# A CYCLIC INEQUALITY

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(Received 20th March 1961)

**1. Introduction.**

For any positive integer $n$ and positive real variables $x_1, x_2, \ldots, x_n$ write

$$S_n(x_1, x_2, \ldots, x_n) = \frac{x_1}{x_2+x_3} + \frac{x_2}{x_3+x_4} + \ldots + \frac{x_{n-1}}{x_n+x_1} + \frac{x_n}{x_1+x_2}$$

$$= \sum_{r=1}^{n} \frac{x_r}{x_{r+1}+x_{r+2}}, \quad \ldots\ldots(1)$$

where $x_r$ is defined for all integers $r$ by the relations

$$x_{r+n} = x_r. \quad \ldots\ldots(2)$$

If

$$\mu(n) = \inf_{\substack{x_r>0 \\ 1 \leqq r \leqq n}} S_n(x_1, x_2, \ldots, x_n), \quad \ldots\ldots(3)$$

and

$$\lambda(n) = \frac{\mu(n)}{n}, \quad \ldots\ldots(4)$$

it is known that (**1, 3, 4**)

$$\lambda(n) = \tfrac{1}{2} \quad (n \leqq 6),$$

and

$$\lambda(n) < \tfrac{1}{2} \quad (\text{even } n \geqq 14, \text{ odd } n \geqq 53).$$

It is also known that (**2**) $\lambda(n)$ tends to a limit $\lambda$ as $n \to \infty$ and that

$$\lambda = \lim_{n\to\infty} \lambda(n) = \inf_{n \geqq 1} \lambda(n).$$

Further (**4**),

$$\lambda \leqq \lambda(24) < 0{\cdot}49950317.$$

No positive lower bound for $\lambda(n)$, and so for $\lambda$, appears to be known, however, other than the lower bound

$$\tfrac{1}{6}(2\sqrt{2}-1) = 0{\cdot}3047. \quad \ldots\ldots(5)$$

stated by me, without proof, in 1957 (**2**), although the problem of estimating $\lambda(n)$ has aroused considerable interest in the last few years. For this reason I give here an account of the method used to obtain the lower bound (5), which I have since improved, so that I can now prove the

**Theorem.** $\lambda(n) > 0{\cdot}33$ *for all* $n \geqq 1$.

The lower bound arises as the minimum of the expression

$$\frac{3}{2}\,\frac{2^{1-\xi}-\xi}{9-10\xi}$$

https://doi.org/10.1017/S0013091500002777 Published online by Cambridge University Press

subject to the condition $0 \leqq \xi \leqq \frac{1}{2}$; a more accurate estimate of this minimum is $0{\cdot}330232$.

As will be realised from its length, the proof of this result is not simple, although the ideas involved are fairly elementary. The details are abbreviated as much as possible.

## 2. Some Lemmas.

We shall make considerable use of the functions $f$ and $g$ which are defined on the non-negative numbers as follows:

$$f(x) = x \quad (0 \leqq x \leqq \tfrac{1}{2}), \qquad f(x) = \sqrt{(2x)} - \tfrac{1}{2} \quad (x \geqq \tfrac{1}{2}),$$
$$g(x) = x \quad (0 \leqq x \leqq 1), \qquad g(x) = 1 + \log x \quad (x \geqq 1).$$

**Lemma 1.** *$f(x)$ and $g(x)$ are convex functions of* $\log x$ *for* $x > 0$; i.e., *for any positive integer $m$ and positive $x_1, x_2, \ldots, x_m$*

$$f(x_1) + f(x_2) + \ldots + f(x_m) \geqq mf\{(x_1 x_2 \ldots x_m)^{1/m}\},$$

*and $g$ satisfies a similar inequality. Further*

$$x \geqq f(x) \geqq \sqrt{(2x)} - \tfrac{1}{2} \textit{ for } x \geqq 0.$$

*Proof.* Both $f(x)$ and $g(x)$ are continuous and have continuous derivatives for $x \geqq 0$. They possess second derivatives $f''(x)$ and $g''(x)$ except for $x = \frac{1}{2}$ and $x = 1$, respectively, and

$$xf''(x) + f'(x) \geqq 0, \quad xg''(x) + g'(x) \geqq 0,$$

except at these points, from which the convexity properties follow. The last inequality for $f$ is easily verified.

