Canad. J. Math. Vol. **62** (3), 2010 pp. 668–720
doi:10.4153/CJM-2010-031-2
© Canadian Mathematical Society 2010

The Supersingular Locus of the Shimura Variety for GU$(1, s)$

Inken Vollaard

Abstract. In this paper we study the supersingular locus of the reduction modulo p of the Shimura variety for GU$(1, s)$ in the case of an inert prime p. Using Dieudonné theory we define a stratification of the corresponding moduli space of p-divisible groups. We describe the incidence relation of this stratification in terms of the Bruhat–Tits building of a unitary group.

In the case of GU$(1, 2)$, we show that the supersingular locus is equidimensional of dimension 1 and is of complete intersection. We give an explicit description of the irreducible components and their intersection behaviour.

Introduction

In this paper we study the supersingular locus of the reduction modulo p of the Shimura variety for GU$(1, s)$ in the case of an inert prime p. For GU$(1, 2)$ this is a purely 1-dimensional variety, and we describe explicitly the irreducible components and their intersection behaviour.

The results of this paper are thus part of the general program of giving an explicit description of the supersingular (or, more generally, the basic) locus of the reduction modulo p of a Shimura variety. Let us review previous work on this problem.

We fix a prime p and denote by $\mathcal{A}_g$ the moduli space over $\overline{\mathbb{F}}_p$ of principally polarized abelian varieties of dimension $g > 0$ with a small enough level structure prime to p. Let $\mathcal{A}_g^{\rm ss}$ be its supersingular locus. If $g = 1$, this is a finite set of points. If $g = 2$, Koblitz [Kb] showed that the irreducible components of $\mathcal{A}_2^{\rm ss}$ are smooth curves that intersect pairwise transversally at the superspecial points, *i.e.*, the points of $\mathcal{A}_2$ where the underlying abelian variety is superspecial (*i.e.*, isomorphic to a power of a supersingular elliptic curve). Each superspecial point is the intersection of $p+1$ irreducible components.

Katsura and Oort [KO1] proved that each irreducible component of $\mathcal{A}_2^{\rm ss}$ is isomorphic to $\mathbb{P}^1$. In [KO2] they calculated the dimension of the irreducible components of $\mathcal{A}_3^{\rm ss}$ and the number of irreducible components of $\mathcal{A}_2^{\rm ss}$ and $\mathcal{A}_3^{\rm ss}$. In the case $g = 3$, Li and Oort [LO] showed that the irreducible components are birationally equivalent to a $\mathbb{P}^1$-bundle over a Fermat curve. Furthermore, they computed for general g the dimension of the supersingular locus and the number of irreducible components.

Received by the editors January 17, 2007.
Published electronically January 26, 2010.
AMS subject classification: **14G35**, 11G18, 14K10.

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

The $\overline{\mathbb{F}}_p$-rational points of $\mathcal{A}_2^{ss}$ are described independently by Kaiser [Ka] and by Kudla and Rapoport [KR2]. They fixed a supersingular principally polarized abelian variety A over $\overline{\mathbb{F}}_p$ of dimension two and studied the $\overline{\mathbb{F}}_p$-rational points of the moduli space of quasi-isogenies of A. They covered the $\overline{\mathbb{F}}_p$-rational points of these moduli spaces with subsets that are in bijection with the $\overline{\mathbb{F}}_p$-rational points of $\mathbb{P}^1$. The incidence relation of these subsets is described by the Bruhat–Tits building of an algebraic group over $\mathbb{Q}_p$. The number of superspecial points of $\mathcal{A}_2^{ss}$ is calculated in [KR2]. Each irreducible component of $\mathcal{A}_2^{ss}$ contains $p^2 + 1$ superspecial points.

Richartz [Ri] described the $\overline{\mathbb{F}}_p$-rational points of the moduli space of quasi-isogenies of a supersingular principally polarized abelian variety of dimension three. In analogy to the case $g = 2$, she defined subsets of the $\overline{\mathbb{F}}_p$-rational points of this moduli space and proved that the incidence relation of these sets is given by the Bruhat–Tits building of an algebraic group over $\mathbb{Q}_p$. She identified some of these sets with the $\overline{\mathbb{F}}_p$-rational points of Fermat curves over $\overline{\mathbb{F}}_p$.

Now consider the supersingular locus of a Hilbert–Blumenthal variety associated with a totally real field extension of degree g of $\mathbb{Q}$ in the case of an inert prime p. For $g = 2$ the supersingular locus was studied by Stamm [St], (*cf.* [KR1]). He showed that the irreducible components of the supersingular locus are isomorphic to $\mathbb{P}^1$ and contain $p^2 + 1$ superspecial points. Two components intersect transversally in at most one superspecial point, and each superspecial point is the intersection of two irreducible components. The number of irreducible components and the number of superspecial points of the supersingular locus were calculated in [KR1].

Bachmat and Goren [BG] analysed the case $g = 2$ for arbitrary prime p. They gave another proof of Stamm's results and proved that in case of a ramified prime p the components of the supersingular locus are smooth rational curves. Furthermore, they calculated the number of components in case of an inert or ramified prime p.

The case of $g = 3$ and an inert prime p was studied by Goren [Gn]. He proved that the irreducible components are smooth rational curves. Each superspecial point is the intersection of three irreducible components and each irreducible component contains only one superspecial point.

Yu [Yu] analysed the case of $g = 4$. He computed the number of irreducible components of the supersingular locus and the completion of the local ring at every superspecial point. Furthermore, he showed that every irreducible component is isomorphic to a ruled surface over $\mathbb{P}^1$.

Goren and Oort [GO] analyzed the Ekedahl–Oort stratification for general g. They proved that the supersingular locus is equidimensional of dimension $[g/2]$.

Finally, consider the supersingular locus of the reduction modulo p of the Shimura variety for GU$(1, s)$. In case of an inert prime p, Bültel and Wedhorn [BW] proved that the dimension of the supersingular locus is equal to $[s/2]$.

We now recall the definition of the moduli space of abelian varieties for GU(r, s). We first review the corresponding PEL-data, (*cf.* [K2]). Let E be an imaginary quadratic extension of $\mathbb{Q}$ such that p is inert in E and let $\mathcal{O}_E$ be its ring of integers. Denote by * the nontrivial Galois automorphism of E. Let V be an E-vector space of dimension $n > 0$ with perfect alternating $\mathbb{Q}$-bilinear form $(\cdot\,,\cdot)\colon V \times V \to \mathbb{Q}$ such that

$$(xv, w) = (v, x^* w)$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

for all $x \in E$ and $v, w \in V$. Denote by G the algebraic group over $\mathbb{Q}$ such that

$$G(R) = \{g \in \mathrm{GL}_{E\otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) \mid (gv, gw) = c(g)(v, w);\ c(g) \in R^\times\}$$

for every $\mathbb{Q}$-algebra R. Let E_p be the completion of E with respect to the p-adic topology. We assume that there exists an $\mathcal{O}_{E_p}$-lattice Λ in $V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ such that the form $(\,\cdot\,,\,\cdot\,)$ induces a perfect $\mathbb{Z}_p$-form on Λ. We fix an embedding φ_0 of E into $\mathbb{C}$, thereby identifying $E \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}$. We assume that there exists an isomorphism of $\mathbb{C}$-vector spaces of $V \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}^n$ such that the form $(\,\cdot\,,\,\cdot\,)$ induces the hermitian form given by the diagonal matrix $\mathrm{diag}(1^r, (-1)^s)$ on $\mathbb{C}^n$. We fix such an isomorphism. Then the group $G_{\mathbb{R}}$ is equal to the group $\mathrm{GU}(r, s)$ of unitary similitudes of $\mathrm{diag}(1^r, (-1)^s)$. The nonnegative integers r, s satisfy $r + s = n$. Let h be the homomorphism of real algebraic groups $h\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \to G_{\mathbb{R}}$ that maps an element $z \in \mathbb{C}^\times$ to the matrix $\mathrm{diag}(z^r, \overline{z}^s)$. Then $(E, \mathcal{O}_E, {}^*, V, (\,\cdot\,,\,\cdot\,), \Lambda, G, h)$ is a PEL-datum. Let K be the reflex field of this PEL-datum. Then K is isomorphic to E if $r \neq s$ and is equal to $\mathbb{Q}$ if $r = s$. Denote by K_p the completion of K with respect to the p-adic topology and by $\mathbb{F}$ the residue field of K.

Let A be an abelian scheme over an $\mathcal{O}_{K_p}$-scheme S of dimension n with $\mathcal{O}_E$-action, *i.e.*, with a morphism $\iota\colon \mathcal{O}_E \to \mathrm{End}\, A$. Let φ_0 and φ_1 be the two $\mathbb{Q}$-embeddings of E into $\overline{K}_p$. Then the polynomial $(T - \varphi_0(a))^r(T - \varphi_1(a))^s$ is an element of $\mathcal{O}_{K_p}[T]$. We say that (A, ι) satisfies the Kottwitz determinant condition of signature (r, s) [K2, Ch. 5] if

$$\mathrm{charpol}(a, \mathrm{Lie}\, A) = (T - \varphi_0(a))^r(T - \varphi_1(a))^s \in \mathcal{O}_S[T]$$

for all $a \in \mathcal{O}_E$.

We recall the definition of the moduli problem defined by Kottwitz [K2] for these PEL-data. Denote by $\mathbb{A}_f^p$ the ring of finite adeles of $\mathbb{Q}$ away from p. We fix a compact open subgroup C^p of $G(\mathbb{A}_f^p)$. We denote by $A^\vee$ the dual abelian scheme of A. Let $\mathcal{M}$ be the moduli problem which associates with any $\mathcal{O}_{K_p}$-scheme S the isomorphism classes of the following data:

- An abelian scheme A over S of dimension n.
- A $\mathbb{Q}$-subspace $\overline{\lambda}$ of $\mathrm{Hom}(A, A^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\overline{\lambda}$ contains a p-principal polarization, *i.e.*, a polarization of order prime to p.
- A homomorphism $\iota\colon \mathcal{O}_E \to \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that the Rosati involution given by $\overline{\lambda}$ on $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ induces the involution * on E.
- A C^p-level structure $\overline{\eta}\colon \mathrm{H}_1(A, \mathbb{A}_f^p) \xrightarrow{\sim} V \otimes_{\mathbb{Q}} \mathbb{A}_f^p \bmod C^p$.

We assume that (A, ι) satisfies the determinant condition of signature (r, s).

Two such data $(A, \overline{\lambda}, \iota, \overline{\eta})$ and $(A', \overline{\lambda}', \iota', \overline{\eta}')$ are isomorphic if there exists an isogeny prime to p from A to A', commuting with the action of $\mathcal{O}_E$, carrying $\overline{\eta}$ into $\overline{\eta}'$ and carrying $\overline{\lambda}$ into $\overline{\lambda}'$. For simplicity we write $(A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \overline{\lambda}, \iota \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \overline{\eta})$ for such an isomorphism class.

The moduli problem $\mathcal{M}$ is represented by a smooth, quasi-projective scheme over $\mathrm{Spec}\, \mathcal{O}_{K_p}$ if C^p is small enough [K2, Ch. 5]. The relative dimension of $\mathcal{M}$ is equal to rs. Denote by $\mathcal{M}^{\mathrm{ss}}$ the supersingular locus of the special fibre $\mathcal{M}_{\mathbb{F}}$ of $\mathcal{M}$. It is a closed subscheme of $\mathcal{M}_{\mathbb{F}}$ which is proper over $\mathrm{Spec}\, \mathbb{F}$. Our goal is to describe the irreducible components of $\mathcal{M}^{\mathrm{ss}}$ and their intersection behaviour.

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

The supersingular locus $\mathcal{M}^{ss}$ contains an $\overline{\mathbb{F}}_p$-rational point [BW, Lemma 5.2]. We will view $\mathcal{M}^{ss}$ as a scheme over $\overline{\mathbb{F}}_p$. Let $x = (A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \iota \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \overline{\lambda}, \overline{\eta}) \in \mathcal{M}^{ss}(\overline{\mathbb{F}}_p)$ and denote by $(\mathbb{X}, \iota)$ the supersingular p-divisible group of height $2n$ corresponding to x with $\mathcal{O}_{E_p}$-action ι. We choose a p-principal polarization $\lambda \in \overline{\lambda}$ and denote again by λ the induced p-principal polarization of $\mathbb{X}$. By construction $(\mathbb{X}, \iota)$ satisfies the determinant condition of type (r, s).

We recall the definition of the moduli space $\mathcal{N}$ of quasi-isogenies of p-divisible groups in characteristic p [RZ, Definition 3.21] in the case of the group GU(r, s). The moduli space $\mathcal{N}$ over Spec $\overline{\mathbb{F}}_p$ is given by the following data up to isomorphism for an $\overline{\mathbb{F}}_p$-scheme S:

- A p-divisible group X over S of height $2n$ with p-principal polarization λ_X and $\mathcal{O}_{E_p}$-action ι_X such that the Rosati involution induced by λ induces the involution * on $\mathcal{O}_E$. We assume that (X, ι) satisfies the determinant condition of type (r, s).
- An $\mathcal{O}_{E_p}$-linear quasi-isogeny $\rho\colon X \to \mathbb{X} \times_{\operatorname{Spec} \overline{\mathbb{F}}_p} S$ such that $\rho^\vee \circ \lambda \circ \rho$ is a $\mathbb{Q}_p$-multiple of λ_X in $\operatorname{Hom}_{\mathcal{O}_{E_p}}(X, X^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The moduli space $\mathcal{N}$ is represented by a separated formal scheme which is locally formally of finite type over $\overline{\mathbb{F}}_p$ [RZ, Theorem 3.25].

We recall the uniformization theorem of Rapoport and Zink [RZ, Theorem 6.30]. We will formulate this theorem only for the underlying schemes, not for the formal schemes. Let $I(\mathbb{Q})$ be the group of quasi-isogenies in $\operatorname{End}_{\mathcal{O}_E}(A) \otimes \mathbb{Q}$ that respect the homogeneous polarization $\overline{\lambda}$. As GU(r, s) satisfies the Hasse principle (see (6.2)), there exists an isomorphism of schemes over $\overline{\mathbb{F}}_p$

$$I(\mathbb{Q}) \backslash (\mathcal{N}^{\mathrm{red}} \times G(\mathbb{A}_f^p)/C^p) \xrightarrow{\sim} \mathcal{M}^{ss}.$$

In Section 6 we will show, in the case of GU$(1, 2)$, that $\mathcal{M}^{ss}$ is locally isomorphic to $\mathcal{N}^{\mathrm{red}}$ if C^p is small enough.

We now state our results. We assume that $p \neq 2$. Let k be an algebraically closed field extension of $\overline{\mathbb{F}}_p$ and let (X, ρ) be an element of $\mathcal{N}(k)$. As the height of the quasi-isogeny ρ is divisible by n (Lemma 1.5), we may define the morphism $\kappa\colon \mathcal{N} \to \mathbb{Z}$ by sending an element of $\mathcal{N}$ to the height of the quasi-isogeny divided by n. The fibres $\mathcal{N}_i$ of κ define a disjoint decomposition of $\mathcal{N}$ into open and closed formal subschemes. In fact, $\mathcal{N}_i$ is empty if ni is odd and $\mathcal{N}_i$ is isomorphic to $\mathcal{N}_0$ if ni is even (Lemma 1.7, Proposition 1.18). For the rest of this introduction, we fix an integer i with ni even.

From now on we assume that r is equal to 1. Let C be a $\mathbb{Q}_{p^2}$-vector space of dimension n. We choose a perfect skew-hermitian form $\{\,\cdot\,,\,\cdot\,\}$ on C such that there exists a self-dual $\mathbb{Z}_{p^2}$-lattice in C if n is odd and such that there exists no self-dual $\mathbb{Z}_{p^2}$-lattice if n is even. Denote by H the special unitary group of $(C, \{\,\cdot\,,\,\cdot\,\})$ over $\mathbb{Q}_p$ and denote by $\mathcal{B}(H, \mathbb{Q}_p)_{\mathrm{simp}}$ the simplicial complex of the Bruhat–Tits building of H. We associate with each vertex Λ of $\mathcal{B}(H, \mathbb{Q}_p)_{\mathrm{simp}}$ a subset $\mathcal{V}(\Lambda)(k)$ of $\mathcal{N}_i(k)$. In Section 3 we attach to each vertex Λ an odd integer l with $1 \leq l \leq n$, the *type* of Λ. The type classifies the different orbits of the action of $H(\mathbb{Q}_p)$ on the set of vertices of $\mathcal{B}(H, \mathbb{Q}_p)_{\mathrm{simp}}$. We call a point of $\mathcal{N}_i(k)$ superspecial if the underlying p-divisible group is superspecial, *i.e.,* if the corresponding Dieudonné module M satisfies $FM = VM$. Vertices of type 1 correspond to superspecial points of $\mathcal{N}_i(k)$. We prove the following theorem (Proposition 2.4, Proposition 3.4, Theorem 3.5).

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Theorem 1 *The sets $\mathcal{V}(\Lambda)(k)$ cover $\mathcal{N}_i(k)$.*

Let Λ and Λ' be two different vertices of $\mathcal{B}(H, \mathbb{Q}_p)_{\mathrm{simp}}$ of types l and l', respectively. Then the intersection of $\mathcal{V}(\Lambda)(k)$ and $\mathcal{V}(\Lambda')(k)$ is nonempty if and only if one vertex is a neighbour of the other or if the corresponding vertices have a common neighbour of type $l'' \leq \min\{l, l'\}$.

We associate with each vertex $\Lambda \in \mathcal{B}(H, \mathbb{Q}_p)_{\mathrm{simp}}$ a variety Y_Λ over $\overline{\mathbb{F}}_p$ such that for each algebraically closed field extension k of $\overline{\mathbb{F}}_p$ we have a bijection of $Y_\Lambda(k)$ with $\mathcal{V}(\Lambda)(k)$. Let l be the type of Λ and let U be the unitary group of an l-dimensional hermitian space over $\mathbb{F}_{p^2}$. We obtain the following theorem (Proposition 2.13, Theorem 2.15).

