(AX,W)g((\nabla'_TA)Y,Z)-g(AX,Z)g((\nabla'_TA)Y,W)\}\\ =g(Y,W)\{&-R(AT,X;Z,\xi)\\ &-R(\xi,X;Z,AT)-(pT\alpha+(n-p)T\beta)g(AX,Z)\\ &\qquad-(p\alpha+(n-p)\beta)g((\nabla'_TA)X,Z)+g\big((\nabla'_TA^2)X,Z\big)\}\\ -g(X,W)\{&-R(AT,Y;Z,\xi)-R(\xi,Y;Z,AT)\\ &-(pT\alpha+(n-p)T\beta)g(AY,Z)\\ &\qquad-(p\alpha+(n-p)\beta)g((\nabla'_TA)Y,Z)+g\big((\nabla'_TA^2)Y,Z\big)\}\\ +g(X,Z)\{&-R(AT,Y;W,\xi)\\ &-R(\xi,Y;W,AT)-(pT\alpha+(n-p)T\beta)g(AY,W)\\ &\qquad-(p\alpha+(n-p)\beta)g((\nabla'_TA)Y,W)+g\big((\nabla'_TA^2)Y,W\big)\}\\ -g(Y,Z)\{&-R(AT,X;W,\xi)\\ &-R(\xi,X;W,AT)-(pT\alpha+(n-p)T\beta)g(AX,W)\\ &\qquad-(p\alpha+(n-p)\beta)g((\nabla'_TA)X,W)+g\big((\nabla'_TA^2)X,W\big)\}\\ -\frac{1}{n-1}&(p(p-1)T\alpha^2+2p(n-p)T\alpha\beta+(n-p)(n-p-1)T\beta^2)\\ \cdot\{&g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\}.\end{aligned}$$

If X, Y, Z, W are vectors in T_α such that $X = W$, $Y = Z$ and X, Y are orthonormal, then by (1.4) and (2.4) we find

(2.5)

$$\begin{aligned}(n-2)\{2\alpha X\alpha g(T,X) + 2\alpha X\alpha g(T,Y) + 2\alpha T\alpha\} \\ = -2\{-2AT\alpha + \alpha Y\alpha g(T,Y) + g((\alpha I - A)\nabla'_Y Y, AT) \\ + \alpha X\alpha g(T,X) + g((\alpha I - A)\nabla'_X X, AT) \\ -(pT\alpha + (n-p)T\beta)\alpha - (p\alpha + (n-p)\beta)T\alpha + T\alpha^2\} \\ -\frac{1}{n-1}(p(p-1)T\alpha^2 + 2p(n-p)T\alpha\beta + (n-p)(n-p-1)T\beta^2).\end{aligned}$$

In particular, for $X = T$, (2.5) implies

$$\begin{aligned}(2.6)\quad 4(n-2)\alpha X\alpha = -2\{-(2p-1)\alpha X\alpha - (n-p)\beta X\alpha - (n-p)\alpha X\beta\} \\ -\frac{1}{n-1}\{2p(p-1)\alpha X\alpha + 2p(n-p)\beta X\alpha \\ + 2p(n-p)\alpha X\beta + 2(n-p)(n-p-1)\beta X\beta\}.\end{aligned}$$

Let $T = X$, $W = \omega$ in T_β and Y, Z in T_α be orthonormal vectors. Then (2.4) gives

$$(2.7)\qquad -(n-2)\beta(\beta-\alpha)g(\nabla'_Z\omega, Y) = 0$$

for orthonormal vectors Y, Z in T_α. By linearization, we find

$$(2.8)\qquad \beta\{g(\nabla'_Y\omega, Y) - g(\nabla'_Z\omega, Z)\} = 0$$

for orthonormal vectors Y, Z in T_α. Similarly, we have

$$(2.9)\qquad \alpha g(\nabla'_{\omega_1}X, \omega_2) = 0,$$

$$(2.10)\qquad \alpha\{g(\nabla'_{\omega_1}X, \omega_1) - g(\nabla'_{\omega_2}X, \omega_2)\} = 0$$

for X in T_α and orthonormal vectors ω_1, ω_2 in T_β.

Let $Y = W$, Z in T_α be orthonormal vectors and $T = \omega_1$, $X = \omega_2$ unit vectors in T_β. Then (2.4) gives

(2.11)

$$\begin{aligned}(n-2)\{-\beta g(\omega_1,\omega_2)Z\alpha - \alpha(\alpha-\beta)g(\nabla'_{\omega_1}Z, \omega_2) \\ = -\beta(\alpha-\beta)g(\nabla'_{\omega_1}Z, \omega_2) + \beta(\alpha-\beta)g(\nabla'_{\omega_2}Z, \omega_1)\} \\ -\beta(\alpha-\beta)g(\nabla'_{\omega_1}Z, \omega_2) + \beta g(\omega_1,\omega_2)Z\beta \\ -(p\alpha + (n-p)\beta)(\alpha-\beta)g(\nabla'_{\omega_1}Z, \omega_2) + (\alpha^2-\beta^2)g(\nabla'_{\omega_1}Z, \omega_2).\end{aligned}$$

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For unit vectors $Y = W = \omega_0$ in T_β, Z in T_α, and $T = \omega_1$, $X = \omega_2$ in T_β which are perpendicular to ω_0