In the following four lemmas we are concerned with finding lower bounds for partial sums of (1); we write, for any positive integer $L$,

$$\phi_L = \phi_L(x_0, x_1, \ldots, x_{L+1}) = \sum_{r=0}^{L-1} \frac{x_r}{x_{r+1} + x_{r+2}}, \qquad \ldots\ldots\ldots\ldots\ldots(6)$$

where $x_0, x_1, \ldots, x_{L+1}$ are any $L+2$ positive numbers, subject only to the restrictions imposed in the lemmas.

**Lemma 2.** *If $L \geqq 1$ and $x_1 \geqq x_2 \geqq \ldots \geqq x_{L+1}$, then $\phi_L \geqq \frac{1}{2}L(x_0/x_L)^{1/L}$.*

*Proof.* By the inequality of the arithmetic and geometric means,

$$\phi_L \geqq L\left\{\frac{x_0 x_1 \ldots x_{L-1}}{(x_1+x_2)(x_2+x_3)\ldots(x_L+x_{L+1})}\right\}^{1/L}$$
$$= L\left\{\frac{x_0}{x_L + x_{L+1}} \cdot \frac{1}{1+\dfrac{x_2}{x_1}} \cdot \ldots \cdot \frac{1}{1+\dfrac{x_L}{x_{L-1}}}\right\}^{1/L}$$
$$\geqq L\left\{\frac{x_0}{2x_L} \cdot \frac{1}{2^{L-1}}\right\}^{1/L} = \tfrac{1}{2}L(x_0/x_L)^{1/L}.$$

**Lemma 3.** *If $L = 2$ and $x_1 \leqq x_2$, $x_3 \leqq x_2$, then $\phi_2 \geqq f(x_0/x_2)$.*

*Proof.* For $x \geqq 0$ and $0 \leqq u \leqq 1$, write

$$h(x, u) = \frac{x}{1+u} + \tfrac{1}{2}u, \qquad (7)$$

so that

$$\phi_2 = \frac{x_0}{x_1+x_2} + \frac{x_1}{x_2+x_3} \geqq \frac{x_0}{x_1+x_2} + \frac{x_1}{2x_2}$$
$$= h\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right). \qquad (8)$$

We prove that

$$h(x, u) \geqq f(x) \quad (x \geqq 0,\ 0 \leqq u \leqq 1). \qquad (9)$$

For $0 \leqq x \leqq \frac{1}{2}$ this follows since

$$h(x, u) - f(x) = \frac{u(1+u-2x)}{2(1+u)}.$$

If $x > \frac{1}{2}$ it is easily verified that, for fixed $x$, $h(x, u)$ has a minimum where $u = \sqrt{(2x)} - 1$, so that

$$h(x, u) \geqq h(x, \sqrt{(2x)} - 1) = f(x).$$

The result follows from (8) and (9).

**Lemma 4.** *If $L = 3$ and $x_1 \leqq x_2 \leqq x_3$, $x_4 \leqq x_3$, then $\phi_3 \geqq g(x_0/x_3)$.*

*Proof.* We have, by (6) and (7) and (9),

$$\phi_3 \geqq \frac{x_0}{x_1+x_3} + \frac{x_1}{x_2+x_3} + \frac{x_2}{2x_3}$$
$$= \frac{x_0}{x_1+x_3} + h\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right)$$
$$\geqq \frac{x_0}{x_1+x_3} + f\left(\frac{x_1}{x_3}\right)$$
$$= H\left(\frac{x_0}{x_3}, \frac{x_1}{x_3}\right), \qquad (10)$$

where

$$H(x, u) = \frac{x}{1+u} + f(u),$$

and we suppose that $x \geqq 0$, $0 \leqq u \leqq 1$. We prove that

$$\psi(x, u) = \frac{x}{1+u} + f(u) - g(x) \geqq 0 \qquad (11)$$

and may clearly suppose that $u > 0$. Since

$$\psi_x(x, u) = \frac{1}{1+u} - g'(x),$$

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there is, for fixed $u$, a minimum where $x = 1+u$, and so

$$\psi(x, u) \geqq \psi(1+u, u) = f(u) - \log(1+u).$$

This last expression has a non-negative derivative for $u \geqq 0$, so that

$$\psi(x, u) \geqq f(0) - \log 1 = 0.$$

This proves (11), and the lemma follows, by (10).