Theorem 2 *The variety Y_Λ is projective, smooth and irreducible and its dimension is equal to $d = (l-1)/2$.*

(i) *There exists a decomposition of Y_Λ into a disjoint union of locally closed subvarieties $Y_\Lambda = \biguplus_{j=0}^{d} X_{P_j}(w_j)$, where each $X_{P_j}(w_j)$ is isomorphic to a Deligne–Lusztig variety with respect to the group U and a parabolic subgroup P_j of U.*

(ii) *For every c with $0 \leq c \leq d$, the locally closed subvariety $X_{P_c}(w_c)$ is equidimensional of dimension c, and its closure in Y_Λ is equal to $\biguplus_{j=0}^{c} X_{P_j}(w_j)$. The variety $X_{P_d}(w_d)$ is open, dense, and irreducible of dimension d in Y_Λ.*

(iii) *For every c with $0 \leq c < d$, the subset $\biguplus_{j=0}^{c} X_{P_j}(w_j)(k)$ of $Y_\Lambda(k)$ corresponds to the subset $\bigcup_{\Lambda'} \mathcal{V}(\Lambda')(k)$ in $\mathcal{N}_i(k)$ where the union is taken over all neighbours Λ' of Λ with type $(2c+1)$.*

In the case of $\mathrm{GU}(1,0)$ and $\mathrm{GU}(1,1)$, the scheme $\mathcal{N}_0^{\mathrm{red}}$ is a disjoint union of infinitely many superspecial points.

In the case of $\mathrm{GU}(1,2)$, we define for each vertex Λ of type 3 a closed embedding of Y_Λ into $\mathcal{N}_0$ such that for every algebraically closed field extension k of $\overline{\mathbb{F}}_p$ the image of $Y_\Lambda(k)$ in $\mathcal{N}_0(k)$ is equal to $\mathcal{V}(\Lambda)(k)$. We denote by $\mathcal{V}(\Lambda)$ the image of Y_Λ. Let $\mathcal{C}$ be the smooth and irreducible Fermat curve in $\mathbb{P}^2_{\overline{\mathbb{F}}_p}$ given by the equation

$$x_0^{p+1} + x_1^{p+1} + x_2^{p+1} = 0.$$

We show that the varieties $\mathcal{V}(\Lambda)$ are the irreducible components of $\mathcal{N}_i$ and prove the following explicit description of $\mathcal{N}^{\mathrm{red}}$ (Theorem 5.10).

Theorem 3 *Let $(r,s) = (1,2)$. The schemes $\mathcal{N}_i^{\mathrm{red}}$, with $i \in \mathbb{Z}$ even, are the connected components of $\mathcal{N}^{\mathrm{red}}$, which are all isomorphic to each other. Each irreducible component of $\mathcal{N}^{\mathrm{red}}$ is isomorphic to $\mathcal{C}$. Two irreducible components intersect transversally in at most one superspecial point. Each irreducible component contains p^3+1 superspecial points and each superspecial point is the intersection of $p+1$ irreducible components.*

The scheme $\mathcal{N}^{\mathrm{red}}$ is equidimensional of dimension 1 and of complete intersection.

Using the uniformization theorem quoted above, we obtain the following conclusions for $\mathcal{M}^{\mathrm{ss}}$ if C^p is small enough (Corollary 6.2).

Theorem 4 *Let $(r,s) = (1,2)$. The supersingular locus $\mathcal{M}^{\mathrm{ss}}$ is equidimensional of dimension 1 and of complete intersection. Its singular points are the superspecial points*

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

of $\mathcal{M}^{ss}$. Each superspecial point is the pairwise transversal intersection of $p + 1$ irreducible components. Each irreducible component is isomorphic to $\mathcal{C}$ and contains $p^3 + 1$ superspecial points. Two irreducible components intersect in at most one superspecial point.

Let J be the group of similitudes of the isocrystal of $(\mathbb{X}, \iota, \lambda)$, or equivalently, the group of similitudes of $(C, \{\,\cdot\,,\,\cdot\,\})$ (1.13). Denote by J^0 the subgroup of all elements $g \in J$ such that the p-adic valuation of the multiplier of g is equal to zero. Let $C_{J,p}$ and $C'_{J,p}$ be maximal compact subgroups of J such that $C_{J,p}$ is hyperspecial and $C'_{J,p}$ is not hyperspecial. We obtain the following corollary (Proposition 6.3).

Corollary 5 *We have*

$$\begin{aligned}
\#\{\textit{irreducible components of } \mathcal{M}^{ss}\} &= \#(I(\mathbb{Q})\backslash(J/C_{J,p} \times G(\mathbb{A}_f^p)/C^p)),\\
\#\{\textit{superspecial points of } \mathcal{M}^{ss}\} &= \#(I(\mathbb{Q})\backslash(J/C'_{J,p} \times G(\mathbb{A}_f^p)/C^p)),\\
\#\{\textit{connected components of } \mathcal{M}^{ss}\} &= \#(I(\mathbb{Q})\backslash(J^0\backslash J \times G(\mathbb{A}_f^p)/C^p))\\
&= \#(I(\mathbb{Q})\backslash(\mathbb{Z} \times G(\mathbb{A}_f^p)/C^p)).
\end{aligned}$$

This paper is organized as follows. In Section 1 we describe the set $\mathcal{N}(k)$ for GU(r, s) using classical Dieudonné theory. From Section 2 on we assume $r = 1$. Section 2 contains the definition of the sets $\mathcal{V}(\Lambda)(k)$ for a lattice Λ in an index set $\mathcal{L}_i$ for every integer i. Furthermore, we prove Theorem 2. In Section 3 we identify the index set $\mathcal{L}_i$ with the set of vertices of $\mathcal{B}(H, \mathbb{Q}_p)$ and analyse the incidence relation of the sets $\mathcal{V}(\Lambda)(k)$ (Theorem 1). Sections 4 and 5 deal with the special case GU$(1, 2)$, and Theorem 3 is proved in Section 5. Here our main tool is Zink's theory of displays [Zi2] which is used to construct a universal display over $\mathcal{N}_0^{red}$. Section 6 contains the transfer of the results on the moduli space $\mathcal{N}$ to the supersingular locus $\mathcal{M}^{ss}$ (Theorem 4, Corollary 5).

We now explain why we restrict ourselves to the signature $(1, s)$. In the case GU(r, s) with $1 < r \leq s$, it is not clear how to obtain a similar decomposition of $\mathcal{N}(k)$ into subsets $\mathcal{V}(\Lambda)(k)$ as above. In particular, one should not expect a linear closure relation order of strata as stated in Theorem 2.

In the case $r = 1$, we expect that the pointwise decomposition of $\mathcal{N}$ given here can be made algebraic. However, it seems not to be promising to construct a universal display over each variety Y_Λ by using a basis of the isodisplay and the equations defining Y_Λ. Indeed, for increasing s, these equations become quite complicated.

1 Dieudonné Lattices in the Supersingular Isocrystal for GU(r, s)

In sections 1 to 5 we depart from the introduction and denote by E an unramified extension of $\mathbb{Q}_p$ of degree 2 with $p \neq 2$. Let $\mathcal{O}_E$ be its ring of integers. We fix a positive integer n and nonnegative integers r and s with $n = r + s$.

Definition 1.1 Let $\mathbb{X}$ be a supersingular p-divisible group of height $2n$ over $\overline{\mathbb{F}}_p$ with $\mathcal{O}_E$-action $\iota\colon \mathcal{O}_E \to \operatorname{End}\mathbb{X}$ such that $(\mathbb{X}, \iota)$ satisfies the determinant condition of

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

signature (r,s), *i.e.*,

$$\operatorname{charpol}_{\overline{\mathbb{F}}_p}(a, \operatorname{Lie}\mathbb{X}) = (T - \varphi_0(a))^r (T - \varphi_1(a))^s \in \overline{\mathbb{F}}_p[T]$$

for all $a \in \mathcal{O}_E$. Here we denote by φ_0 and φ_1 the different $\mathbb{Z}_p$-morphisms of $\mathcal{O}_E$ to $\overline{\mathbb{F}}_p$. Let λ be a p-principal quasi-polarization of $\mathbb{X}$ such that the Rosati involution induced by λ induces the involution * on $\mathcal{O}_E$.

We define the moduli space $\mathcal{N}$ over $\operatorname{Spec}\overline{\mathbb{F}}_p$ given by the following data up to isomorphism for an $\overline{\mathbb{F}}_p$-scheme S:

- A p-divisible group X over S of height $2n$ with p-principal polarization λ_X and $\mathcal{O}_E$-action ι_X such that the Rosati involution induced by λ_X induces the involution * on $\mathcal{O}_E$. We assume that (X,ι) satisfies the determinant condition of signature (r,s).
- An $\mathcal{O}_E$-linear quasi-isogeny $\rho\colon X \to \mathbb{X} \times_{\operatorname{Spec}\overline{\mathbb{F}}_p} S$ such that $\rho^\vee \circ \lambda \circ \rho$ is a $\mathbb{Q}_p$-multiple of λ_X in $\operatorname{Hom}_{\mathcal{O}_E}(X, X^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 1.2 The moduli space $\mathcal{N}$ is represented by a separated formal scheme which is locally formally of finite type over $\overline{\mathbb{F}}_p$ [RZ, Theorem 2.16].

Our goal is to describe the irreducible components of $\mathcal{N}$ and their intersection behaviour.

We will now study the set $\mathcal{N}(k)$ for any algebraically closed field extension k of $\overline{\mathbb{F}}_p$. Let $W(k)$ be the ring of Witt vectors over k, let $W(k)_{\mathbb{Q}}$ be its quotient field and let σ be the Frobenius automorphism of $W(k)$. We write W instead of $W(\overline{\mathbb{F}}_p)$. Denote by $\mathbb{M}$ the Dieudonné module of $\mathbb{X}$ and by

$$N = \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Q} \tag{1.1}$$

the associated supersingular isocrystal of dimension $2n$ with Frobenius F and Verschiebung V. The $\mathcal{O}_E$-action ι on $\mathbb{X}$ induces an E-action on N. The p-principal polarization λ of $\mathbb{X}$ induces a perfect alternating form $\langle\cdot\,,\cdot\rangle\colon N \times N \to W_{\mathbb{Q}}$ such that for all $a \in E$ and x, y of N

$$\langle Fx, y\rangle = \langle x, Vy\rangle^\sigma \tag{1.2}$$

and $\langle ax, y\rangle = \langle x, a^* y\rangle$. Denote by N_k the isocrystal $N \otimes_{W_{\mathbb{Q}}} W(k)_{\mathbb{Q}}$. For a lattice $M \subset N_k$, we denote by

$$M^\perp = \{ y \in N_k \mid \langle y, M\rangle \subset W(k)\}$$

the dual lattice of M in N_k with respect to the form $\langle\cdot\,,\cdot\rangle$. By covariant Dieudonné theory, the tangent space $\operatorname{Lie}\mathbb{X}$ can be identified with M/VM and we obtain the following proposition.

Proposition 1.3 *For any algebraically closed field extension k of $\overline{\mathbb{F}}_p$ we obtain the following identification.*

$$\begin{aligned}\mathcal{N}(k) = \bigl\{ M \subset N_k \text{ a } W(k)\text{-lattice} \mid M &\text{ is } F\text{-}, V\text{- and } O_E\text{-invariant},\\ \operatorname{charpol}_k(a, M/VM) &= (T-\varphi_0(a))^r (T-\varphi_1(a))^s \\ &\text{for all } a \in \mathcal{O}_E,\ M = p^i M^\perp \text{ for some } i \in \mathbb{Z} \bigr\}.\end{aligned} \tag{1.3}$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

We will now analyze the set $\mathcal{N}(k)$ in the form of (1.3). Consider the decomposition

$$E \otimes_{\mathbb{Q}_p} W(k)_{\mathbb{Q}} \cong W(k)_{\mathbb{Q}} \times W(k)_{\mathbb{Q}} \\ a \otimes x \mapsto (\varphi_0(a)x, \varphi_1(a)x), \tag{1.4}$$

given by the two embeddings $\varphi_i \colon E \hookrightarrow W(k)_{\mathbb{Q}}$. It induces a $\mathbb{Z}/2\mathbb{Z}$-grading

$$N_k = N_{k,0} \oplus N_{k,1} \tag{1.5}$$

of N_k into free $W(k)_{\mathbb{Q}}$-modules of rank n. Furthermore, each $N_{k,i}$ is totally isotropic with respect to $\langle \cdot , \cdot \rangle$ and F induces a σ-linear isomorphism $F \colon N_{k,i} \to N_{k,i+1}$. We obtain $\mathcal{O}_E \otimes_{\mathbb{Z}_p} W(k) \cong W(k) \times W(k)$ analogous to (1.4). Therefore, every $\mathcal{O}_E$-invariant Dieudonné module $M \subset N_k$ has a decomposition $M = M_0 \oplus M_1$ such that F and V are operators of degree 1 and $M_i \subset N_{k,i}$. For an $\mathcal{O}_E$-lattice M in N_k we will always denote such a decomposition by $M_0 \oplus M_1$. Furthermore, for M_i we define the dual lattice of M_i with respect to $\langle \cdot , \cdot \rangle$ as

$$M_i^{\perp} = \{ y \in N_{k,i+1} \mid \langle y, M_i \rangle \subset W(k) \}.$$

For $W(k)$-lattices L and L' in a finite dimensional $W(k)_{\mathbb{Q}}$-vector space, we denote by $[L' \colon L]$ the index of L in L'. If $L \subset L'$, the index is defined as the length of the $W(k)$-module L'/L. If $[L' \colon L] = m$, we write $L \overset{m}{\subset} L'$. In general, we define

$$[L' \colon L] = [L' \colon L \cap L'] - [L \colon L \cap L'].$$

Lemma 1.4 *Let $M = M_0 \oplus M_1$ be an $\mathcal{O}_E$-invariant lattice of N_k. Assume that M is invariant under F and V. Then M satisfies the determinant condition of signature* (r, s) *if and only if*

$$pM_0 \overset{s}{\subset} FM_1 \overset{r}{\subset} M_0, \tag{1.6}$$

$$pM_1 \overset{r}{\subset} FM_0 \overset{s}{\subset} M_1. \tag{1.7}$$

Proof Consider the decomposition $M/VM = M_0/VM_1 \oplus M_1/VM_0$. The determinant condition is equivalent to the condition that VM_1 is of index r in M_0 and VM_0 is of index s in M_1. Since $FV = VF = p\,\mathrm{id}_M$, we obtain that pM_1 is of index r in FM_0 and pM_0 is of index s in FM_1. ■

Let $\mathbb{M}_k = \mathbb{M}_{k,0} \oplus \mathbb{M}_{k,1}$ be the Dieudonné module of $\mathbb{X}_k$. For a Dieudonné lattice $M \in \mathcal{N}(k)$, denote by $\mathrm{vol}(M) = [\mathbb{M}_k \colon M]$ the volume of M.

Lemma 1.5 *Let $M \in \mathcal{N}(k)$. If $M = p^i M^{\perp}$ for some integer i, then* $\mathrm{vol}(M) = ni$.

Proof Since both vector spaces $N_{k,0}$ and $N_{k,1}$ are maximal totally isotropic with respect to $\langle \cdot , \cdot \rangle$, the condition $M = p^i M^{\perp}$ is equivalent to the two conditions

$$p^i M_0^{\perp} = M_1, \quad p^i M_1^{\perp} = M_0.$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

By duality, the last two conditions are equivalent. We obtain

$$\begin{aligned}\mathrm{vol}(M) &= [\mathbb{M}_{k,0}:M_0] + [\mathbb{M}_{k,1}:M_1] = [M_0^{\perp}:\mathbb{M}_{k,0}^{\perp}] + [\mathbb{M}_{k,1}:M_1]\\ &= [p^{-i}M_1:\mathbb{M}_{k,1}] + [\mathbb{M}_{k,1}:M_1] = ni,\end{aligned}$$

which proves the claim. ■

Let $M \in \mathcal{N}(k)$ and let (X,ρ) be the corresponding p-divisible group and quasi-isogeny. Denote the height of ρ by $\mathrm{ht}(\rho)$. Then $\mathrm{ht}(\rho) = \mathrm{vol}(M)$. As the height of a quasi-isogeny of p-divisible groups over S is locally constant over S, we obtain by Lemma 1.5 a morphism

$$\begin{aligned}\kappa : \mathcal{N} &\to \mathbb{Z},\\ (X,\rho) &\mapsto \mathrm{ht}(\rho)/n.\end{aligned}$$

Remark 1.6 For $i \in \mathbb{Z}$ the fibre $\mathcal{N}_i = \kappa^{-1}(i)$ is the open and closed formal subscheme of $\mathcal{N}$ of quasi-isogenies of height ni.

Lemma 1.7 *If ni is odd, the formal scheme $\mathcal{N}_i$ is empty.*

Proof Let M be an element of $\mathcal{N}(k)$. Since both $\mathbb{M}_k$ and M satisfy the determinant condition of signature (r,s), we obtain by Lemma 1.4 that

$$\begin{aligned}\mathrm{vol}(M) &= [\mathbb{M}_{k,0}:M_0] + [\mathbb{M}_{k,1}:M_1] = [\mathbb{M}_{k,0}:M_0] + [F\mathbb{M}_{k,1}:FM_1]\\ &= 2[\mathbb{M}_{k,0}:M_0] + [F\mathbb{M}_{k,1}:\mathbb{M}_{k,0}] + [M_0:FM_1] = 2[\mathbb{M}_{k,0}:M_0].\end{aligned}$$

Hence $\mathrm{vol}(M)$ is even, *i.e.*, ni is even. ■

Let $N_k = N_{k,0} \oplus N_{k,1}$ be as in (1.5). To describe the set $\mathcal{N}(k)$, it is convenient to express $\mathcal{N}(k)$ in terms of $N_{k,0}$. Let τ be the σ^2-linear operator $V^{-1}F$ on N_k. Then $N_{k,0}$ and $N_{k,1}$ are both τ-invariant. Denote by $\mathbb{Q}_{p^2}$ the unique unramified extension of degree 2 of $\mathbb{Q}_p$ in $W_{\mathbb{Q}}$ and denote by $\mathbb{Z}_{p^2}$ its ring of integers. Since the isocrystal N_k is supersingular, (N_k,τ) is isoclinic with slope zero. Thus $(N_{k,i},\tau)$ is isoclinic of slope zero for $i = 0,1$, i.e., there exists a τ-invariant lattice in $N_{k,i}$. For every τ-invariant lattice $M_i \subset N_{k,i}$, there exists a τ-invariant basis of M_i [Zi1, §6.26]. Let C be the $\mathbb{Q}_{p^2}$-vector space of all τ-invariant elements of $N_{k,0}$ and let M_0^{τ} be the $\mathbb{Z}_{p^2}$-module of τ-invariant elements of M_0.

Remark 1.8 We obtain $M_0 = M_0^{\tau} \otimes_{\mathbb{Z}_{p^2}} W(k)$, the set of τ-invariants M_0^{τ} is a lattice in C and $N_{k,0} = C \otimes_{\mathbb{Q}_{p^2}} W(k)_{\mathbb{Q}}$. Note that the $\mathbb{Q}_{p^2}$-vector space C does not depend on k.

We write C_k for the base change $C \otimes_{\mathbb{Q}_{p^2}} W(k)_{\mathbb{Q}}$.

Definition 1.9 We define a new form on C_k by $\{x,y\} := \langle x, Fy\rangle$.