(2.12)

$$\begin{aligned}(n-2)\{&-\beta g(\omega_1,\omega_2)Z\beta + \beta(\alpha-\beta)g(\omega_1,\omega_2)g(\nabla'_{\omega_0}Z,\omega_0)\\ &-\beta(\alpha-\beta)g(\nabla'_{\omega_1}Z,\omega_2)\}\\ = &-\beta(\alpha-\beta)g(\nabla'_{\omega_1}Z,\omega_2) + \beta g(\omega_1,\omega_2)Z\beta\\ &-\beta(\alpha-\beta)g(\nabla'_{\omega_1}Z,\omega_2) + \beta(\alpha-\beta)g(\nabla'_{\omega_2}Z,\omega_1)\\ &-(p\alpha+(n-p)\beta)(\alpha-\beta)g(\nabla'_{\omega_1}Z,\omega_2) + (\alpha^2-\beta^2)g(\nabla'_{\omega_1}Z,\omega_2).\end{aligned}$$

Subtracting (2.12) from (2.11), we obtain

$$(2.13)\quad \begin{aligned}\alpha\{(-\beta Z\alpha + \beta Z\beta)g(\omega_1,\omega_2) - \alpha(\alpha-\beta)g(\nabla'_{\omega_1}Z,\omega_2)\}\\ = \alpha\beta(\alpha-\beta)\{g(\omega_1,\omega_2)g(\nabla'_{\omega_0}Z,\omega_0) - g(\nabla'_{\omega_1}Z,\omega_2)\}\end{aligned}$$

Putting $\omega_1 = \omega_2$ and using (2.10), we find

$$(2.13)' \qquad \alpha\{-\beta Z\alpha + \beta Z\beta - \alpha 1.g(\nabla'_{\omega_1}Z,\omega_1)\} = 0$$

Let $X_1,\dots,X_p$, $\omega_1,\dots,\omega_{n-p}$ be an orthonormal basis of T_xN such that $X_1,\dots,X_p$ (resp. $\omega_1,\dots,\omega_{n-p}$) forms an orthonormal basis of T_α (resp. T_β). Since M is Einstein, we have

$$(2.14)\quad \begin{aligned}0 &= S(X_i,\xi)\\ &= \sum_{j=1}^{p} R(X_i,X_j;X_j,\xi) + \sum_{k=1}^{n-p} R(X_i,\omega_k;\omega_k,\xi)\\ &= pX_i\alpha + (n-p)X_i\beta - (n-p)(\alpha-\beta)g(\nabla'_{\omega_k}X_i,\omega_k),\end{aligned}$$

using (2.10) for all i, k. From (2.13)′ and (2.14) we obtain

$$(2.15)\qquad \alpha\{(p\alpha + (n-p)\beta)X_i\alpha + (n-p)(\alpha-\beta)X_i\beta\} = 0.$$

Now, we assume that $\dim T_\alpha \geqslant 3$. Let X, $Y = Z$, $T = W$ be orthonormal vectors in T_α. Then (2.4) gives

$$(2.16)\qquad (n-1)\alpha X\alpha = 0.$$

If $\alpha \neq 0$, then from (2.6) we obtain $(n-p-1)(\alpha-\beta)X\beta = 0$. Since we may assume $p \neq n-1$, we have $X\beta = 0$. Therefore from (2.9), (2.10) and (2.13)′ we obtain $g(\nabla'_{\omega_1}Z,\omega_2) = 0$ for all ω_1, ω_2 in T_β. If $\alpha \equiv 0$, then (2.6) gives $X\beta = 0$. Then (2.11) and (2.12) imply

$$(2.11)' \qquad \beta^2\{(n-p+1)g(\nabla'_{\omega_1}Z,\omega_2) - g(\nabla'_{\omega_2}Z,\omega_1)\} = 0$$

(2.12)′

$$\beta^2(n-2)\left\{-g(\omega_1,\omega_2)g(\nabla'_{\omega_0}Z,\omega_0)+g(\nabla'_{\omega_1}Z,\omega_1)\right\}$$
$$=\beta^2\left\{(n-p+1)g(\nabla'_{\omega_1}Z,\omega_2)-g(\nabla'_{\omega_2}Z,\omega_1)\right\}=0.$$

Putting $\omega_1=\omega_2$ in (2.12)′, we have

$$g(\nabla'_{\omega_0}Z,\omega_0)=g(\nabla'_{\omega_1}Z,\omega_1) \tag{2.17}$$

for orthonormal vectors ω_0, ω_1 in T_β. Combining (2.14) and (2.17), we obtain $g(\nabla'_\omega Z,\omega)=0$ for all ω in T_β. By linearization, we find

$$g(\nabla'_{\omega_1}Z,\omega_2)+g(\nabla'_{\omega_2}Z,\omega_1)=0. \tag{2.18}$$

Summing up (2.11)′ and (2.18), we have

$$(n-p+2)g(\nabla'_{\omega_1}Z,\omega_2)=0,$$

that is,

$$g(\nabla'_{\omega_1}Z,\omega_2)=0$$

for all ω_1, ω_2 in T_β. If $\dim T_\alpha=2$, then we have only to show $X\alpha=X\beta=0$ for all unit vectors X in T_α, since we can make use of the above argument. Then from (2.6) and (2.15)

$$\alpha\{(2\alpha+(n-2)\beta)X_i\alpha+(n-2)(\alpha-\beta)X_i\beta\}=0 \tag{2.19}$$

(2.20)

$$\{(2n^2-9n+9)\alpha-(n-2)(n-3)\beta\}X_i\alpha-(n-2)(n-3)(\alpha-\beta)X_i\beta=0.$$

Hence we obtain $X_i\alpha=X_i\beta=0$. Therefore we have $R(X,Y;Z,\xi)=0$ for all X, Y, Z in TU. From Lemmas 1.1, 1.2 and 1.3 we obtain the conclusion.

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Department of Mathematics
Chuo University
1-13-27 Kasuga, Bunkyo-ku
Tokyo 112, Japan