**Lemma 5.** *If* $L \geqq 4$ *and* $x_1 \leqq x_2 \leqq \ldots \leqq x_L$, $x_{L+1} \leqq x_L$, *then*

$$\phi_L \geqq \tfrac{1}{2}(L-1) f\{(x_0/x_L)^{2/(L-1)}\}.$$

*Proof.* We have, by (7), (9) and Lemma 1,

$$\begin{aligned}\phi_L &\geqq \frac{x_0}{x_1+x_3} + \frac{x_1}{2x_3} + 0 + \frac{x_3}{2x_5} + \frac{x_4}{2x_6} + \ldots + \frac{x_{L-2}}{2x_L} + \frac{x_{L-1}}{2x_L} \\ &\geqq h(x_0/x_3, x_1/x_3) + \tfrac{1}{2}(L-3)\left\{\frac{x_3 x_4}{x_L^2}\right\}^{1/(L-3)} \\ &\geqq f(x_0/x_3) + \tfrac{1}{2}(L-3)(x_3/x_L)^{2/(L-3)} \\ &\geqq f(x_0/x_3) + \tfrac{1}{2}(L-3) f\{(x_3/x_L)^{2/(L-3)}\} \\ &\geqq \tfrac{1}{2}(L-1) f\{(x_0/x_L)^{2/(L-1)}\}.\end{aligned}$$

The functions $f$ and $g$, whose values appear as lower bounds in Lemmas 3, 4 and 5, can be replaced by functions taking larger values over certain parts of their domains; thus, for $1 \leqq x \leqq 9/4$, $1+\log x$ can be replaced by $2\sqrt{x}-1$. These improvements, however, produce no corresponding improvements in the main lemma, which is Lemma 7, and so are not given.

**Lemma 6.** *Let* $a, b, c, d, A, B$ *and* $C$ *be non-negative constants with* $A+B+C = 1$. *Then, if* $x, y$ *and* $z$ *are any non-negative numbers such that*

$$x^A y^B z^C = d,$$

*we have*

$$aAx + bBy + cCz \geqq d a^A b^B c^C,$$

*where* $0^0$ *is always to be interpreted as* 1.

The proof is straightforward and is omitted.

**Lemma 7.** *Let*

$$F(x, y, z, t) = px + q'(y\sqrt{2}-\tfrac{1}{2}) + r(z\sqrt{2}-\tfrac{1}{2}) + 2sg(t),$$

*where* $x, y, z$ *and* $t$ *are positive numbers satisfying*

$$x^p y^{q'} z^r t^s = 1$$

*and* $p, q, q', r$ *and* $s$ *are non-negative numbers such that*

$$q' \geqq \tfrac{3}{4}q, \quad p+q+r+3s = n > 0.$$

*Then*

$$F(x, y, z, t) > 0{\cdot}66n.$$

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*Proof.* We put $m = p+q'+r$. If $m = 0$, then $s = \frac{1}{3}n$, $t = 1$ and

$$F(x, y, z, t) = \tfrac{2}{3}ng(1) = \tfrac{2}{3}n.$$

We may therefore suppose that $m>0$, and put

$$p = mu, \quad q' = mv, \quad s = mw,$$

so that

$$m(1+\tfrac{1}{3}v+3w)\geqq n. \qquad \text{.................................(12)}$$

By Lemma 6, with $a = m$, $b = c = m\sqrt{2}$, $A = u$, $B = v$,

$$F(x, y, z, t)\geqq m\{t^{-w}2^{\frac{1}{2}(1-u)}-\tfrac{1}{2}(1-u)+2wg(t)\} = F_1,$$

say. We find the minimum of $F_1$ subject to (12) and the inequalities

$$t\geqq 0, \quad 0\leqq u\leqq 1-v, \quad w\geqq 0.$$

We have

$$\frac{\partial F_1}{\partial u} = \tfrac{1}{2}mt^{-w}\{t^{w}-2^{\frac{1}{2}(1-u)}\log 2\}. \qquad \text{...........................(13)}$$

It is convenient to write

$$\alpha = \log 2 = 0{\cdot}69315, \quad \beta = -\text{loglog } 2 = 0{\cdot}36651,$$

all decimals being correct to the number of places given. There are three cases to be considered: (i) $\alpha 2^{\frac{1}{2}v}\leqq t^{w}\leqq \alpha 2^{\frac{1}{2}}$, (ii) $t^{w}\leqq \alpha 2^{\frac{1}{2}v}$, and (iii) $t^{w}\geqq \alpha 2^{\frac{1}{2}}$.