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

This is a perfect form on C_k linear in the first and σ-linear in the second variable. By (1.2) we obtain the following property of $\{\,\cdot\,,\,\cdot\,\}$

$$\{x, y\} = -\{y, \tau^{-1}(x)\}^{\sigma}, \tag{1.8}$$

which in turn implies

$$\{\tau(x), \tau(y)\} = \{x, y\}^{\sigma^2}. \tag{1.9}$$

For a $W(k)$-lattice L in C_k, denote by $L^\vee$ the dual of L with respect to the form $\{\cdot,\cdot\}$ defined by $L^\vee = \{y \in C_k \mid \{y, L\} \subset W(k)\}$. We obtain by (1.8) that

$$(L^\vee)^\vee = \tau(L). \tag{1.10}$$

In particular, taking the dual is not an involution on the set of lattices in C_k. The identity (1.9) implies that

$$\tau(L^\vee) = \tau(L)^\vee. \tag{1.11}$$

The form $\{\,\cdot\,,\,\cdot\,\}$ on C_k induces by restriction to C a perfect skew-hermitian form on C with respect to $\mathbb{Q}_{p^2}/\mathbb{Q}_p$ which we will again denote by $\{\,\cdot\,,\,\cdot\,\}$. For the perfect form to be skew-hermitian means that it is linear in the first and σ-linear in the second variable and we have $\{x, y\} = -\{y, x\}^\sigma$, where the Frobenius σ is an involution on $\mathbb{Q}_{p^2}$.

It is clear that for each τ-invariant lattice A of C_k we obtain $(A^\vee)^\tau = (A^\tau)^\vee$, where the second dual is taken with respect to the skew-hermitian form $\{\,\cdot\,,\,\cdot\,\}$ on C.

Proposition 1.10 *There is a bijection between* $\mathcal{N}(k)$ *and*

$$\mathcal{D}(C)(k) = \{A \subset C_k \mid A \textit{ is a lattice and } p^{i+1}A^\vee \overset{r}{\subset} A \overset{s}{\subset} p^iA^\vee \textit{ for some } i \in \mathbb{Z}\}.$$

The bijection is obtained by associating with $M = M_0 \oplus M_1 \in \mathcal{N}(k)$ *the lattice* M_0 *in* C_k.

Remark 1.11 Note that by duality and (1.10) the chain condition

$$p^{i+1}A^\vee \overset{r}{\subset} A \overset{s}{\subset} p^iA^\vee$$

is equivalent to the chain condition $p^{-i}\tau(A) \overset{s}{\subset} A^\vee \overset{r}{\subset} p^{-i-1}\tau(A)$, which is equivalent to $p^{i+1}A^\vee \overset{r}{\subset} \tau(A) \overset{s}{\subset} p^iA^\vee$.

Definition 1.12 An element $M \in \mathcal{N}(k)$ is called *superspecial* if $F(M) = V(M)$, *i.e.*, if M is τ-invariant. A lattice $A \subset C$ is superspecial if and only if A is τ-invariant.

Note that M is superspecial if and only if the corresponding lattice $A = M_0$ is superspecial.

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Proof of Proposition 1.10 Let $M = M_0 \oplus M_1$ be an $\mathcal{O}_E$-invariant lattice that is stable under F and V.

Claim $M = p^i M^{\perp}$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle$ if and only if $FM_1 = p^{i+1}M_0^{\vee}$.

Indeed, the lattice M is equal to $p^i M^{\perp}$ if and only if $p^i M_0^{\perp} = M_1$ (proof of Lemma 1.5). We have

$$\begin{aligned} F(M_0^{\perp}) &= \{y \in N_{k,0} \mid \langle F^{-1}y, M_0\rangle \subset W(k)\} \\ &= \{y \in N_{k,0} \mid \{p^{-1}y, M_0\} \subset W(k)\} = pM_0^{\vee} \end{aligned}$$

which proves the claim.

Let M be an element of the set $\mathcal{N}(k)$. Since FM_1 is equal to $p^{i+1}M_0^{\vee}$ for some $i \in \mathbb{Z}$, we obtain from (1.6) with $M_0 = A$ the chain condition $p^{i+1}A^{\vee} \overset{r}{\subset} A \overset{s}{\subset} p^i A^{\vee}$. Hence A is an element of $\mathcal{D}(C)(k)$. Conversely, associate with a lattice A of $\mathcal{D}(C)(k)$ the lattice $A \oplus F^{-1}(p^{i+1}A^{\vee}) \subset N_{k,0} \oplus N_{k,1}$. It is an element of $\mathcal{N}(k)$ by the same arguments. ■

Lemma 1.13 *Let $t \in \mathbb{Z}_{p^2}^{\times}$ with $t^{\sigma} = -t$ and let V be a $\mathbb{Q}_{p^2}$-vector space of dimension n. Let I_n be the identity matrix of rank n and let J_n be the matrix*

$$J_n = \begin{pmatrix} p & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

There exist two perfect skew-hermitian forms on V up to isomorphism. These forms correspond to tI_n and to tJ_n, respectively. Furthermore, if M is a lattice in V and $i \in \mathbb{Z}$ with

$$p^{i+1}M^{\vee} \overset{r}{\subset} M \overset{s}{\subset} p^i M^{\vee}, \tag{1.12}$$

then $s \equiv ni \bmod 2$ in the first case and $s \not\equiv ni \bmod 2$ in the second case. In particular, the form $t\begin{pmatrix} I_r & \\ & pI_s \end{pmatrix}$ is isomorphic to the form tI_n if s is even and is isomorphic to tJ_n if s is odd.

Proof Let $\{\,\cdot\,,\,\cdot\,\}$ be a perfect skew-hermitian form on V. Let U be the unitary group over $\mathbb{Q}_p$ associated with the pair $(V, \{\,\cdot\,,\,\cdot\,\})$. As $H^1(\mathbb{Q}_p, U) \cong \mathbb{Z}/2\mathbb{Z}$ [K1, §6], there exist two non isomorphic skew-hermitian forms on V. Choose a basis of V.

Claim The skew-hermitian forms on V corresponding to the matrix tI_n and to the matrix tJ_n are not isomorphic.

Indeed, assume that these forms are isomorphic. Then there exists a lattice M in V with $M = M^{\vee}$ and a lattice L in V with

$$pL^{\vee} \overset{n-1}{\subset} L \overset{1}{\subset} L^{\vee}.$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Let k be an integer such that $L^\vee$ is contained in p^kM. We obtain $p^{-k}M \overset{l}{\subset} L \overset{1}{\subset} L^\vee \overset{l}{\subset} p^kM$. Thus $2kn = 2l + 1$, which is a contradiction. Therefore, the two forms are not isomorphic.

Let M be a lattice in V which satisfies (1.12). A similar index argument as above shows that $s \equiv ni \bmod 2$ if the skew-hermitian form is isomorphic to tI_n and $s \not\equiv ni \bmod 2$ in the other case. ■

Lemma 1.14 *Let V be a $\mathbb{Q}_{p^2}$-vector space of dimension n and let $\{\cdot\,,\cdot\}$ be a perfect skew-hermitian form on V. Let $M \subset V$ be a $\mathbb{Z}_{p^2}$-lattice with*

$$pM^\vee \overset{r}{\subset} M \overset{s}{\subset} M^\vee.$$

Then there exists a basis of M such that the form $\{\cdot\,,\cdot\}$ with respect to this basis is given by the matrix $t\begin{pmatrix} I_r & \\ & pI_s \end{pmatrix}$.

Proof The lemma is proved by an analogue of the Gram–Schmidt orthogonalization. ■

Proposition 1.15 *There exists a basis of C such that the form $\{\cdot\,,\cdot\}$ with respect to this basis is given by the matrix (tI_n) if s is even and by $(t\,J_n)$ if s is odd.*

In particular, the isocrystal N with $\mathcal{O}_E$-action and perfect form $\langle\cdot\,,\cdot\rangle$ as in (1.1) *is uniquely determined up to isomorphism.*

Proof The image of the Dieudonné lattice $\mathbb{M}_k \in \mathcal{N}(k)$ under the bijection of Proposition 1.10 satisfies $p\mathbb{M}_{k,0}^\vee \overset{r}{\subset} \mathbb{M}_{k,0} \overset{s}{\subset} \mathbb{M}_{k,0}^\vee$. Thus the proposition follows from Lemma 1.14. ■

Let J be the group of isomorphisms of the isocrystal N with additional structure, i.e.,

$$(1.13)\quad J = \Big\{ g \in \mathrm{GL}_{E\otimes_{\mathbb{Q}_p} W(\overline{\mathbb{F}}_p)_\mathbb{Q}}(N) \;\Big|\; Fg = gF;\ \langle gx, gy\rangle = c(g)\langle x, y\rangle \text{ with } c(g) \in (W(\overline{\mathbb{F}}_p)_\mathbb{Q})^\times \Big\}.$$

Then J is the group $\mathrm{End}^\circ_{\mathcal{O}_E,\lambda}(\mathbb{X})^\times$ of $\mathcal{O}_E$-linear quasi-isogenies ρ of the p-divisible group $\mathbb{X}$ of Definition 1.1 such that $\lambda \circ \rho$ is a $\mathbb{Q}_p^\times$-multiple of $\rho^\vee \circ \lambda$.

An element $g \in J$ acts on $\mathcal{N}$ by sending $(X, \iota_X, \lambda_X, \rho)$ to $(X, \iota_X, \lambda_X, g\circ\rho)$. Consider the decomposition $N = N_0 \oplus N_1$ of the isocrystal N as in (1.5).

Remark 1.16 (i) The group J can be identified with the group of unitary similitudes of the hermitian space $(C, \{\cdot\,,\cdot\})$. Indeed, let $g \in J$. As g is $\mathcal{O}_E$-linear, it respects the grading of N. Since g commutes with F, the action of g on N is uniquely determined by its action on N_0 and we obtain $\{gx, gy\} = c(g)\{x, y\}$ for all elements $x, y \in N_0$. As g commutes with τ, it is defined over $\mathbb{Q}_{p^2}$, *i.e.*, an automorphism of $N_0^\tau = C$. In particular, $c(g) \in \mathbb{Q}_{p^2}$.

(ii) Let v_p be the p-adic valuation on $\mathbb{Q}_{p^2}$. It defines a morphism $\theta\colon J \to \mathbb{Z}$ by sending an element $g \in J$ to $v_p(c(g))$. Denote by J^0 the kernel of θ.

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

(iii) For $\rho \in \mathrm{End}^{\circ}_{\mathcal{O}_E,\lambda}(\mathbb{X})^{\times}$, let $g \in J$ be the corresponding automorphism of the isocrystal and let $\alpha = v_p(c(g))$. We obtain $g\mathbb{M} = c(g)(g\mathbb{M})^{\perp}$, hence by Lemma 1.5 the height of ρ is equal to $n\alpha$. By Remark 1.6 the element g defines an isomorphism of $\mathcal{N}_i$ with $\mathcal{N}_{i+\alpha}$ for every integer i.

Lemma 1.17 *The image of θ is equal to $\mathbb{Z}$ if n is even and equal to $2\mathbb{Z}$ if n is odd. In particular, there exists a quasi-isogeny $\rho \in \mathrm{End}_{\mathcal{O}_E,\lambda}(\mathbb{X})$ of height $h \in \mathbb{Z}$ if and only if h is divisible by n if n is even and if h is divisible by $2n$ if n is odd.*

Proof For $g = p\,\mathrm{id}_N$ we have $c(g) = p^2$. If n is odd, the same argument as in Lemma 1.7 shows that the image of θ is contained in $2\mathbb{Z}$. Thus we may assume that n is even. It is sufficient to show that there exists an element $g \in \mathrm{GL}(C)$ such that $\{gx, gy\} = p\{x, y\}$ for all $x, y \in C$.

Let T_1 and T_2 be the matrices in $\mathrm{GL}(C)$ given by

$$T_1 = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \qquad T_2 = \begin{pmatrix} & & & p \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}.$$

By Lemma 1.13 the perfect skew-hermitian form on C induced by tT_1 is isomorphic to tI_n and the perfect skew-hermitian form induced by tT_2 is isomorphic to tJ_n. We set $g = \mathrm{diag}(p^{n/2}, 1^{n/2})$. Then g satisfies the claim. ∎

For $i \in \mathbb{Z}$ we denote by $\mathcal{D}_i(C)(k)$ the image of $\mathcal{N}_i(k)$ (see Remark 1.6) under the bijection of Proposition 1.10, *i.e.*,

$$\mathcal{D}_i(C)(k) = \{A \in \mathcal{D}(C)(k) \mid p^{i+1}A^{\vee} \overset{r}{\subset} A \overset{s}{\subset} p^i A^{\vee}\}. \tag{1.14}$$

We have a decomposition of $\mathcal{D}(C)(k)$ into a disjoint union of the sets $\mathcal{D}_i(C)(k)$,

$$\mathcal{D}(C)(k) = \biguplus_{i\in\mathbb{Z}} \mathcal{D}_i(C)(k).$$

The sets $\mathcal{D}_i(C)(k)$ are invariant under the action of τ on $\mathcal{D}(C)(k)$. By Lemma 1.7 we know that $\mathcal{D}_i(C)(k)$ is empty if ni is odd.

Proposition 1.18 *Let i be an integer such that ni is even. If n is even, let g be an element of J such that $v_p(c(g)) = -1$ (Lemma 1.17). There exists an isomorphism*

$$\Psi_i \colon \mathcal{N}_i \xrightarrow{\sim} \mathcal{N}_0 \tag{1.15}$$

such that the following holds. If i is even, Ψ_i induces on k-rational points the bijection

$$\Psi_i \colon \mathcal{D}_i(C)(k) \xrightarrow{\sim} \mathcal{D}_0(C)(k),$$
$$A \longmapsto p^{-\frac{i}{2}}A,$$

and if i is odd, Ψ_i *induces the bijection*

$$\Psi_i \colon \mathcal{D}_i(C)(k) \xrightarrow{\sim} \mathcal{D}_0(C)(k),$$
$$A \longmapsto p^{\frac{-i+1}{2}} g(A).$$

Proof If i is even, the multiplication $p^{-i/2} \colon \mathbb{X} \to \mathbb{X}$ defines an isomorphism of $\mathcal{N}_i$ with $\mathcal{N}_0$ that satisfies the claim. If i is odd, the quasi-isogeny $p^{(-i+1)/2} g$ induces an isomorphism of $\mathcal{N}_i$ with $\mathcal{N}_0$ that satisfies the claim. ■

2 The Set Structure of $\mathcal{N}$ for GU$(1, s)$

From now on, we will restrict ourselves to the case of GU$(1, s)$. Our goal is to describe the irreducible components of $\mathcal{N}_i$ for any integer i with ni even. In this section we will define irreducible varieties over $\overline{\mathbb{F}}_p$ of dimension equal to the dimension of $\mathcal{N}_i$ such that the k-rational points of these varieties cover $\mathcal{N}_i(k)$ for every algebraically closed field extension k of $\overline{\mathbb{F}}_p$.

We will always denote by k an algebraically closed field extension of $\overline{\mathbb{F}}_p$. Let C_k be the $W(k)_{\mathbb{Q}}$-vector space of dimension $n = s + 1$ with σ^2-linear operator τ as in Remark 1.8. By Definition 1.9 the vector space C_k is equipped with a perfect form $\{\,\cdot\,,\,\cdot\,\}$, linear in the first variable and σ-linear in the second variable, that satisfies $\{x, y\} = -\{y, \tau^{-1}(x)\}^{\sigma}$ for all $x, y \in C_k$. By Proposition 1.15 the restriction of $\{\,\cdot\,,\,\cdot\,\}$ to C is a perfect skew-hermitian form equivalent to the skew-hermitian form induced by tI_n if n is odd and equivalent to tJ_n if n is even.

Let $i \in \mathbb{Z}$. The set $\mathcal{D}_i(C)(k)$ as in (1.14) consists of all lattices A in C_k such that

$$p^{i+1}A^{\vee} \overset{1}{\subset} A \overset{n-1}{\subset} p^{i}A^{\vee}. \tag{2.1}$$

or equivalently,

$$p^{i+1}A^{\vee} \overset{1}{\subset} \tau(A) \overset{n-1}{\subset} p^{i}A^{\vee}. \tag{2.2}$$

By Lemma 1.7 we know that $\mathcal{D}_i(C)(k)$ is empty if ni is odd. If ni is even, the set $\mathcal{D}_i(C)(k)$ is nonempty by Proposition 1.18. Therefore, we will always assume that ni is even.

We need the following crucial lemma.

Lemma 2.1 *Let A be a lattice in* $\mathcal{D}_i(C)(k)$. *There exists an integer d with* $0 \leq d \leq s/2$ *and such that*

$$\Lambda = A + \tau(A) + \cdots + \tau^{d}(A)$$

is a τ*-invariant lattice. If d is minimal with this condition, we have*

$$p^{i+1}\Lambda^{\vee} \subset p^{i+1}A^{\vee} \overset{1}{\subset} A \overset{d}{\subset} \Lambda \overset{n-2d-1}{\subset} p^{i}\Lambda^{\vee} \subset p^{i}A^{\vee}$$

and $p^{i+1}\Lambda^{\vee}$ *is of index* $(2d + 1)$ *in* Λ.

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

The proof of Lemma 2.1 will use explicitly that the index of $p^{i+1}A^\vee$ in A is equal to 1. It does not work in the case of general index.

Proof For every nonnegative integer j, denote by T_j the lattice

$$T_j = A + \tau(A) + \cdots + \tau^j(A). \tag{2.3}$$

We have $T_{j+1} = T_j + \tau(T_j)$. Let $d \geq 0$ be minimal with $T_d = \tau(T_d)$, *i.e.*, $\tau(T_j) \neq T_j$ for every $0 \leq j < d$. Such an integer exists [RZ, Proposition 2.17]. If $d = 0$, there is nothing to prove, hence we may assume that $d \geq 1$.

Claim If $d \geq 2$, then for $2 \leq j \leq d$

$$\tau(T_{j-2}) \overset{1}{\subset} T_{j-1} \overset{1}{\subset} T_j \quad \text{and} \quad \tau(T_{j-2}) \overset{1}{\subset} \tau(T_{j-1}) \overset{1}{\subset} T_j. \tag{2.4}$$

In particular, $A = T_0$ is of index j in T_j.

Indeed, if $j = 2$ we obtain $p^{i+1}A^\vee \overset{1}{\subset} A$ and $p^{i+1}A^\vee \overset{1}{\subset} \tau(A)$ by (2.1) and (2.2). Hence either $A = \tau(A)$ or A and $\tau(A)$ are both of index 1 in $T_1 = A + \tau(A)$. Since $d \geq 1$, the second possibility occurs. As A is of index 1 in T_1, the lattice $\tau(A)$ is of index 1 in $\tau(T_1)$. We obtain that either $T_1 = \tau(T_1)$ or that T_1 and $\tau(T_1)$ are both of index 1 in $T_2 = T_1 + \tau(T_1)$. Since $d \geq 2$, the second possibility occurs, which proves the claim for $j = 2$. The general case follows by induction on j.

We will now show that T_d is contained in $T_d^\vee$. By (2.1) and (2.2), we have

$$A + \tau(A) \subset p^i A^\vee \subset p^{-1}\tau(A), \quad \tau(A) + \tau^2(A) \subset p^i \tau(A)^\vee \subset p^{-1}\tau(A),$$

hence

$$A + \tau(A) + \tau^2(A) \subset p^{-1}\tau(A). \tag{2.5}$$

Using (2.5) we obtain

$$T_d = A + \cdots + \tau^d(A) \subset p^{-1}\tau(A) + \cdots + p^{-1}\tau^{d-1}(A) \subset p^{-1}T_{d-1}.$$

Since T_d is τ-invariant, T_d is contained in $p^{-1}\tau^l(T_{d-1})$ for every integer l, thus

$$T_d \subset p^{-1} \bigcap_{l \in \mathbb{Z}} \tau^l(T_{d-1}).$$

Now

$$\bigcap_{l \in \mathbb{Z}} \tau^l(T_{d-1}) = \bigcap_{l \in \mathbb{Z}} \tau^l(A). \tag{2.6}$$

Indeed, this is clear if $d = 1$, since $T_0 = A$. If $d \geq 2$, we obtain by (2.4) that $T_{d-1} \cap \tau(T_{d-1}) = \tau(T_{d-2})$, hence

$$\bigcap_{l \in \mathbb{Z}} \tau^l(T_{d-1}) = \bigcap_{l \in \mathbb{Z}} \tau^l(T_{d-2}),$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

and (2.6) follows by induction. Since $d \geq 1$, we have $A \neq \tau(A)$, and hence $A \cap \tau(A) = p^{i+1}A^\vee$ by (2.1) and (2.2). We obtain

$$T_d \subset p^i \bigcap_{l \in \mathbb{Z}} \tau^l(A)^\vee. \tag{2.7}$$

Dualizing (2.3) for $j = d$ shows that $T_d^\vee = A^\vee \cap \cdots \cap \tau^d(A^\vee)$, hence by (2.7)

$$T_d \subset p^i \bigcap_{l \in \mathbb{Z}} \tau^l(A)^\vee = p^i \bigcap_{l \in \mathbb{Z}} \tau^l(T_d)^\vee = p^i T_d^\vee.$$

The last equality is satisfied because T_d, and hence $T_d^\vee$, are τ-invariant. We obtain

$$p^{i+1}T_d^\vee \subset p^{i+1}A^\vee \overset{1}{\subset} A \subset T_d \subset p^i T_d^\vee \subset p^i A^\vee. \tag{2.8}$$

Using (2.8) and $A \overset{s}{\subset} p^i A^\vee$ we see that $d \leq s/2$, which proves the claim. ■

Definition 2.2 For $i \in \mathbb{Z}$ with ni even, let $\mathcal{L}_i$ be the set of all lattices Λ in C satisfying

$$p^{i+1}\Lambda^\vee \subsetneqq \Lambda \subset p^i \Lambda^\vee. \tag{2.9}$$

Let $\mathcal{L} = \biguplus_{i \in \mathbb{Z}} \mathcal{L}_i$ be the disjoint union of the sets $\mathcal{L}_i$. We say that $\Lambda \in \mathcal{L}_i$ is of type l, if $p^{i+1}\Lambda^\vee$ is of index l in Λ. Denote by $\mathcal{L}_i^{(l)}$ the set of all lattices of type l in $\mathcal{L}_i$. For $\Lambda \in \mathcal{L}_i$ let $\Lambda_k = \Lambda \otimes_{\mathbb{Z}_{p^2}} W(k)$. We define

$$\mathcal{V}(\Lambda)(k) = \{A \subset \Lambda_k \mid p^{i+1}A^\vee \overset{1}{\subset} A\}.$$

Remark 2.3 (i) Let $\Lambda \in \mathcal{L}_i$. The type l of Λ is always an odd integer with $1 \leq l \leq n$. Indeed, it is clear that $1 \leq l \leq n$. Since ni is even, the integer $n - l$ is even if and only if n is odd (Lemma 1.13).

(ii) There is a bijection between $\mathcal{L}_i$ and the set of all τ-invariant lattices in C_k satisfying (2.9). Via this bijection the superspecial lattices in $\mathcal{D}_i(C)(k)$ correspond to the lattices $\Lambda \in \mathcal{L}_i$ of type 1.

(iii) Let $\Lambda \in \mathcal{L}_i$. By duality a lattice A in $\mathcal{V}(\Lambda)(k)$ satisfies

$$p^{i+1}\Lambda_k \subset p^{i+1}\Lambda_k^\vee \subset p^{i+1}A^\vee \subset A \subset \Lambda_k. \tag{2.10}$$

(iv) For every i the isomorphism Ψ_i of Proposition 1.18 induces an isomorphism Ψ_i of $\mathcal{L}_i^{(l)}$ with $\mathcal{L}_0^{(l)}$ such that $\mathcal{V}(\Psi_i(\Lambda))(k) = \Psi_i(\mathcal{V}(\Lambda)(k))$.

The sets $\mathcal{V}(\Lambda)(k)$ will be identified with the k-rational points of an irreducible smooth variety. In Section 5 we will prove that for GU$(1, 2)$ the varieties corresponding to lattices Λ of maximal type are isomorphic to the irreducible components of $\mathcal{N}$. We will start with some basic properties of $\mathcal{V}(\Lambda)(k)$.

Proposition 2.4 (i) *We have* $\mathcal{D}_i(C)(k) = \bigcup_{\Lambda \in \mathcal{L}_i} \mathcal{V}(\Lambda)(k)$. *In particular,* $\mathcal{L}_i \neq \varnothing$.
(ii) *For* $\Lambda \in \mathcal{L}_i$ *and* $\Lambda' \in \mathcal{L}_j$ *with* $i \neq j$, *we have* $\mathcal{V}(\Lambda)(k) \cap \mathcal{V}(\Lambda')(k) = \varnothing$.

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

(iii) *Let* Λ *and* Λ' *be elements of* $\mathcal{L}_i$.

(a) *If* $\Lambda \subset \Lambda'$, *then* $\mathcal{V}(\Lambda)(k) \subset \mathcal{V}(\Lambda')(k)$.

(b) *We have*

$$\mathcal{V}(\Lambda)(k) \cap \mathcal{V}(\Lambda')(k) = \begin{cases} \mathcal{V}(\Lambda \cap \Lambda')(k) & \text{if } \Lambda \cap \Lambda' \in \mathcal{L}_i, \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof Statements (i), (ii), and (iii)(a) follow from the definition and Lemma 2.1.

To prove (iii)(b), let A be an element of $\mathcal{V}(\Lambda)(k) \cap \mathcal{V}(\Lambda')(k)$. By (2.10) we have

$$p^{i+1}\Lambda^\vee \subset p^{i+1}A^\vee \subsetneqq A \subset \Lambda_k \subset p^i\Lambda_k^\vee \subset p^iA^\vee$$

and similarly for Λ_k' instead of Λ_k. We obtain

$$\begin{aligned} p^{i+1}(\Lambda_k \cap \Lambda_k')^\vee \subset p^{i+1}A^\vee \subsetneqq A \subset (\Lambda_k \cap \Lambda_k') \subset \Lambda_k \\ \subset p^i\Lambda_k^\vee \subset p^i(\Lambda_k \cap \Lambda_k')^\vee \subset p^iA^\vee, \end{aligned}$$

hence $\Lambda \cap \Lambda' \in \mathcal{L}_i$. A similar calculation shows the equality

$$\mathcal{V}(\Lambda)(k) \cap \mathcal{V}(\Lambda')(k) = \mathcal{V}(\Lambda \cap \Lambda')(k). \qquad \blacksquare$$

Proposition 2.5 *Let* $\Lambda, \Lambda' \in \mathcal{L}_i$.

(i) $\mathcal{V}(\Lambda)(k)$ *contains a superspecial lattice. Furthermore,* $\#\mathcal{V}(\Lambda)(k) = 1$ *if and only if* Λ *is of type* 1. *In this case* $\mathcal{V}(\Lambda)(k) = \{\Lambda_k\}$.

(ii) $\mathcal{V}(\Lambda')(k) \subset \mathcal{V}(\Lambda)(k)$ *if and only if* $\Lambda' \subset \Lambda$. *In particular,* $\mathcal{V}(\Lambda')(k) = \mathcal{V}(\Lambda)(k)$ *if and only if* $\Lambda' = \Lambda$.

(iii) *Let* l *be the type of* Λ. *For every odd integer* $1 \le l' \le n$, *there exists a lattice* $\Lambda' \in \mathcal{L}_i$ *of type* l' *with* $\Lambda' \subset \Lambda$ *if* $l' \le l$ *and* $\Lambda \subset \Lambda'$ *if* $l \le l'$.

In particular, the maximal sets $\mathcal{V}(\Lambda)(k)$ *are the sets with* Λ *of type* n *if* n *is odd and of type* $n-1$ *if* n *is even.*

For the proof of Proposition 2.5, we first need a description of the sets $\mathcal{V}(\Lambda)(k)$ in terms of linear algebra. We first consider the case $i = 0$. Let $\Lambda \in \mathcal{L}_0$ be of type l. We associate with Λ the $\mathbb{F}_{p^2}$-vector spaces

(2.11) $$V = \Lambda/p\Lambda^\vee \quad \text{and} \quad V' = \Lambda^\vee/\Lambda.$$

By (2.9) the vector space V is of dimension l and V' is of dimension $n-l$. For $z \in \mathbb{Z}_{p^2}$ denote by $\bar{z}$ its image in $\mathbb{F}_{p^2}$. The skew-hermitian form $\{\cdot\,,\cdot\}$ on C induces a perfect skew-hermitian form $(\cdot\,,\cdot)$ on V by

(2.12) $$(\bar{x}, \bar{y}) = \overline{\{x, y\}} \in \mathbb{F}_{p^2}$$

for $\bar{x}, \bar{y} \in V$ and lifts $x, y \in \Lambda$. Similarly, we obtain a perfect skew-hermitian form on V' by

$$(\bar{x}, \bar{y})' = \overline{(p\{x, y\})} \in \mathbb{F}_{p^2}$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

for $\bar{x}, \bar{y} \in V'$ and lifts $x, y \in \Lambda^\vee$.

Let τ be the operator on $V_k = V \otimes_{\mathbb{F}_{p^2}} k$ defined by the Frobenius of k over $\mathbb{F}_{p^2}$. We denote again by $(\,\cdot\,,\,\cdot\,)$ the induced form on V_k given by

$$\begin{aligned} V_k \times V_k &\to k, \\ (v \otimes x, w \otimes y) &\mapsto xy^\sigma (v, w). \end{aligned} \tag{2.13}$$

This form is linear in the first variable and σ-linear in the second variable and satisfies

$$(x, y) = -(y, \tau^{-1}(x))^\sigma. \tag{2.14}$$

For a subspace U of V_k, we denote by $U^\perp$ the orthogonal complement

$$U^\perp = \{x \in V_k \mid (x, U) = 0\}.$$

By (2.14) we obtain

$$(U^\perp)^\perp = \tau(U) \quad \text{and} \quad \tau(U)^\perp = (\tau(U))^\perp \tag{2.15}$$

analogous to (1.11).

Remark 2.6 Let G be the unitary group associated with $(V, (\,\cdot\,,\,\cdot\,))$. Since $\mathrm{H}^1(\mathbb{F}_p, G) = 0$, there exists up to isomorphism only one skew-hermitian form on V, and similarly for V'.

Proposition 2.7 *There exists an inclusion preserving bijection*

$$\begin{aligned} &\{T \subset \Lambda_k \mid T \textit{ is a lattice and } pT^\vee \overset{j}{\subset} T \subset \Lambda_k\} \\ &\qquad \to \{U \subset V_k \mid U \textit{ is a } k\textit{-subspace and } \dim U = \frac{l+j}{2},\ U^\perp \subset U\}, \\ &T \mapsto \overline{T}, \end{aligned}$$

where $\overline{T}$ is equal to $T/p\Lambda_k^\vee$.

In particular, the τ-invariant lattices on the left-hand side correspond to the τ-invariant subspaces on the right-hand side.

Proof For a lattice T contained in the set on the left-hand side, we obtain from (2.9)

$$pT \subset p\Lambda_k \overset{n-l}{\subset} p\Lambda_k^\vee \subset pT^\vee \overset{j}{\subset} T \subset \Lambda_k. \tag{2.16}$$

Since $\overline{pT^\vee} = \overline{T}^\perp$, we obtain from (2.16) the inclusions $\{0\} \subset \overline{T}^\perp \overset{j}{\subset} \overline{T} \subset V_k$, which proves the claim. ■

Corollary 2.8 *Let $\Lambda \in \mathcal{L}_0^{(l)}$.*

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

(i) *The set of lattices* $\Lambda_1 \in \mathcal{L}_0^{(l_1)}$ *with* $\Lambda_1 \subset \Lambda$ *is equal to the set of* $\mathbb{F}_{p^2}$*-subspaces* $U \subset V$ *of dimension* $(l+l_1)/2$ *with* $U^\perp \subset U$. *In particular, superspecial points in* $\mathcal{V}(\Lambda)(k)$ *correspond to subspaces* $U \subset V$ *of dimension* $(l+1)/2$ *with* $U^\perp \subset U$.

(ii) *The lattices* $\Lambda_1 \in \mathcal{L}_0^{(l_1)}$ *with* $\Lambda \subset \Lambda_1$ *correspond to the* $\mathbb{F}_{p^2}$*-subspaces* $U \subset V'$ *of dimension* $n-(l+l_1)/2$ *with* $U^{\perp'} \subset U$.

Proof Part (i) follows from Proposition 2.7. The superspecial points are lattices in $\mathcal{L}_0$ of type 1.

To prove (ii), let Λ_1 be an element of $\mathcal{L}_0^{(l_1)}$ with $\Lambda \subset \Lambda_1$. We have

$$\Lambda \subset \Lambda_1 \overset{n-l_1}{\subset} \Lambda_1^\vee \subset \Lambda^\vee. \tag{2.17}$$

For a lattice $L \subset C$ the dual $L^{\vee'}$ with respect to $p\{\cdot,\cdot\}$ is equal to $p^{-1}L^\vee$. Thus (2.17) is equivalent to

$$\Lambda = p(\Lambda^\vee)^{\vee'} \subset p(\Lambda_1^\vee)^{\vee'} \overset{n-l_1}{\subset} \Lambda_1^\vee \subset \Lambda^\vee.$$

Now (ii) follows from Proposition 2.7 with V' instead of V and $p\{\cdot,\cdot\}$ instead of $\{\cdot,\cdot\}$, as the dimension of V' is equal to $n-l$. ∎

Proof of Proposition 2.5 We first prove the proposition in the case $i=0$.

Let Λ be an element of $\mathcal{L}_0$ of type l. Let V be as in (2.11). By Corollary 2.8 the superspecial points of $\mathcal{V}(\Lambda)(k)$ correspond to $\mathbb{F}_{p^2}$-subspaces $U \subset V$ of dimension $(l+1)/2$ with $U^\perp \subset U$. Such subspaces always exist, hence $\mathcal{V}(\Lambda)(k)$ always contains superspecial points. Furthermore, if $\dim V = l \geq 3$, there exists more than one subspace with these properties. This shows that $\mathcal{V}(\Lambda)(k)$ consists of only one element if and only if $l=1$, hence proves (i).

To prove (ii), let Λ' and Λ be two elements of $\mathcal{L}_0$ such that $\mathcal{V}(\Lambda')(k) \subset \mathcal{V}(\Lambda)(k)$. We want to prove that $\Lambda' \subset \Lambda$. First note that $\mathcal{V}(\Lambda')(k) = \mathcal{V}(\Lambda')(k) \cap \mathcal{V}(\Lambda)(k)$ is not empty. Hence by Proposition 2.4(iii)(b), we obtain $\mathcal{V}(\Lambda')(k) = \mathcal{V}(\Lambda \cap \Lambda')(k)$ and $\Lambda \cap \Lambda' \in \mathcal{L}_0$. Therefore, it is sufficient to prove that for $\Lambda' \subsetneqq \Lambda$, the set $\mathcal{V}(\Lambda')(k)$ is strictly contained in $\mathcal{V}(\Lambda)(k)$.

Let V be as in (2.11) and let V_1 be the subspace of V corresponding to Λ' (Corollary 2.8). Since $V_1 \subsetneqq V$, there exists a subspace $U \not\subseteq V_1$ of V with $U^\perp \overset{1}{\subset} U$. Thus there exists an element of $\mathcal{V}(\Lambda)(k) \setminus \mathcal{V}(\Lambda')(k)$ (Proposition 2.7).

Part (iii) follows from Corollary 2.8.

By Remark 2.3(iv) the case of arbitrary i follows from the case $i=0$. ∎

Let $\Lambda \in \mathcal{L}_0$ be of type l and let

$$d = \frac{l-1}{2}.$$

Let $V = \Lambda/p\Lambda^\vee$ as in (2.11). By Proposition 2.7 we have

$$\mathcal{V}(\Lambda)(k) = \{U \subset V_k \mid \dim U = d+1, U^\perp \subset U\}.$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Denote by $\mathrm{Grass}_{d+1}(V)$ the Grassmannian over $\mathbb{F}_{p^2}$ of $(d+1)$-dimensional subspaces of V. The set $\mathcal{V}(\Lambda)(k)$ can naturally be endowed with the structure of a closed subscheme of $\mathrm{Grass}_{d+1}(V)$. For every $\mathbb{F}_{p^2}$-algebra R, let V_R be the base change $V \otimes_{\mathbb{F}_{p^2}} R$. By abuse of notation, we denote again by σ the Frobenius on R. Let U be a locally direct summand of V_R of rank m. We define the dual module $U^{\perp} \subset V_R$ as follows. For an R-module M, let $M^{(p)} = M \otimes_{R,\sigma} R$ be the Frobenius twist of M and let $M^* = \mathrm{Hom}_R(M, R)$. Then $(\,\cdot\,,\,\cdot\,)$ induces an R-linear isomorphism

$$\phi\colon (V_R)^{(p)} \xrightarrow{\sim} (V_R)^*.$$

Thus $\phi(U^{(p)})$ is a locally direct summand of $(V_R)^*$ of rank m. Let ψ_U be the composition $V_R \xrightarrow{\sim} (V_R)^{**} \twoheadrightarrow \phi(U^{(p)})^*$. The orthogonal complement

$$U^{\perp} := \ker(\psi_U) \tag{2.18}$$

is a locally direct summand of V_R of rank $l - m$. Over an algebraically closed field, this definition coincides with the usual definition.