(i) In this case $t<1$, since $\alpha 2^{\frac{1}{2}} = 0{\cdot}98026<1$, and the inequalities for $t^{w}$ may be expressed in the form

$$\frac{\alpha-2\beta}{2\log t}\leqq w\leqq \frac{v\alpha-2\beta}{2\log t}. \qquad \text{..............................(14)}$$

It follows from (13) that $F_1$ has a minimum where $\alpha 2^{\frac{1}{2}(1-u)} = t^{w}$, so that

$$F_1\geqq F_2 = m\alpha^{-1}\{1-\beta+w(2\alpha t-\log t)\}>0.$$

We suppose first that $0<t\leqq\frac{1}{2}$; since $2\alpha t-\log t$ decreases with $t$,

$$\begin{aligned} F_2 &\geqq m\alpha^{-1}(1-\beta+2\alpha w) \\ &\geqq \frac{n}{\alpha}\frac{1-\beta+2\alpha w}{\frac{4}{3}+3w}\geqq \tfrac{2}{3}n, \end{aligned}$$

since $9(1-\beta)-8\alpha>0$.

We may therefore suppose that $\frac{1}{2}<t<1$. Then

$$F_2\geqq F_3 = n\frac{1-\beta+w(2\alpha t-\log t)}{\alpha(1+\frac{1}{3}v+3w)},$$

and $\partial F_3/\partial w<0$, since

$$(1+\tfrac{1}{3}v)(2\alpha t-\log t)-3(1-\beta)<2\alpha(1+\tfrac{1}{3}v)-3(1-\beta)\leqq\tfrac{8}{3}\alpha-3(1-\beta)<0.$$

Hence, by (14),

$$F_3\geqq F_4 = \frac{n}{\alpha}\,\frac{A-Bv}{C-Dv},$$

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where

$$A = 2\alpha\beta t - \log t, \quad \beta = \tfrac{1}{2}\alpha(2\alpha t - \log t),$$
$$C = 3\beta - \log t, \quad D = \tfrac{3}{2}\alpha + \tfrac{1}{3}\log t.$$

We show that $F_4 \geqq \frac{2}{3}n$; this will follow if we prove that

$$3A - 2\alpha C \geqq v(3B - 2\alpha D). \quad \text{.............................(15)}$$

Now

$$3B - 2\alpha D = -3\alpha\{\alpha(1-t) + \tfrac{13}{18}\log t\},$$

which decreases for $\frac{1}{2} < t < 1$, since $18\alpha < 13$, and so exceeds zero. Hence we can take $v = 1$ in (15) and must prove that

$$3(A - B) \geqq 2\alpha(C - D),$$

i.e. that

$$\psi(t) = -(18 - 25\alpha)\log t - 18\alpha(2\beta - \alpha)(1 - t) > 0.$$

Now

$$\psi'(t) = 18\alpha(2\beta - \alpha) - (18 - 25\alpha)t^{-1}$$
$$= 0{\cdot}49755 - 0{\cdot}67132t^{-1} < 0,$$

so that $\psi(t) \geqq \psi(1) = 0$, as required. Thus we have shown that, if case (i) holds, then $F_1 \geqq \frac{2}{3}n$.

(ii) In this case $0 < t < 1$ and

$$w \geqq \frac{v\alpha - 2\beta}{2\log t} > 0. \quad \text{.................................(16)}$$

Then $\partial F_1/\partial u \leqq 0$, by (13), so that, since $u \leqq 1 - v$,

$$F_1 \geqq nF_5 = n\frac{t^{-w}2^{\frac{1}{2}v} - \frac{1}{2}v + 2wt}{1 + \frac{1}{3}v + 3w}. \quad \text{..........................(17)}$$