We denote by Y_Λ the closed subscheme of $\mathrm{Grass}_{d+1}(V)$ defined over $\mathbb{F}_{p^2}$ given by

$$Y_\Lambda(R) = \{U \subset V_R \text{ a locally direct summand} \mid \mathrm{rk}_R\, U = d+1,\ U^{\perp} \subset U\} \tag{2.19}$$

for every $\mathbb{F}_{p^2}$-algebra R.

Remark 2.9 (i) Note that for every algebraically closed field extension k of $\overline{\mathbb{F}}_p$ we obtain $Y_\Lambda(k) = \mathcal{V}(\Lambda)(k)$. By abuse of notation we will again denote by Y_Λ the corresponding scheme over $\overline{\mathbb{F}}_p$.

(ii) Since there exists only one skew-hermitian form on V up to isomorphism, Y_Λ depends up to isomorphism only on the dimension of V, *i.e.,* the type of Λ. We will show that Y_Λ is a smooth irreducible variety over $\mathbb{F}_{p^2}$.

(iii) Let $\Lambda \in \mathcal{L}_i^{(l)}$. By Remark 2.3(iv) the lattice $\Lambda' = \Psi_i(\Lambda)$ is contained in $\mathcal{L}_0^{(l)}$ and Ψ_i induces a bijection between $\mathcal{V}(\Lambda)(k)$ and $\mathcal{V}(\Lambda')(k)$. Thus the set $\mathcal{V}(\Lambda)(k)$ can be identified with the k-rational points of the variety $Y_{\Lambda'}$. We set $Y_\Lambda = Y_{\Lambda'}$.

For every $\mathbb{F}_p$-algebra R we define a form $\langle\,\cdot\,,\,\cdot\,\rangle$ on $V \otimes_{\mathbb{F}_p} R$ by

$$\begin{aligned}\langle\,\cdot\,,\,\cdot\,\rangle\colon (V \otimes_{\mathbb{F}_p} R) \times (V \otimes_{\mathbb{F}_p} R) &\to \mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} R,\\ (v \otimes a, w \otimes b) &\mapsto (v, w) \otimes ab,\end{aligned}$$

where $(\,\cdot\,,\,\cdot\,)$ is the skew-hermitian form defined in (2.12). The form $\langle\,\cdot\,,\,\cdot\,\rangle$ is linear in R in both components, whereas the form $(\,\cdot\,,\,\cdot\,)_R$, extended to $V \otimes_{\mathbb{F}_{p^2}} R$ analogously to (2.13), is linear in R in the first component and σ-linear in R in the second component. Let G be the unitary group over $\mathbb{F}_p$ of Remark 2.6. Then for every $\mathbb{F}_p$-algebra R the set $G(R)$ is defined as

$$G(R) = \{g \in \mathrm{GL}(V \otimes_{\mathbb{F}_p} R) \mid \langle gx, gy\rangle = \langle x, y\rangle \text{ for all } x, y \in V \otimes_{\mathbb{F}_p} R\}.$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Let T be the matrix

$$T = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \in \mathrm{GL}_l(\mathbb{F}_p)$$

and let $\bar{t}$ be an element of $\mathbb{F}_{p^2}^\times$ with $\bar{t}^p = -\bar{t}$.

Remark 2.10 As there exists only one perfect skew-hermitian form on V up to isomorphism, we can choose an $\mathbb{F}_{p^2}$-basis $e_1, \dots, e_l$ of V such that the form $(\cdot, \cdot)$ with respect to this basis is given by the matrix $\bar{t}T$. We identify V via this basis with $(\mathbb{F}_{p^2})^l$.

Lemma 2.11 *Let $G_{\overline{\mathbb{F}}_p}$ be the base change of G over $\overline{\mathbb{F}}_p$. Then $G_{\overline{\mathbb{F}}_p}$ is isomorphic to $\mathrm{GL}_{l,\overline{\mathbb{F}}_p}$ with Frobenius action given by $\Phi(h) = T^t(h^{(p)})^{-1}T$ for an element $h \in \mathrm{GL}_{l,\overline{\mathbb{F}}_p}$.*

Proof Let σ be the Frobenius of $\overline{\mathbb{F}}_p$. Furthermore, denote by $V_{\overline{\mathbb{F}}_p,\mathrm{id}}$ the vector space $V \otimes_{\mathbb{F}_{p^2}} \overline{\mathbb{F}}_p$ and denote by $V_{\overline{\mathbb{F}}_p,\sigma}$ the vector space $V \otimes_{\mathbb{F}_{p^2}} \overline{\mathbb{F}}_p$, where in the second case the morphism $\mathbb{F}_{p^2} \hookrightarrow \overline{\mathbb{F}}_p$ is given by the Frobenius σ. We obtain an isomorphism

$$V \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \xrightarrow{\sim} V_{\overline{\mathbb{F}}_p,\mathrm{id}} \oplus V_{\overline{\mathbb{F}}_p,\sigma},$$
$$v \otimes a \mapsto (v \otimes a, v \otimes a).$$

Both $V_{\overline{\mathbb{F}}_p,\mathrm{id}}$ and $V_{\overline{\mathbb{F}}_p,\sigma}$ are totally isotropic with respect to the form $\langle \cdot, \cdot \rangle$, and the form defines a perfect $\overline{\mathbb{F}}_p$-linear pairing between the two $\overline{\mathbb{F}}_p$-vektor spaces $V_{\overline{\mathbb{F}}_p,\mathrm{id}}$ and $V_{\overline{\mathbb{F}}_p,\sigma}$.

We now identify both $V_{\overline{\mathbb{F}}_p,\mathrm{id}}$ and $V_{\overline{\mathbb{F}}_p,\sigma}$ with $\overline{\mathbb{F}}_p^l$ using the $\mathbb{F}_{p^2}$-basis $e_1, \dots, e_l$ of V of Remark 2.10. Then each $g \in G(\overline{\mathbb{F}}_p)$ corresponds to a pair $(g_{\mathrm{id}}, g_\sigma) \in \mathrm{GL}_l(\overline{\mathbb{F}}_p) \times \mathrm{GL}_l(\overline{\mathbb{F}}_p)$ with

$$ {}^t g_{\mathrm{id}} T g_\sigma = T. \tag{2.20}$$

In particular, we obtain an isomorphism

$$\begin{aligned} G_{\overline{\mathbb{F}}_p} &\to \mathrm{GL}_{l,\overline{\mathbb{F}}_p}, \\ g &\mapsto g_{\mathrm{id}}. \end{aligned} \tag{2.21}$$

For an element $g \in G(\overline{\mathbb{F}}_p)$ the Frobenius Φ of G is defined as $(\mathrm{id}_V \otimes \sigma) \circ g \circ (\mathrm{id}_V \otimes \sigma)^{-1}$. It is easy to see that for the corresponding $(g_{\mathrm{id}}, g_\sigma)$ we obtain

$$\Phi(g_{\mathrm{id}}, g_\sigma) = (g_\sigma^{(p)}, g_{\mathrm{id}}^{(p)}),$$

where we denote for a matrix $h \in \mathrm{GL}_{l,\overline{\mathbb{F}}_p}$ by $h^{(p)}$ the matrix obtained by application of the Frobenius σ to every entry of h. Thus by (2.20) and (2.21) the Frobenius Φ

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

may be identified with the morphism

$$\begin{aligned} \mathrm{GL}_{l,\overline{\mathbb{F}}_p} &\to \mathrm{GL}_{l,\overline{\mathbb{F}}_p}, \\ h &\mapsto T^t(h^{(p)})^{-1}T, \end{aligned}$$

which proves the claim. ■

The diagonal torus S and the standard Borel B of upper triangular matrices in $G_{\overline{\mathbb{F}}_p}$ are defined over $\mathbb{F}_p$. Note that T corresponds to the longest Weyl group element of G with respect to S. Let $I = (i_1, \dots, i_c)$ be an ordered partition of l. Denote by P_I the standard parabolic subgroup of $G_{\overline{\mathbb{F}}_p}$ containing B such that $\mathrm{GL}_{i_1,\overline{\mathbb{F}}_p} \times \cdots \times \mathrm{GL}_{i_c,\overline{\mathbb{F}}_p}$ is the Levi subgroup of P containing S. We write $\Phi(I)$ for the partition $(i_c, \dots, i_1)$ and obtain $\Phi(P_{(i_1,\dots,i_c)}) = P_{\Phi(I)}$. Hence Φ induces a morphism $\Phi\colon G_{\overline{\mathbb{F}}_p}/P_I \to G_{\overline{\mathbb{F}}_p}/P_{\Phi(I)}$. To simplify notation, we write G instead of $G_{\overline{\mathbb{F}}_p}$. For a flag $\mathcal{F} \in (G/P_I)(R)$, the dual flag $\mathcal{F}^\perp$ with respect to $(\,\cdot\,,\,\cdot\,)$ is defined analogously to (2.18).

Lemma 2.12 *Let $\mathcal{F}$ be a flag in G/P_I. The Frobenius Φ and the duality morphism $\mathcal{F} \mapsto \mathcal{F}^\perp$ define the same morphism $G/P_I \to G/P_{\Phi(I)}$,* i.e., *the dual flag $\mathcal{F}^\perp$ is equal to $\Phi(\mathcal{F})$.*

Proof Let $\mathcal{F}$ be a flag in $G/P_I(R)$. Without loss of generality, we may assume that the constituents of $\mathcal{F}$ are free. Let $\mathcal{F}_{I,0}$ be the standard flag corresponding to the parabolic subgroup P_I. Then there exists an element $g \in G(R)$ such that $\mathcal{F} = g\mathcal{F}_{I,0}$. We have $\Phi(gP_I) = \Phi(g)P_{\Phi(I)}$, hence $\Phi(\mathcal{F}) = \Phi(g)\mathcal{F}_{\Phi(I),0} = \Phi(g)(\mathcal{F}_{I,0})^\perp$. Furthermore, $(\Phi(g)x, gy) = (x, y)$ for all $x, y \in R^l$, hence $\Phi(g)(\mathcal{F}_{I,0})^\perp = (g\mathcal{F}_{I,0})^\perp$, which proves the claim. ■

Proposition 2.13 *The closed subscheme Y_Λ of* $\mathrm{Grass}_{d+1}(V)$ *is smooth of dimension d.*

Proof It is sufficient to prove the claim after base change to $\overline{\mathbb{F}}_p$. By Lemma 2.12 it is clear that Y_Λ is the intersection of the graph of the Frobenius

$$\begin{aligned} G/P_{(d+1,d)} &\hookrightarrow G/P_{(d+1,d)} \times G/P_{(d,d+1)}, \\ g &\mapsto (g, \Phi(g)) \end{aligned}$$

with the morphism

$$\begin{aligned} G/P_{(d,1,d)} &\hookrightarrow G/P_{(d+1,d)} \times G/P_{(d,d+1)}, \\ g &\mapsto (g, g). \end{aligned}$$

It is easy to see that this intersection is transversal of dimension d, hence Y_Λ is smooth. ■

Our next goal is to relate Y_Λ with generalized Deligne–Lusztig varieties. Let S_l be the symmetric group in l elements and denote by W_I the Weyl group of the Levi subgroup $\mathrm{GL}_{i_1,\overline{\mathbb{F}}_p} \times \cdots \times \mathrm{GL}_{i_c,\overline{\mathbb{F}}_p}$ of P_I. We identify W_I with the corresponding subgroup

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

of S_l. Let

$$\mathcal{F} = [0 \subsetneqq \mathcal{F}_1 \subsetneqq \cdots \subsetneqq \mathcal{F}_c \subsetneqq V],$$
$$\mathcal{G} = [0 \subsetneqq \mathcal{G}_1 \subsetneqq \cdots \subsetneqq \mathcal{G}_c \subsetneqq V]$$

be two flags in $G/P_I(R)$. Then $\mathcal{F}$ and $\mathcal{G}$ are in standard position if all submodules $\mathcal{F}_i + \mathcal{G}_j$ are locally direct summands of R^l. Equivalently, the stabilizers of $\mathcal{F}$ and $\mathcal{G}$ contain a common maximal torus. For $\mathcal{F}$ and $\mathcal{G}$ in standard position, we recall the definition of the relative position $\operatorname{inv}(\mathcal{F}, \mathcal{G})$. Consider the diagonal action of G on $G/P_I \times G/P_I$. By the Bruhat decomposition, the orbits of this action are classified by the quotient $W_{P_I} \backslash S_l / W_{P_I}$. Then $\operatorname{inv}(\mathcal{F}, \mathcal{G})$ is defined as the element of the constant sheaf associated with $W_{P_I} \backslash S_l / W_{P_I}$ such that locally on R the element $(\mathcal{F}, \mathcal{G}) \in G/P_I \times G/P_I$ is contained in the corresponding orbit.

For a partition I with $I = \Phi(I)$ and an element $w \in W_{P_I} \backslash S_l / W_{P_I}$, we recall the generalized Deligne–Lusztig variety $X_{P_I}(w)$ over $\overline{\mathbb{F}}_p$,

$$X_{P_I}(w) = \{\mathcal{F} \in G/P_I \mid \operatorname{inv}(\Phi(\mathcal{F}), \mathcal{F}) = w\}.$$

We call this variety a generalized Deligne–Lusztig variety instead of a Deligne–Lusztig variety, as in [DL] this variety is only defined in the case of a Borel subgroup. As noted in [Lu2], this construction works for arbitrary parabolic subgroups defined over $\mathbb{F}_p$.

The variety $X_{P_I}(w)$ is the transversal intersection of the graph of the Frobenius with the orbit of $(1, w)$ in $G/P_I \times G/P_I$ under the diagonal action of G. Therefore, the variety $X_{P_I}(w)$ is smooth, and its dimension is equal to the dimension of the subvariety $P_I w P_I / P_I$ of G/P_I. In particular, if $P_I = B$, the dimension is equal to the length of w.

For $i = 0, \dots, d$ denote by $w_i = (d+1, \dots, d+i+1) \in S_l$ the cycle that maps $d+1$ to $d+2$, etc. and denote by I_i the partition $(d-i, 1^{2i+1}, d-i)$. To simplify notation, we write P_i and W_i instead of P_{I_i} and W_{P_i}.

We have $w_0 = \mathrm{id}$ and $I_0 = (d, 1, d)$. If $i = d$, we obtain

$$w_d = (d+1, d+2, \dots, l-1, l) \quad \text{and} \quad I_d = (1^l),$$

i.e., $P_d = B$. Note that $I_{d-1} = I_d = (1^l)$, but $w_{d-1} \neq w_d$.

Lemma 2.14 *For $0 \leq i \leq d$ the dimension of $X_{P_i}(w_i)$ is equal to i.*

Proof We have $\dim P_i w_i P_i / P_i = \dim P_i w_i P_i / B - \dim P_i / B$. Since I_i is equal to $(d-i, 1^{2i+1}, d-i)$, the Levi subgroup L of P_i containing S is isomorphic to

$$\mathrm{GL}_{d-i} \times \mathbb{G}_m^{2i+1} \times \mathrm{GL}_{d-i}.$$

Let B_{d-i} be the standard Borel of upper triangular matrices in GL_{d-i}. Then

$$\dim P_i/B = \dim L/(B \cap L) = 2 \dim \mathrm{GL}_{d-i} / B_{d-i} = (d-i-1)(d-i).$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

On the other hand, $P_i w_i P_i = BW_i w_i W_i B$. Let $w_i' \in S_l$ be the longest representative of the residue class of w_i in $W_i \backslash S_l / W_i$. Then the dimension of $P_i w_i P_i / B$ is equal to the length of w_i'.

Suppose first that $i = d$, *i.e.*, $P_d = B$. We obtain $W_d = 1$. Thus $w_d' = w_d$ and

$$\dim X_B(w_d) = \mathrm{length}(w_d) = d.$$

Now suppose $i < d$. Then

$$\begin{aligned} W_i &= \langle (12), \dots, (d-i-1, d-i), (d+i+2, d+i+3), \dots, (n-1, n) \rangle \\ &\cong S_{d-i} \times S_{d-i}. \end{aligned}$$

Since $w_i = (d+1, \dots, d+i+1)$, we obtain that w_i' is equal to the product of w_i with the longest element in W_i. Therefore, the length of w_i' is equal to $i + (n-i)(n-i-1)$, which proves the claim. ■

The next theorem establishes a link between Y_Λ and generalized Deligne–Lusztig varieties.

Theorem 2.15 *There exists a decomposition of Y_Λ over $\overline{\mathbb{F}}_p$ into a disjoint union of locally closed subvarieties*

$$Y_\Lambda = \biguplus_{i=0}^{d} X_{P_i}(w_i), \tag{2.22}$$

such that for every j with $0 \leq j \leq d$ the subset $\biguplus_{i=0}^{j} X_{P_i}(w_i)$ is closed in Y_Λ.

The variety $X_B(w_d)$ is open, dense and irreducible of dimension d in Y_Λ. In particular, Y_Λ is irreducible of dimension d.

Proof As in Remark 2.10 we identify V with $(\mathbb{F}_{p^2})^l$ such that the form $(\,\cdot\,,\,\cdot\,)$ is given by the matrix tT. For an algebraically closed field extension k of $\overline{\mathbb{F}}_p$ and a subspace $U \subset V_k$, we have $(U^\perp)^\perp = \tau(U)$ by (2.15). For an arbitrary $\overline{\mathbb{F}}_p$-algebra R, we do not have an operator τ on V_R, but to simplify notation, we write $\tau(U)$ for $(U^\perp)^\perp$ for all locally direct summands U of V_R.

Let U be an element of $Y_\Lambda(R)$. We may assume that $\operatorname{Spec} R$ is connected. The module U is a locally direct summand of R^l of rank $d+1$ such that $U^\perp \subset U$ with $U/U^\perp$ locally free of rank 1. Let $\mathcal{F}$ be the flag $\mathcal{F} = [0 \subsetneqq U^\perp \subsetneqq U \subsetneqq V]$. By Lemma 2.12 the flag $\Phi(\mathcal{F})$ is equal to $\Phi(\mathcal{F}) = [0 \subsetneqq U^\perp \subsetneqq \tau(U) \subsetneqq V]$. If $U + \tau(U)$ is a locally direct summand of R^l, *i.e.*, if $\mathcal{F}$ and $\Phi(\mathcal{F})$ are in standard position, we obtain that $\mathrm{inv}(\Phi(\mathcal{F}), \mathcal{F})$ is either the identity or w_1 in $W_0 \backslash S_l / W_0$. Hence we obtain a disjoint decomposition of $\mathcal{V}(\Lambda)$ into the open subvariety $X_{P_0}(w_1)$ and the closed subvariety $X_{P_0}(\mathrm{id})$:

$$Y_\Lambda = X_{P_0}(\mathrm{id}) \uplus X_{P_0}(w_1). \tag{2.23}$$

Let i be an integer with $0 \leq i \leq d-1$, and let $\mathcal{F}$ be a flag in $X_{P_i}(w_i) \uplus X_{P_i}(w_{i+1})$. Then $\mathcal{F}$ is of the form

$$\mathcal{F} = [0 \subset \mathcal{F}_{-i-1} \subsetneqq \cdots \subsetneqq \mathcal{F}_{-1} \subsetneqq \mathcal{F}_1 \subsetneqq \cdots \subsetneqq \mathcal{F}_{i+1} \subset V].$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

For an integer j with $1 \leq j \leq i+1$, the modules $\mathcal{F}_{-j}$ and $\mathcal{F}_j$ are locally direct summands of R^l of rank $(d-j+1)$ and $(d+j)$, respectively.

Claim: The flag $\mathcal{F}$ is uniquely determined by $\mathcal{F}_1$. Indeed, since $\Phi(\mathcal{F})$ and $\mathcal{F}$ are in relative position w_i or w_{i+1}, we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{F}^{\perp}_{i+1} \arrow[d, equal] & \subsetneqq \cdots \subsetneqq & \mathcal{F}^{\perp}_{1} \arrow[d, equal] & \subsetneqq & \mathcal{F}^{\perp}_{-1} \arrow[d, phantom, "\nparallel"] \arrow[drr, hook] & \subsetneqq & \mathcal{F}^{\perp}_{-2} \arrow[d, phantom, "\nparallel"] & \subsetneqq \cdots \subsetneqq & \mathcal{F}^{\perp}_{-i} \arrow[d, phantom, "\nparallel"] \arrow[drr, hook] & \subsetneqq & \mathcal{F}^{\perp}_{-i-1} \\
\mathcal{F}_{-i-1} & \subsetneqq \cdots \subsetneqq & \mathcal{F}_{-1} & \subsetneqq & \mathcal{F}_{1} & \subsetneqq & \mathcal{F}_{2} & \subsetneqq \cdots \subsetneqq & \mathcal{F}_{i} & \subsetneqq & \mathcal{F}_{i+1}.
\end{tikzcd}$$

Furthermore, we have

$$\begin{aligned}
\mathcal{F}^{\perp}_{-i-1} &= \mathcal{F}_{i+1} && \text{if } \operatorname{inv}(\Phi(\mathcal{F}), \mathcal{F}) = w_i, \\
\mathcal{F}^{\perp}_{-i-1} &\neq \mathcal{F}_{i+1} && \text{if } \operatorname{inv}(\Phi(\mathcal{F}), \mathcal{F}) = w_{i+1}.
\end{aligned}$$

As $\Phi(\mathcal{F})$ and $\mathcal{F}$ are in standard position, we obtain that for j with $2 \leq j \leq i+1$,

$$\mathcal{F}_j = \mathcal{F}_{j-1} + \mathcal{F}^{\perp}_{-j+1} = \mathcal{F}_{j-1} + \tau(\mathcal{F}_{j-1}).$$

By induction we obtain $\mathcal{F}_j = \sum_{m=1}^{j} \tau^m(\mathcal{F}_1)$. Therefore, $\mathcal{F}$ is uniquely determined by $\mathcal{F}_1$, and the claim is proved.

Let i with $0 \leq i \leq d-2$. The claim shows that the natural morphisms

$$X_{P_{i+1}}(w_{i+1}) \longrightarrow X_{P_i}(w_{i+1}), \qquad X_{P_{i+1}}(w_{i+2}) \longrightarrow X_{P_i}(w_{i+1})$$

are monomorphisms. We now want to show that we have a decomposition of $X_{P_i}(w_{i+1})$ into the closed subvariety $X_{P_{i+1}}(w_{i+1})$ and the open subvariety $X_{P_{i+1}}(w_{i+2})$,

$$X_{P_i}(w_{i+1}) = X_{P_{i+1}}(w_{i+1}) \uplus X_{P_{i+1}}(w_{i+2}). \tag{2.24}$$

Indeed, let $\mathcal{F}$ be a flag in $X_{P_i}(w_{i+1})(R)$. Since $\Phi(\mathcal{F})$ and $\mathcal{F}$ are in standard position, $\mathcal{F}_{i+2} := \mathcal{F}_{i+1} + \tau(\mathcal{F}_{i+1})$ and $\mathcal{F}_{-i-2} := \mathcal{F}^{\perp}_{i+2}$ are locally direct summands of R^l of rank $(d+i+2)$ and $(d-i-1)$, respectively. We extend $\mathcal{F}$ to the flag

$$\mathcal{F}' = [0 \subset \mathcal{F}_{-i-2} \subsetneqq \cdots \subsetneqq \mathcal{F}_{-1} \subsetneqq \mathcal{F}_1 \subsetneqq \cdots \subsetneqq \mathcal{F}_{i+2} \subset V].$$

If $\Phi(\mathcal{F}')$ and $\mathcal{F}'$ are in standard position, we obtain that $\operatorname{inv}(\Phi(\mathcal{F}'), \mathcal{F}')$ is either w_{i+1} or w_{i+2} in $W_{i+1} \backslash S_l / W_{i+1}$. Thus, $\mathcal{F}'$ is either an element of $X_{P_{i+1}}(w_{i+1})$ or of $X_{P_{i+1}}(w_{i+2})$. This proves (2.24).

The disjoint sum decomposition in (2.22) follows by induction from (2.23) and (2.24). The subset $\biguplus_{i=0}^{j} X_{P_i}(w_i)$ is closed in Y_Λ by construction. By Lemma 2.14 the variety $X_{P_i}(w_i)$ is of dimension i. Since Y_Λ is smooth of dimension d, the open subvariety $X_B(w_d)$ is dense. It remains to show that $X_B(w_d)$ is irreducible.

Consider the set $D = \{(1,2), \dots, (l-1,l)\}$ of simple reflections of G with respect to B. The Frobenius action on D with respect to G is given by $\sigma((i,i+1)) = (l-i, l-i+1)$. Since l is odd, there are d orbits of the action of σ on D. Each orbit is of the form $\{(i,i+1),(l-i,l-i+1)\}$ with $d+1 \leq i \leq l-1$. A Coxeter element of S_l is a product of the form $\nu_1 \cdots \nu_d \in S_l$, where $\nu_1, \dots, \nu_d \in D$ are representatives of the different orbits [Lu1]. The element $w_d = (d+1, d+2) \cdots (l-1, l) \in S_l$ is a Coxeter element. It was shown in [Lu1, 4.8] that $X_B(w)$ is irreducible for each Coxeter element w. In particular, $X_B(w_d)$ is irreducible. This proves the theorem. ■

Proposition 2.16 *Let $\Lambda' \in \mathcal{L}_0$ with $\Lambda' \subset \Lambda$ and denote by V' the $\mathbb{F}_{p^2}$-vector space $\Lambda'/p(\Lambda')^\vee$. Let l' be the type of Λ' and let $d' = (l'-1)/2$. Then the variety $Y_{\Lambda'}$ is a closed subvariety of Y_Λ of dimension equal to d'.*

Proof By definition (2.19) we have

$$Y_{\Lambda'}(R) = \{U \subset V'_R \mid U \text{ is a locally direct summand}, \operatorname{rk}_R U = d'+1,\ U^\perp \subset U\},$$

hence by Proposition 2.7

$$Y_{\Lambda'}(R) = \{U \in Y_\Lambda(R) \mid U \subset (\Lambda'/p\Lambda^\vee)_R\}.$$

Thus $Y_{\Lambda'}$ is a closed subvariety of Y_Λ. By Proposition 2.13 the dimension of $Y_{\Lambda'}$ is equal to d'. ■

For $U \in Y_\Lambda(k)$ let $0 \leq i_U \leq d$ be the minimal integer such that $U + \cdots + \tau^i(U)$ is τ-invariant. For a lattice $A \in \mathcal{V}(\Lambda)(k)$, let i_A be the minimal integer such that $A + \cdots + \tau^i(A)$ is τ-invariant. Suppose that A corresponds to U via the bijection of Proposition 2.7. Then $i_A = i_U$.

Denote by $\mathcal{V}(\Lambda)(k)^\circ \subset \mathcal{V}(\Lambda)(k)$ the set of lattices $A \in \mathcal{V}(\Lambda)(k)$ with $i_A = d$, *i.e.*, $A + \cdots + \tau^d(A) = \Lambda_k$.

Corollary 2.17 *For every integer i with $0 \leq i \leq d$ we have*

$$X_{P_i}(w_i)(k) = \{U \in Y_\Lambda(k) \mid i_U = i\} = \{A \in \mathcal{V}(\Lambda)(k) \mid i_A = i\}.$$

In particular, $X_B(w_d)(k) = \mathcal{V}(\Lambda)(k)^\circ$. Hence

$$\biguplus_{j=0}^{i} X_{P_j}(w_j)(k) = \bigcup_{\substack{\Lambda' \subset \Lambda \\ \Lambda' \in \mathcal{L}_0^{(l')} \\ l' \leq 2i+1}} \mathcal{V}(\Lambda')(k) = \bigcup_{\substack{\Lambda' \subset \Lambda \\ \Lambda' \in \mathcal{L}_0^{(2i+1)}}} \mathcal{V}(\Lambda')(k).$$

There exists a scheme theoretical decomposition

$$Y_\Lambda = \bigcup_{\substack{\Lambda' \subsetneq \Lambda \\ \Lambda' \in \mathcal{L}_0}} Y_{\Lambda'} \uplus X_B(w_d),$$

where the first summand is closed and the second summand is open.

Proof The corollary follows from the proof of Theorem 2.15 and from Proposition 2.16. ■

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

3 The Combinatorial Intersection Behaviour of the Sets $\mathcal{V}(\Lambda)$

Let C be a $\mathbb{Q}_{p^2}$-vector space of dimension n with perfect skew-hermitian form $\{\cdot\,,\cdot\}$ as in Section 2. Denote by t an element of $\mathbb{Z}_{p^2}^{\times}$ with $t^\sigma = -t$. Again we assume that the skew-hermitian form $\{\cdot\,,\cdot\}$ is equivalent to the form induced by the matrix T with $T = tI_n$ if n is odd and equivalent to the form induced by $T = t\,J_n$ if n is even. Denote by H the special unitary group over $\mathbb{Q}_p$ with respect to $(C, \{\cdot\,,\cdot\})$, *i.e.,* for a $\mathbb{Q}_p$-algebra R, we have

$$H(R) = \{g \in \mathrm{SL}_{\mathbb{Q}_{p^2}\otimes_{\mathbb{Q}_p} R}(C \otimes_{\mathbb{Q}_p} R) \mid \{gx, gy\} = \{x, y\} \text{ for all } x, y \in C \otimes_{\mathbb{Q}_p} R\}.$$

Let k be an algebraically closed field extension of $\overline{\mathbb{F}}_p$. In this section we will prove that the incidence relation of the sets $\mathcal{V}(\Lambda)(k)$ of $\mathcal{D}_i(C)(k)$ can be read off from the combinatorial simplicial structure of the Bruhat–Tits building $\mathcal{B}(H, \mathbb{Q}_p)$ associated with H. As in Section 2, we assume that ni is even.

Propositions 2.4 and 2.5 show that the intersection behaviour of the sets $\mathcal{V}(\Lambda)(k)$ only depends on $\Lambda \in \mathcal{L}_i$. Therefore, we write $\mathcal{V}(\Lambda)$ and $\mathcal{D}_i(C)$ instead of $\mathcal{V}(\Lambda)(k)$ and $\mathcal{D}_i(C)(k)$.

Definition 3.1 Let $\mathcal{B}_i$ be the abstract simplicial complex given by the following data. Let m be a nonnegative integer. An m-simplex is a subset $S \subset \mathcal{L}_i$ of $m + 1$ elements that satisfies the following condition. There exists an ordering $\Lambda_0, \dots, \Lambda_m$ of the elements of S such that

$$p^{i+1}\Lambda_m^\vee \subsetneqq \Lambda_0 \subsetneqq \Lambda_1 \subsetneqq \cdots \subsetneqq \Lambda_m. \tag{3.1}$$

A vertex is defined as a 0-simplex.

Remark 3.2 Let S be an m-simplex of $\mathcal{B}_i$, and let $\Lambda_0, \dots, \Lambda_m$ be an ordering of the elements of S that satisfies (3.1). Since all Λ_j are in $\mathcal{L}_i$, we obtain from (3.1) the more precise chain of inclusions

$$p^{i+1}\Lambda_0^\vee \subsetneqq \Lambda_0 \subsetneqq \cdots \subsetneqq \Lambda_m \subset p^i\Lambda_m^\vee \subsetneqq \cdots \subsetneqq p^i\Lambda_0^\vee. \tag{3.2}$$

Obviously, we have $0 \le m \le (n-1)/2$.

Definition 3.3 Let $\{\Lambda\}$ be a vertex of $\mathcal{B}_i$. A vertex $\{\Lambda'\} \in \mathcal{B}_i$ is a neighbour of $\{\Lambda\}$ if $\Lambda \neq \Lambda'$ and if there exists a simplex $S \in \mathcal{B}_i$ such that Λ and Λ' are contained in S.

The above definition is equivalent to the condition that $\{\Lambda, \Lambda'\}$ is a 1-simplex of $\mathcal{B}_i$, i.e., $\Lambda \subsetneqq \Lambda'$ or $\Lambda' \subsetneqq \Lambda$. We say that $\{\Lambda\}$ is of type l if Λ is of type l. Let l and l' be the types of Λ and Λ' respectively. In the first case we obtain $l < l'$ and in the second case $l' > l$.

Proposition 3.4 (i) *Let Λ be a lattice in $\mathcal{L}_i$ of type l. Then the set*

$$\{\Lambda' \in \mathcal{L}_i \mid \mathcal{V}(\Lambda') \subsetneqq \mathcal{V}(\Lambda)\}$$

corresponds to the set of neighbours of $\{\Lambda\}$ of type $l' < l$.

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

(ii) *Let* Λ *and* Λ' *be in* $\mathcal{L}_i$ *of type* l *and* l' *respectively such that* $\Lambda \neq \Lambda'$. *Then*

$$\mathcal{V}(\Lambda) \cap \mathcal{V}(\Lambda') \neq \varnothing$$

if and only if $\{\Lambda\}$ *is a neighbour of* $\{\Lambda'\}$ *or if there exists a vertex* $\{\widetilde{\Lambda}\}$ *in* $\mathcal{B}_i$ *of type* $c < \min\{l, l'\}$ *such that* $\{\widetilde{\Lambda}\}$ *is a common neighbour of* $\{\Lambda\}$ *and* $\{\Lambda'\}$.

Proof The proposition follows from Proposition 2.4(iii)(b). ■

Theorem 3.5 *Let* $\mathcal{B}(H, \mathbb{Q}_p)_{\mathrm{simp}}$ *be the abstract simplicial complex of the Bruhat–Tits building of* H. *Then there exists a simplicial bijection between* $\mathcal{B}_i$ *and* $\mathcal{B}(H, \mathbb{Q}_p)_{\mathrm{simp}}$.

Proof We choose a basis of C such that the form $\{\cdot, \cdot\}$ is given by T. For a matrix g, denote by $g^{(\sigma)}$ the matrix obtained by applying the Frobenius σ of $\mathbb{Q}_{p^2}/\mathbb{Q}_p$ to every entry of g. Then H is isomorphic to $\mathrm{SL}(C)$ over $\mathbb{Q}_{p^2}$ with Frobenius

$$\Phi(g) = T({}^t g^{(\sigma)})^{-1} T$$

for $g \in \mathrm{SL}(C)$ (the proof is analogous to the proof of Lemma 2.11). The simplicial complex $\mathcal{B}(H, \mathbb{Q}_p)_{\mathrm{simp}}$ is equal to the fixed points of Φ on the simplicial complex $\mathcal{B}(\mathrm{SL}(C), \mathbb{Q}_{p^2})_{\mathrm{simp}}$ [Ti, 2.6.1].

An m-simplex of $\mathcal{B}(\mathrm{SL}(C), \mathbb{Q}_{p^2})_{\mathrm{simp}}$ is a set $\{[\Lambda_0], \dots, [\Lambda_m]\}$ of homothety classes of lattices with the following property. There exist representatives $\Lambda_j \in [\Lambda_j]$ which, after renumbering the lattices $\Lambda_0, \dots, \Lambda_m$, form an infinite lattice chain

$$\cdots \subsetneqq p\Lambda_m \subsetneqq \Lambda_0 \subsetneqq \cdots \subsetneqq \Lambda_m \subsetneqq p^{-1}\Lambda_0 \subsetneqq \cdots \tag{3.3}$$

We first consider the case $i = 0$. We define a simplicial morphism

$$\varphi\colon \mathcal{B}_0 \to \mathcal{B}(\mathrm{SL}(C), \mathbb{Q}_{p^2})_{\mathrm{simp}}$$
$$\{\Lambda_0, \dots, \Lambda_m\} \mapsto \{[\Lambda_0], \dots, [\Lambda_m], [\Lambda_m^\vee] \dots [\Lambda_0^\vee]\}.$$

This morphism is well defined as (3.3) follows from (3.2). To show that φ induces an isomorphism onto $\mathcal{B}(H, \mathbb{Q}_p)_{\mathrm{simp}}$, it is sufficient to prove the following claim.

Claim: Each simplex of $\mathcal{B}(H, \mathbb{Q}_p)_{\mathrm{simp}}$ can be written uniquely as

$$\{[\Lambda_0], \dots, [\Lambda_a], [\Lambda_a^\vee], \dots, [\Lambda_0^\vee]\}$$

such that there exist representatives $\Lambda_0, \dots, \Lambda_a$ satisfying

$$p\Lambda_0^\vee \subsetneqq \Lambda_0 \subsetneqq \cdots \subsetneqq \Lambda_a \subset \Lambda_a^\vee \subsetneqq \cdots \subsetneqq \Lambda_0^\vee.$$

Indeed, a simplex $\{[\Lambda_0], \dots, [\Lambda_m]\}$ of $\mathcal{B}(\mathrm{SL}(C), \mathbb{Q}_{p^2})_{\mathrm{simp}}$ is a fixed point under the action of Φ if the lattice chain (3.3) coincides with its dual chain

$$\cdots \subsetneqq p\Lambda_0^\vee \subsetneqq \Lambda_m^\vee \subsetneqq \cdots \subsetneqq \Lambda_0^\vee \subsetneqq p^{-1}\Lambda_m^\vee \subsetneqq \cdots.$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

This means that there exist integers j and a with $0 \leq a \leq m$ such that

$$\begin{array}{ccccccccc} p^j\Lambda_{a+1}^\vee & \subsetneqq & p^j\Lambda_a^\vee & \subsetneqq \cdots \subsetneqq & p^j\Lambda_0^\vee & \subsetneqq & p^{j-1}\Lambda_m^\vee & \subsetneqq \cdots \subsetneqq & p^{j-1}\Lambda_{a+1}^\vee \\ \| & & \| & & \| & & \| & & \| \\ p\Lambda_m & \subsetneqq & \Lambda_0 & \subsetneqq \cdots \subsetneqq & \Lambda_a & \subsetneqq & \Lambda_{a+1} & \subsetneqq \cdots \subsetneqq & \Lambda_m. \end{array}$$

An easy index calculation shows that there exists an ordering of the homothety classes and representatives such that we obtain a lattice chain

$$p\Lambda_0^\vee \subsetneqq \Lambda_0 \subsetneqq \Lambda_1 \subsetneqq \cdots \subsetneqq \Lambda_{[\frac{m+1}{2}]} \subset \Lambda_{[\frac{m+1}{2}]}^\vee \subsetneqq \cdots \subsetneqq \Lambda_1^\vee \subsetneqq \Lambda_0^\vee.$$

This proves the claim in the case of $i = 0$.