The numerator is positive since $2^{\frac{1}{2}v} > \frac{1}{2}v$. Regarded as a function of $t$, $F_5$ has a minimum at

$$t = 2^{-\xi}, \text{ where } \xi = \frac{1 - \frac{1}{2}v}{1 + w}.$$

We put

$$\eta = \frac{1}{1 + w},$$

so that

$$v = 2\left(1 - \frac{\xi}{\eta}\right), \quad w = \frac{1}{\eta} - 1.$$

Now $0 \leqq v \leqq 1$ and so, by (16),

$$0 < \eta \leqq 2\xi \leqq 2\eta \leqq 2$$

and

$$\xi \leqq 1 - \frac{\beta}{\alpha} = 0{\cdot}47123.$$

We have

$$F_5 = F_5(\xi, \eta) \mathrel{\dot{=}} 3\frac{2^{1-\xi} + \xi - \eta}{9 - 2\xi - 4\eta},$$

and

$$(9-2\xi-4\eta)^2 \frac{\partial F_5}{\partial \eta} = 3(2^{3-\xi}+6\xi-9)<0;$$

for $2^{3-\xi}+6\xi-9<0$ for $\xi<\frac{1}{2}$. Thus $F$ decreases with $\eta$ and so, since $\eta \leqq 2\xi<1$,

$$F_5 \geqq F_5(\xi, 2\xi) = F_6(\xi) = 3\frac{2^{1-\xi}-\xi}{9-10\xi}. \qquad (18)$$

It therefore suffices to show that

$$F_6(\xi)>0{\cdot}66 \text{ for } 0 \leqq \xi \leqq \tfrac{1}{2},$$

i.e. that

$$F_7(\xi) = 2^{1-\xi}+1{\cdot}2\xi-1{\cdot}98>0.$$

This holds since $F_7(\xi)$ has a minimum where $2^\xi = \frac{5}{3}\log 2$, so that

$$F_7(\xi) \geqq \tfrac{6}{5}\frac{\log(\frac{5}{3}e\log 2)}{\log 2}-1{\cdot}98 = 0{\cdot}00107>0.$$

(iii) In this case we can again show that $F_1 \geqq \frac{2}{3}n$. We have $\partial F_1/\partial u \geqq 0$, so that

$$F_1 \geqq F_8 = m\{t^{-w}2^{\frac{1}{2}}-\tfrac{1}{2}+2wg(t)\}$$

and

$$\frac{\partial F_8}{\partial t} = mw\{2g'(t)-2^{\frac{1}{2}}t^{-w-1}\}.$$

If $t \geqq 1$, $g'(t) = t^{-1}$ so that $\partial F_8/\partial t>0$ and

$$F_8 \geqq m(2^{\frac{1}{2}}-\tfrac{1}{2}+2w)$$

$$\geqq n\frac{2^{\frac{1}{2}}-\frac{1}{2}+2w}{\frac{4}{3}+3w} \geqq \frac{2}{3}n,$$

since $9(2^{\frac{1}{2}}-\frac{1}{2})>8$.

We may therefore suppose that $t<1$ and we then have

$$F_1 \geqq n\frac{t^{-w}2^{\frac{1}{2}}-\frac{1}{2}+2wt}{\frac{4}{3}+3w}.$$

By (17), the right-hand side is the same as $nF_5$ with $v = 1$ and so has a minimum for $t = 2^{-\xi}$, where $\xi = 1/\{2(1+w)\}$. Now $v = 1$ gives $\eta = 2\xi$, so that, by (18),

$$F_1 \geqq nF_6(\xi).$$

The conditions on $\xi$ are different in this case, however. They are

$$1-\frac{\beta}{\alpha} \leqq \xi \leqq \tfrac{1}{2}.$$

It is easily verified that $9{\cdot}2^{1-\xi}>18-11\xi$ in this range, so that $F_6(\xi)>\frac{2}{3}$. This completes the proof of the lemma.

**3. Proof of the Theorem.** Let $x_1, x_2, \ldots, x_n$ be any $n$ positive numbers. We prove that

$$S_n(x_1, x_2, \ldots, x_n)>0{\cdot}33n.$$

We may suppose that the $n$ numbers are not all equal, since otherwise $S_n = \frac{1}{2}n$. We divide them into disjoint subsets of numbers with consecutive suffixes by the following method, which we first describe in general terms.

Because of our convention (2), we may regard the $n$ numbers as being arranged in cyclic order, for example, on the circumference of a circle. This circumference we divide into subsets by making a cut before every minimum and after every maximum. This yields subsets of two types, (i) *increasing* subsets running from a minimum to a maximum, and (ii) *decreasing* subsets running from just after a maximum to just before a minimum.