Now consider the general case. By Remark 2.3(iv) the isomorphism Ψ_i of Proposition 1.18 induces an inclusion and type preserving isomorphism of $\mathcal{L}_i$ with $\mathcal{L}_0$. Thus Ψ_i induces a simplicial bijection

$$\theta_i \colon \mathcal{B}_i \xrightarrow{\sim} \mathcal{B}_0, \tag{3.4}$$

which proves the theorem. ∎

Proposition 3.6 *Let $\Lambda, \Lambda' \in \mathcal{L}_i$. There exist a positive integer u and elements $\Lambda_1 = \Lambda, \Lambda_2, \dots, \Lambda_{u-1}, \Lambda_u = \Lambda'$ in $\mathcal{L}_i$ such that $\mathcal{V}(\Lambda_j) \cap \mathcal{V}(\Lambda_{j+1}) \neq \varnothing$ for every j with $1 \leq j \leq u-1$.*

Proof The building $\mathcal{B}(H, \mathbb{Q}_p)$ is connected, and thus the simplicial complex $\mathcal{B}(H, \mathbb{Q}_p)_{\mathrm{simp}}$ is connected. Then the proposition follows from Theorem 3.5 and Proposition 3.4. ∎

Proposition 3.7 *In the case of* $\mathrm{GU}(1,2)$, i.e., *if $n = 3$, the simplicial complex $\mathcal{B}_i$ is a tree. It has two different kind of vertices. Vertices of type* 1 *correspond to lattices in $\mathcal{L}_i^{(1)}$,* i.e., *superspecial lattices in $\mathcal{D}_i(C)$, and have $p+1$ neighbours of type* 3*. Vertices of type* 3 *correspond to lattices in $\mathcal{L}_i^{(3)}$ and have $p^3 + 1$ neighbours of type* 1*.*

Proof As n is equal to 3, Remark 3.2 shows that there exist only 0-simplices and 1-simplices. The building $\mathcal{B}(H, \mathbb{Q}_p)$ is contractible, hence its simplicial complex is a tree. Thus by Theorem 3.5, the simplicial complex $\mathcal{B}_i$ is a tree. The type of a lattice $\Lambda \in \mathcal{L}_i$ is equal to 1 or 3 (Remark 2.3). By construction of $\mathcal{B}_i$, each vertex of type 1 has only neighbours of type 3 and each vertex of type 3 has only neighbours of type 1.

Now consider the case $i = 0$. Let Λ be in $\mathcal{L}_0^{(3)}$. Denote by V the $\mathbb{F}_{p^2}$-vector space $V = \Lambda/p\Lambda$ with perfect skew-hermitian form $(\,\cdot\,,\,\cdot\,)$ as in (2.11) and (2.12). By Corollary 2.8(i), the neighbours of $\{\Lambda\}$ correspond to $\mathbb{F}_{p^2}$-subspaces U of V of dimension 2 with $U^\perp \subset U$. As all skew-hermitian forms on V are equivalent, there exists a basis of V such that $(\,\cdot\,,\,\cdot\,)$ is given by the matrix $\bar{\iota}I_3$. By duality the neighbours of $\{\Lambda\}$ correspond to the totally isotropic subspaces of V of dimension 1, *i.e.,* to the $\mathbb{F}_{p^2}$-rational points of the Fermat curve $\mathcal{C}$ in $\mathbb{P}^2_{\mathbb{F}_{p^2}}$ given by the equation $x_0^{p+1} + x_1^{p+1} +$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

x_2^{p+1}. An easy calculation shows that the number of $\mathbb{F}_{p^2}$-rational points of $\mathcal{C}$ is equal to $p^3 + 1$.

Analogously, the neighbours of a 0-simplex $\{M\}$ correspond by Corollary 2.8(ii) to the totally isotropic 1-dimensional subspaces of the 2-dimensional $\mathbb{F}_{p^2}$-vector space $V' = M^\vee/M$. An easy calculation shows the claim.

Now consider the general case. By (3.4) there exists a simplicial bijection between $\mathcal{B}_i$ and B_0, hence the claim follows from the case $i = 0$. ■

Remark 3.8 Let $n = 3$ and let $\Lambda, \Lambda' \in \mathcal{L}_i$. As $\mathcal{B}_i$ is a tree, there exists a unique lattice chain for Λ and Λ' as in Proposition 3.6 of minimal length. We call its length u the distance $u(\Lambda, \Lambda')$ of Λ and Λ'. The distance of two lattices Λ and Λ' of the same type is even.

4 The Local Structure of $\mathcal{N}^{\mathrm{red}}$ for GU$(1, 2)$

In the next two sections, we will describe the scheme theoretic structure of $\mathcal{N}^{\mathrm{red}}$ in the case of the unitary group GU$(1, 2)$. We again assume that $p \neq 2$. We will first describe the scheme-theoretic structure of the open and closed subscheme $\mathcal{N}_0^{\mathrm{red}}$ of quasi-isogenies of height 0.

We will always denote by k an algebraically closed field extension of $\overline{\mathbb{F}}_p$. By abuse of notation, we will mostly identify the elements of $\mathcal{N}(k)$ with their corresponding Dieudonné modules as in (1.3). We say that a p-divisible group in $\mathcal{N}_0(k)$ is superspecial if the corresponding Dieudonné module is superspecial. Let $N = N_0 \oplus N_1$ be the isocrystal over $W(\overline{\mathbb{F}}_p)_{\mathbb{Q}}$ with perfect alternating form $\langle\,\cdot\,,\,\cdot\,\rangle$ as in Section 1. In Sections 2 and 3, we have denoted by C the $\mathbb{Q}_{p^2}$-vector space N_0^τ of τ-invariant elements of N_0 (Remark 1.8). By abuse of notation, we will identify the sets $\mathcal{N}_0(k)$ and $\mathcal{D}_0(C)(k)$ using the bijection of Proposition 1.10. By Remark 2.3 we will identify the superspecial points of $\mathcal{N}_0(k)$ with the lattices in $\mathcal{L}_0^{(1)}$.

We fix some notation. As in Lemma 1.13, we fix an element $t \in \mathbb{Z}_{p^2}^\times$ with $t^\sigma = -t$ and denote by $\bar{t}$ its image in $\mathbb{F}_{p^2}^\times$. For an element x in a ring R, we write $[x]$ for the Teichmüller representative $(x, 0, \dots) \in W(R)$. The map $[\,\cdot\,]\colon R \to W(R)$ is multiplicative and injective. As $p \neq 2$, we have $1 + [-1] = 0$, which we will frequently use in the sequel.

For every scheme S over $\mathbb{F}_{p^2}$, we denote again by S the corresponding scheme over $\overline{\mathbb{F}}_p$.

For $\Lambda \in \mathcal{L}_0$ we have defined a closed subset $\mathcal{V}(\Lambda)(k)$ of $\mathcal{N}_0(k)$. By Remark 2.9(i) and Proposition 2.13 there exists a smooth, irreducible and proper variety Y_Λ over $\mathbb{F}_{p^2}$ such that $Y_\Lambda(k) = \mathcal{V}(\Lambda)(k)$ for every algebraically closed field extension k of $\overline{\mathbb{F}}_p$. The varieties Y_Λ depend up to isomorphism only on the type l of Λ (Remark 2.9(ii)). By Remark 2.3 the type l is equal to 1 or 3. If $l = 1$, the variety Y_Λ consists of only one point. If $l = 3$, the variety Y_Λ is smooth and irreducible of dimension 1 (Proposition 2.13, Theorem 2.15).

Remark 4.1 By Proposition 2.4 we know that $\mathcal{N}_0(k) = \bigcup_{\Lambda \in \mathcal{L}_0^{(3)}} \mathcal{V}(\Lambda)(k)$. If the intersection of two different sets $\mathcal{V}(\Lambda)(k)$ and $\mathcal{V}(\Lambda')(k)$ is nonempty, they intersect at

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

one superspecial point, the lattice $\Lambda \cap \Lambda'$ (Proposition 2.4, Proposition 2.5). Each set $\mathcal{V}(\Lambda)(k)$ contains $p^3 + 1$ superspecial points and each superspecial point is contained in $p + 1$ sets $\mathcal{V}(\Lambda)(k)$ (Proposition 3.7).

Let $\mathbf{M} \in \mathcal{N}_0(k)$ be a superspecial Dieudonné lattice. The following lemma follows directly from Sections 1 and 2.

Lemma 4.2 *The lattice $\mathbf{M}$ is already defined over $\mathbb{Z}_{p^2}$. There exists a τ-invariant basis $e_1, e_2, e_3, f_1, f_2, f_3$ of $\mathbf{M}$ such that $\mathbf{M}_0 = \langle e_1, e_2, e_3\rangle_{W(k)}$ and $\mathbf{M}_1 = \langle f_1, f_2, f_3\rangle_{W(k)}$. The matrix of F with respect to the above basis of $\mathbf{M}$ is given by*

$$\begin{pmatrix} & & & 1 & & \\ & & & & p & \\ & & & & & 1 \\ p & & & & & \\ & 1 & & & & \\ & & p & & & \end{pmatrix}.$$

The form $\{\cdot,\cdot\}$ on $\mathbf{M}_0$ is given by the matrix

$$t\begin{pmatrix} p & & \\ & 1 & \\ & & p \end{pmatrix}. \tag{4.1}$$

In particular, the only nonzero values of the alternating form $\langle\cdot,\cdot\rangle$ on the basis of $\mathbf{M}$ are given by $\langle e_i, f_j\rangle = -\langle f_i, e_j\rangle = t\delta_{ij}$.

Proof As $\mathbf{M}$ is superspecial, it is τ-invariant, hence defined over $\mathbb{Z}_{p^2}$. By Lemma 1.14 there exists a basis e_1, e_2, e_3 of $\mathbf{M}_0^\tau$ such that the form $\{\cdot,\cdot\}$ is given by the matrix (4.1). We have $p\mathbf{M}_0^\vee = \langle e_1, pe_2, e_3\rangle$. The proof of Proposition 1.10 shows that $\mathbf{M}_1$ is equal to $F^{-1}(p\mathbf{M}_0^\vee)$. For $i = 1, 3$ we define $f_i = F^{-1}e_i$ and set $f_2 = F^{-1}(pe_2)$. This basis satisfies the conditions of the lemma. ∎

Let $\mathcal{J}$ be the set $\mathcal{J} = \{[\lambda:\mu] \in \mathbb{P}^1(\mathbb{F}_{p^2}) \mid \lambda^{p+1} + \mu^{p+1} = 0\}$. Note that $\mathcal{J}$ has $p + 1$ elements.

Lemma 4.3 *There exists a bijection between the sets $\mathcal{V}(\Lambda)(k)$ with $\Lambda \in \mathcal{L}_0^{(3)}$ that contain $\mathbf{M}$ and the elements $[\lambda:\mu] \in \mathcal{J}$. For $[\lambda:\mu] \in \mathcal{J}$ the corresponding lattice $\Lambda \in \mathcal{L}_0^{(3)}$ is given by*

$$\Lambda = \langle e_1, e_2, e_3, p^{-1}([\lambda]e_1 + [\mu]e_3)\rangle_{W(\overline{\mathbb{F}}_p)}. \tag{4.2}$$

In particular, $\mathbf{M}$ is contained in $p + 1$ sets $\mathcal{V}(\Lambda)(k)$.

Proof Let $\Lambda \in \mathcal{L}_0^{(3)}$ be a lattice with $\mathbf{M}_0^\tau \subset \Lambda$. We must prove that Λ is of the form (4.2) for unique $[\lambda:\mu] \in \mathcal{J}$. We have

$$p\Lambda \overset{1}{\subset} p(\mathbf{M}_0^\tau)^\vee \overset{1}{\subset} \mathbf{M}_0^\tau \overset{1}{\subset} \Lambda = \Lambda^\vee \overset{1}{\subset} (\mathbf{M}_0^\tau)^\vee.$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Since $\mathbf{M}_0^\tau$ is of index 1 in Λ, there exist elements $\lambda, \mu, \nu \in \mathbb{F}_{p^2}$ such that

$$\Lambda = \langle e_1, e_2, e_3, p^{-1}([\lambda]e_1 + [\nu]e_2 + [\mu]e_3)\rangle_{W(\mathbb{F}_{p^2})}.$$

As Λ is totally isotropic, $\nu = 0$ and $\lambda^{p+1} + \mu^{p+1} = 0$. The element $[\lambda:\mu] \in \mathbb{P}^1(\mathbb{F}_{p^2})$ is uniquely determined by Λ.

On the other hand, it is clear that every lattice defined by (4.2) is contained in $\mathcal{L}_0^{(3)}$ and contains $\mathbf{M}_0^\tau$. ∎

The Frobenius σ acts on the set $\mathcal{J}$. We choose a set

$$\tilde{\mathcal{J}} = \{(\lambda_i, \mu_i)\}_{0 \le i \le p} \tag{4.3}$$

of representatives of the different elements of $\mathcal{J}$ such that $(\lambda_i^p, \mu_i^p) \in \tilde{\mathcal{J}}$ for every i. We denote by $\sigma(i)$ the unique integer j with $0 \le j \le p$ and $(\lambda_i^p, \mu_i^p) = (\lambda_j, \mu_j)$.

We fix an integer i with $0 \le i \le p$. Let Λ_i be the lattice corresponding to (λ_i, μ_i) and let

$$e_{\lambda_i,\mu_i} = p^{-1}([\lambda_i]e_1 + [\mu_i]e_3) \in \Lambda_i.$$

Then $\{e_1, e_2, e_{\lambda_i,\mu_i}\}$ is a $W(\mathbb{F}_{p^2})$-basis of Λ_i. Let V be the $\overline{\mathbb{F}}_p$-vector space $\Lambda_i/p\Lambda_i$ with induced basis $\{\bar{e}_1, \bar{e}_2, \bar{e}_{\lambda_i,\mu_i}\}$. Denote by $(\,\cdot\,,\,\cdot\,)$ the perfect form on V induced by $\{\,\cdot\,,\,\cdot\,\}$. With respect to the above basis the form $(\,\cdot\,,\,\cdot\,)$ is given by the matrix

$$t\begin{pmatrix} & & \lambda_i^p \\ & 1 & \\ \lambda_i & & \end{pmatrix} \in \mathrm{GL}_3(\mathbb{F}_{p^2}).$$

In (2.19) the variety Y_{Λ_i} is defined as the closed subvariety of $\mathrm{Grass}_2(V)$ over $\mathbb{F}_{p^2}$ given by

$$Y_{\Lambda_i}(R) = \{U \subset V_R \mid U \text{ is a locally direct summand, } \mathrm{rk}_R\, U = 2,\ U^\perp \subset U\}.$$

The superspecial lattice $\mathbf{M}$ corresponds via the bijection of Proposition 2.7 to $\mathbf{U} = \langle \bar{e}_1, \bar{e}_2\rangle$. Let $\mathcal{U}_{\mathbf{M},i}$ be the open neighbourhood of $\mathbf{U}$ in Y_{Λ_i} given for each $\mathbb{F}_{p^2}$-algebra R by

$$\mathcal{U}_{\mathbf{M},i}(R) = \{U_{a,b} = \langle \bar{e}_1 + a\bar{e}_{\lambda_i,\mu_i}, \bar{e}_2 + b\bar{e}_{\lambda_i,\mu_i}\rangle \mid a, b \in R\} \cap \mathcal{V}(\Lambda_i)(R).$$

Lemma 4.4 *We have*

$$\mathcal{U}_{\mathbf{M},i}(R) = \{U_{a,b} = \langle \bar{e}_1 + a\bar{e}_{\lambda_i,\mu_i}, \bar{e}_2 + b\bar{e}_{\lambda_i,\mu_i}\rangle \mid a^p\lambda_i^p + a\lambda_i - b^{p+1}\lambda_i^{p+1} = 0;\ a, b \in R\}.$$

In particular, $\mathcal{U}_{\mathbf{M},i}$ *is isomorphic to* $T_i = \operatorname{Spec} R_i$*, where*

$$R_i = \mathbb{F}_{p^2}[a, b]/(a^p\lambda_i^p + a\lambda_i - b^{p+1}\lambda_i^{p+1}). \tag{4.4}$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Proof For $U_{a,b}$ to be contained in $\mathcal{V}(\Lambda_i)(R)$, it is necessary and sufficient that $U_{a,b}^{\perp} \subset U_{a,b}$. An easy computation shows that $U_{a,b} \cap U_{a,b}^{\perp} \neq 0$ if and only if

$$a^p \lambda_i^p + a\lambda_i - b^{p+1}\lambda_i^{p+1} = 0.$$

In this case we obtain $U_{a,b}^{\perp} = \langle \bar{e}_1 - b^p \lambda_i^p \bar{e}_2 - a^p \lambda_i^{p-1} \bar{e}_{\lambda_i,\mu_i} \rangle$. ∎

Remark 4.5 As the Frobenius twist $T_i^{(p)}$ of T_i is isomorphic to $T_{\sigma(i)}$, we will always identify $T_i^{(p)}$ with $T_{\sigma(i)}$. We denote by $\mathrm{Fr}_{T_i}\colon T_i \to T_{\sigma(i)}$ the relative Frobenius.