More precisely, if $L \leqq n$, a subset $x_{q+1}, x_{q+2}, \ldots, x_{q+L}$ is called an increasing subset of length $L$, if $L \geqq 2$ and

$$x_q > x_{q+1},\ x_{q+1} \leqq x_{q+2} \leqq \cdots \leqq x_{q+L},\ x_{q+L} > x_{q+L+1}.$$

It is called a decreasing subset of length $L$, if $L \geqq 1$ and

$$x_q > x_{q+1} \geqq x_{q+2} \geqq \cdots \geqq x_{q+L} > x_{q+L+1}.$$

In either case $x_q/x_{q+L}$ is called the *ratio* of the subset.

In this way the set of $n$ numbers $x_1, x_2, \ldots, x_n$ is divided into a number of disjoint subsets, the product of whose ratios is 1. Two increasing subsets may follow one after the other, but two or more consecutive decreasing subsets can always be combined into a single decreasing subset, and we shall suppose that this is done.

The sum $S(x_1, x_2, \ldots, x_n)$ also splits up into a number of *subset sums*, one from each subset; for example, the contribution to $S(x_1, x_2, \ldots, x_n)$ from the subset $x_{q+1}, x_{q+2}, \ldots, x_{q+L}$ is

$$\sum_{r=0}^{L-1} \frac{x_{q+r}}{x_{q+1+r}+x_{q+2+r}},$$

and is a sum of the form (6). Lemmas 2 to 5 are applicable to these subset sums. In fact Lemma 2 applies to decreasing subsets of any length $L \geqq 1$, while Lemmas 3, 4 and 5 apply to increasing subsets of lengths 2, 3 and $L \geqq 4$, respectively. We can split $S = S(x_1, x_2, \ldots, x_n)$ into four corresponding parts, namely

$$S = D + I_2 + I_3 + I_4,$$

where, for example, $I_4$ denotes the sum of all increasing subset sums of length $L \geqq 4$.

If $p$ is the total length of all decreasing subsets, Lemma 2 combined with the inequality of the arithmetic means shows that

$$D \geqq \tfrac{1}{2}px,$$

where $x^p$ is the product of all the ratios associated with these subsets. Similarly, Lemmas 1 and 3 show that

$$I_2 \geqq \tfrac{1}{2}rf(z^2),$$

where $r$ is the total length of all increasing subsets of length 2 and $z^r$ is the product of their ratios. Also, Lemmas 1 and 4 show that

$$I_3 \geqq sg(t),$$

where $3s$ is the total length of all increasing subsets of length 3 and $t^s$ is the product of their ratios. Finally, if

$$q' = \Sigma(L-1), \quad q = \Sigma L,$$

the summations being over all lengths $L \geqq 4$ of increasing subsets, Lemmas 1 and 5 show that

$$I_4 \geqq \tfrac{1}{2}q' f(y^2),$$

where $y^{q'}$ is the product of their ratios. By the last part of Lemma 1 we have

$$2S \geqq px + q'(y\sqrt{2} - \tfrac{1}{2}) + r(z\sqrt{2} - \tfrac{1}{2}) + 2sg(t)$$

and, since $L-1 \geqq \frac{3}{4}L$ for $L \geqq 4$,

$$q' \geqq \tfrac{3}{4}q \text{ and } p+q+r+3s = n.$$

Also, since the product of all the ratios is 1,

$$x^p y^{q'} z^r t^s = 1.$$

The Theorem now follows from Lemma 7.

## REFERENCES

(1) L. J. Mordell, On the inequality $\sum_{r=1}^{n} x_r/(x_{r+1}+x_{r+2}) \geqq \frac{1}{2}n$ and some others, *Abh. Math. Sem. Univ. Hamburg*, **22** (1958), 229-240.

(2) R. A. Rankin, An inequality, *Math. Gaz.* **42** (1958), 39-40.

(3) A. Zulauf, Note on a conjecture of L. J. Mordell, *Abh. Math. Sem. Univ. Hamburg*, **22** (1958), 241.

(4) A. Zulauf, On a conjecture of L. J. Mordell II, *Math. Gaz.*, **43** (1959), 182-184.

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