Lemma 4.6 *The variety Y_{Λ_i} is isomorphic to the smooth and projective curve C_{λ_i} in $\mathbb{P}^2_{\mathbb{F}_{p^2}}$ given by the equation $a^p\lambda_i^p d + a\lambda_i d^p - b^{p+1}\lambda_i^{p+1} = 0$ for $[a{:}b{:}d] \in \mathbb{P}^2_{\mathbb{F}_{p^2}}$.*

Proof By Lemma 4.4 the projective nonsingular curves Y_{Λ_i} and C_{λ_i} are isomorphic over the open subvarieties $\mathcal{U}_{a,b}$ and $\{d \neq 0\}$ respectively. Therefore, they are isomorphic. ∎

Remark 4.7 (i) The complement of $\mathcal{U}_{\mathbf{M},i}$ in Y_{Λ_i} contains only one point. This point is $\mathbb{F}_{p^2}$-rational. Indeed, this follows from the same result for C_{λ_i}.

(ii) The curve Y_{Λ_i} is isomorphic to the Fermat curve $\mathcal{C}$ in $\mathbb{P}^2_{\mathbb{F}_{p^2}}$ given by the equation $x_0^{p+1}+x_1^{p+1}+x_2^{p+1} = 0$. Indeed, as there exists up to isomorphism only one perfect skew-hermitian form on $(\mathbb{F}_{p^2})^3$, the Fermat curve $\mathcal{C}$ is isomorphic to the curve C_{λ_i} of Lemma 4.6.

For $a, b \in k$ denote by $c \in k$ the element $c = -\lambda_i\mu_i^{-1}a$. Let

$$f_{\lambda_i,\mu_i} = p^{-1}([\lambda_i^p]f_1 + [\mu_i^p]f_3) \in N_1.$$

We define the following elements of the isocrystal N_k, which depend on a, b.

$$\begin{aligned}
\tilde{e}_1 &= e_1 - [b^{p^{-1}}\lambda_i^p]e_2 + ([a] - [b^{\frac{p+1}{p}}\lambda_i^p])e_{\lambda_i,\mu_i},\\
\tilde{e}_2 &= e_2 + [b]e_{\lambda_i,\mu_i},\\
\tilde{e}_3 &= e_3 - [b^{p^{-1}}\mu_i^p]e_2 + ([c] - [b^{\frac{p+1}{p}}\mu_i^p])e_{\lambda_i,\mu_i},\\
\tilde{f}_1 &= f_1 - [b\lambda_i]p^{-1}f_2 - [a\lambda_i^{1-p}]f_{\lambda_i,\mu_i},\\
\tilde{f}_2 &= f_2 + [b^{p^{-1}}]pf_{\lambda_i,\mu_i},\\
\tilde{f}_3 &= f_3 - [b\mu_i]p^{-1}f_2 - [c\mu_i^{1-p}]f_{\lambda_i,\mu_i}.
\end{aligned} \tag{4.5}$$

Denote by $\delta_{\mathbf{M},i}$ the open immersion

$$\begin{aligned}
\delta_{\mathbf{M},i}\colon T_i &\hookrightarrow Y_{\Lambda_i},\\
(a,b) &\mapsto U_{a,b}
\end{aligned} \tag{4.6}$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

of Lemma 4.4. By Proposition 2.7 we have a bijection $\theta_i \colon Y_{\Lambda_i}(k) \leftrightarrow \mathcal{V}(\Lambda_i)(k)$. Consider the following diagram

$$\begin{tikzcd}
 & Y_{\Lambda_i}(k) \arrow[r, leftrightarrow, "\theta_{\Lambda_i}"'] & \mathcal{V}(\Lambda_i)(k) \\
T_i(k) \arrow[ur, hook, "\delta_{\mathbf{M},i}"'] \arrow[urr, bend left, "\Psi_{\mathbf{M},i}(k)"] \arrow[r, "\sim"'] & \mathcal{U}_{\mathbf{M},i}(k) \arrow[u, hook] \arrow[r, leftrightarrow, "1:1"] & \mathcal{S}_{\mathbf{M},i}(k) \arrow[u, hook]
\end{tikzcd} \tag{4.7}$$

By Remark 4.7(i), the complement of the set $\mathcal{S}_{\mathbf{M},i}(k) \subset \mathcal{V}(\Lambda_i)(k)$ defined in the above diagram consists of only one superspecial point.

Proposition 4.8 *The map $\Psi_{\mathbf{M},i}(k)$ is given by*

$$\Psi_{\mathbf{M},i}(k) \colon T_i(k) \xrightarrow{1:1} \mathcal{S}_{\mathbf{M},i}(k) \subset \mathcal{V}(\Lambda)(k),$$
$$(a, b) \mapsto M_{a,b} = M_0 \oplus M_1,$$

with

$$M_0 = \langle \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \rangle_{W(k)}, \tag{4.8}$$

$$M_1 = \langle \tilde{f}_1, \tilde{f}_2, \tilde{f}_3 \rangle_{W(k)}, \tag{4.9}$$

and $V(M_{a,b}) = \langle \tilde{e}_1, p\tilde{e}_2, \tilde{e}_3, p\tilde{f}_1, \tilde{f}_2, p\tilde{f}_3 \rangle_{W(k)}$.

Proof For $(a, b) \in T_i(k)$ we have $\delta_{\mathbf{M},i}(a, b) = U_{a,b}$. Let $\pi \colon \Lambda_i \to \Lambda_i / p\Lambda_i$ be the natural projection, and let π_k be the base change of π with $W(k)$. Then by Proposition 2.7 we obtain $M_0 = \pi_k^{-1}(U_{a,b})$ and $M_1 = F^{-1}(pM_0^\vee)$. We have

$$\begin{aligned} M_0 &= \langle e_1 + [a]e_{\lambda_i,\mu_i}, e_2 + [b]e_{\lambda_i,\mu_i}, pe_{\lambda_i,\mu_i} \rangle \\ &= \langle e_1 + [a]e_{\lambda_i,\mu_i}, e_2 + [b]e_{\lambda_i,\mu_i}, e_3 + [c]e_{\lambda_i,\mu_i} \rangle. \end{aligned} \tag{4.10}$$

Similarly, we obtain for $pM_0^\vee = \pi_k^{-1}(U_{a,b}^\perp)$

$$\begin{aligned} pM_0^\vee &= \langle e_1 - [b^p \lambda_i^p]e_2 - [a^p \lambda_i^{p-1}]e_{\lambda_i,\mu_i}, pe_2, pe_{\lambda_i,\mu_i} \rangle \\ &= \Big\langle e_1 - [b^p \lambda_i^p]e_2 - [a^p \lambda_i^{p-1}]e_{\lambda_i,\mu_i}, pe_2, \\ &\qquad e_3 - [b^p \mu_i^p]e_2 - [c^p \mu_i^{p-1}]e_{\lambda_i,\mu_i} \Big\rangle. \end{aligned} \tag{4.11}$$

Note that for the equalities (4.10) and (4.11) we again use $p \neq 2$. From (4.11) we obtain

$$\begin{aligned} M_1 = \Big\langle & f_1 - [b\lambda_i]p^{-1} f_2 - [a\lambda_i^{1-p}] f_{\lambda_i,\mu_i}, f_2, \\ & f_3 - [b\mu_i]p^{-1} f_2 - [c\mu_i^{1-p}] f_{\lambda_i,\mu_i} \Big\rangle. \end{aligned} \tag{4.12}$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

As $pf_{\lambda_i,\mu_i} = [\lambda_i^p]\tilde{f}_1 + [\mu_i^p]\tilde{f}_3$, equality (4.9) follows from (4.12). The equality (4.8) follows from the equations

$$\tilde{e}_1 = (e_1 + [a]e_{\lambda_i,\mu_i}) - [b^{p^{-1}}\lambda_i^p]\tilde{e}_2, \qquad \tilde{e}_3 = (e_3 + [c]e_{\lambda_i,\mu_i}) - [b^{p^{-1}}\mu_i^p]\tilde{e}_2.$$

An easy calculation shows that $V(M_{a,b})$ has the desired basis. ■

Remark 4.9 The map $\Psi_{\mathbf{M},i}(k)$ is not a morphism. Indeed, the formulas (4.5) show that the module $M_{a,b}$ is not defined over a non perfect ring.

Our goal is to find an affine scheme $T_{\mathbf{M}}$ such that locally at $\mathbf{M}$ the variety $\mathcal{N}_0^{\mathrm{red}}$ is isomorphic to $T_{\mathbf{M}}$. Consider the polynomial ring in a_i, b_i for $0 \le i \le p$,

$$A = \mathbb{F}_{p^2}[a_i, b_i]_{0\le i\le p}.$$

For $0 \le i \le p$ let $h_i \in A$ be the polynomial $h_i = a_i^p\lambda_i^p + a_i\lambda_i - b_i^{p+1}\lambda_i^{p+1}$ and let $\mathfrak{a} \subset A$ be the ideal $\mathfrak{a} = (h_i, a_ia_j, a_ib_j, b_ib_j)_{0\le i\ne j\le p}$. Let $A' = A/\mathfrak{a}$ and let Z be the affine scheme $\operatorname{Spec} A'$. Denote by Z_i the closed subscheme $\operatorname{Spec} A'/(a_j, b_j)_{j\ne i}$ of Z.

Lemma 4.10 *The closed subschemes Z_i are the irreducible components of Z. They intersect transversally at the origin 0 and each Z_i is isomorphic to $\mathcal{U}_i$. Furthermore, Z is reduced and its tangent space at the origin has dimension $p+1$.*

Proof For $0 \le i \le p$ the closed subscheme Z_i is isomorphic to $\mathcal{U}_i$. We obtain

$$Z \setminus \{0\} = \biguplus_{i=0}^{p} Z_i \setminus \{0\},$$

hence the Z_i are the irreducible components of Z. In particular, Z is smooth away from the origin.

We now prove that A' is reduced. Since $\mathcal{U}_i$ is irreducible, it is clear that h_i is irreducible. Let $f \in A$ such that $f^r \in \mathfrak{a}$ for an integer $r \ge 1$. We want to show that $f \in \mathfrak{a}$. By the definition of $\mathfrak{a}$, we may assume that $f = \sum_{i=0}^p f_i$, with polynomials $f_i \in A$ that depend only on a_i, b_i. Then

$$f^r \equiv \sum_{i=0}^{p} f_i^r \bmod \mathfrak{a}.$$

As $f^r \in \mathfrak{a}$, we obtain that $f_i^r \in \mathfrak{a}$ for every i by the definition of $\mathfrak{a}$. Therefore, f_i^r is divisible by h_i. Since h_i is irreducible, it divides f_i, which proves that A' is reduced.

The tangent space at the origin is given by the equations $da_i = 0$ for $i = 0, \dots, p$, where da_i and db_i are the differentials of a_i and b_i, respectively. Therefore, the tangent space is of dimension $p+1$. In particular, all irreducible components intersect transversally at the origin. ■

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

The moduli space $\mathcal{M}$ of abelian varieties defined in the introduction is smooth of dimension 2. By the uniformization theorem of Rapoport and Zink, the moduli space $\mathcal{N}$ is locally isomorphic to the closed subvariety $\mathcal{M}^{\mathrm{ss}}$ of the special fibre of $\mathcal{M}$ if C^p is small enough (*cf.* Section 6). Thus the tangent space of $\mathcal{N}^{\mathrm{red}}$ at each closed point is at most of dimension 2. Therefore, the local structure of $\mathcal{N}_0^{\mathrm{red}}$ at a supersingular point cannot be given by Z. We will define a modification of the ring A' such that the tangent space at the origin has dimension 2 and prove that the local structure of $\mathcal{N}_0^{\mathrm{red}}$ is given by this modification.

Consider $R = \mathbb{F}_{p^2}[a_i, x, y]_{0 \leq i \leq p}$. For an integer k with $0 \leq k \leq p$ let $g_k \in R$ be the polynomial

$$g_k = \sum_{i=0}^{p} (a_i^p \lambda_i^{p-k} \mu_i^k + a_i \lambda_i^{1-k} \mu_i^k) - x^{p+1-k} y^k.$$

Define

$$R_{\mathbf{M}} = R/(x^{p+1} + y^{p+1}, a_i a_j, a_i(\lambda_i y - \mu_i x), g_k)_{0 \leq i \neq j \leq p,\ 0 \leq k \leq p} \tag{4.13}$$

and denote by $T_{\mathbf{M}} = \operatorname{Spec} R_{\mathbf{M}}$ the corresponding affine scheme. Let

$$R_{\mathbf{M},i} = R_{\mathbf{M}}/(a_j, \lambda_i y - \mu_i x)_{j \neq i} \tag{4.14}$$

and let $T_{\mathbf{M},i}$ be the corresponding closed subscheme of $T_{\mathbf{M}}$.

Let R_i be as in (4.4). We write a_i, b_i instead of a, b for the indeterminates of R_i. We have the following equality in R_i

$$(\lambda_i^{-1} \mu_i)^k h_i = a_i^p \lambda_i^{p-k} \mu_i^k + a_i \lambda_i^{1-k} \mu_i^k - (b_i \lambda_i)^{p+1-k} (b_i \mu_i)^k.$$

Therefore, the morphism

$$\begin{aligned} \eta_i \colon R_i &\longrightarrow R_{\mathbf{M},i}, \\ a_i &\longmapsto a_i, \\ b_i &\longmapsto \lambda_i^{-1} x \end{aligned} \tag{4.15}$$

is an isomorphism.

Proposition 4.11 *The closed subschemes $T_{\mathbf{M},i}$ are the $p+1$ irreducible components of $T_{\mathbf{M}}$, which intersect pairwise transversally at the origin 0. Furthermore, $T_{\mathbf{M},i}$ is isomorphic to $\mathcal{U}_i$ for $0 \leq i \leq p$, hence is smooth of dimension 1.*

In particular, $T_{\mathbf{M}}$ is of dimension 1 and smooth away from the origin. The tangent space of $T_{\mathbf{M}}$ at the origin is 2-dimensional.

Proof As $R_{\mathbf{M},i}$ is isomorphic to R_i by (4.15), the subscheme $T_{\mathbf{M},i}$ is isomorphic to $\mathcal{U}_i$. Thus it is smooth of dimension 1. The schemes $T_{\mathbf{M}}$ and $T_{\mathbf{M},i}$ coincide on the open locus $\{a_i \neq 0\}$. We have

$$T_{\mathbf{M}} \setminus \{0\} = \biguplus_{i=0}^{p} T_{\mathbf{M},i} \setminus \{0\}.$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

This shows that $T_{\mathbf{M}}$ is of dimension 1 and smooth away from the origin.

We now compute the tangent space of $T_{\mathbf{M}}$ at the origin. By (4.13) the tangent space at the origin is given in terms of the differentials $da_0, \dots, da_p, dx, dy$ by the equations $\sum_{i=0}^{p} \lambda_i^{1-k} \mu_i^k da_i = 0$ for $0 \leq k \leq p$. This system of equations has rank $p+1$ (Vandermonde determinant), hence we obtain $da_i = 0$ for all i. This proves that the tangent space has dimension 2. Furthermore, we see that any two irreducible components intersect transversally at the origin. ■

Proposition 4.12 *The scheme $T_{\mathbf{M}}$ is reduced.*

To prove Proposition 4.12, we first consider the homomorphism

$$\begin{aligned} \psi\colon R = \mathbb{F}_{p^2}[a_i, x, y]_{0\leq i\leq p} &\to A = \mathbb{F}_{p^2}[a_i, b_i]_{0\leq i\leq p}, \\ a_i &\mapsto a_i, \\ x &\mapsto \sum_{i=0}^{p} b_i \lambda_i, \\ y &\mapsto \sum_{i=0}^{p} b_i \mu_i. \end{aligned}$$

It is easy to see that the homomorphism ψ induces a homomorphism $\psi'\colon R_{\mathbf{M}} \to A'$. We will show that ψ' is injective. This will prove that $R_{\mathbf{M}}$ is reduced, as A' is reduced by Lemma 4.10.

Let $\tilde{A}$ be the ring $\tilde{A} = A/(a_i a_j, b_i b_j, a_i b_j)_{0\leq i\neq j\leq p}$ and denote by $\tilde{R}$ the ring

$$\tilde{R} = R/(x^{p+1} + y^{p+1}, a_i a_j, a_i(\lambda_i y - \mu_i x))_{0\leq i\neq j\leq p}.$$

We obtain a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\psi"] \arrow[d, "\alpha_1"] \arrow[dd, bend right, "\alpha_3"'] & A \arrow[d, "\beta_1"'] \arrow[dd, bend left, "\beta_3"] \\ \tilde{R} \arrow[r, "\tilde{\psi}"] \arrow[d, "\alpha_2"] & \tilde{A} \arrow[d, "\beta_2"'] \\ R_{\mathbf{M}} \arrow[r, "\psi'"] & A' \end{tikzcd}$$

Note that all rings in this diagram are graded rings with respect to the grading induced by the indeterminates a_i, b_i, x, and y. All homomorphisms respect the gradings.

Lemma 4.13 *The morphism $\tilde{\psi}\colon \tilde{R} \to \tilde{A}$ is injective.*

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Proof Let $f \in R$ such that $\beta_1 \circ \psi(f) = 0$. We want to show that $\alpha_1(f) = 0$. We may assume that f is of the form

$$f = \sum_{i=0}^{p} a_i f_i(a_i, x, y) + \tilde{f}(x, y). \tag{4.16}$$

Here $f_i, \tilde{f}$ are polynomials in R such that f_i depends only on a_i, x, y and $\tilde{f}$ depends only on x, y. Since $\beta_1 \circ \psi(f) = 0$, we obtain for all i

$$\tilde{f}(b_i\lambda_i, b_i\mu_i) = 0 \in A, \tag{4.17}$$

$$a_i f_i(a_i, b_i\lambda_i, b_i\mu_i) = 0 \in A. \tag{4.18}$$

Let $\tilde{f}_m(x, y)$ be the homogeneous component of degree m of $\tilde{f}$. Then

$$\tilde{f}_m(\lambda_i b_i, \mu_i b_i) = b_i^m \tilde{f}_m(\lambda_i, \mu_i) = 0$$

for all i. Therefore, $\tilde{f}_m(x, y) = 0$ for all $(x, y) \in \operatorname{Proj} \mathbb{F}_{p^2}[x, y]/(x^{p+1} + y^{p+1})$. Since this scheme is reduced, we obtain that $x^{p+1} + y^{p+1}$ divides $\tilde{f}_m$. Thus $\alpha_1(\tilde{f}) = 0$.

Now consider the equation (4.18). We may write

$$f_i(a_i, x, y) = \sum_{r,m} a_i^r f_{r,m}(x, y),$$

where $f_{r,m}$ is a polynomial in x, y homogeneous of degree m. By (4.18) we obtain $f_{r,m}(\lambda_i, \mu_i) = 0$. An analogous argument as above shows that the polynomial $\mu_i x - \lambda_i y$ divides $f_{r,m}$, hence $a_i(\mu_i x - \lambda_i y)$ divides f_i. Therefore, $\alpha_1(f_i) = 0$ and $\tilde{\psi}$ is injective. ■

Proof of Proposition 4.12 As A' is reduced by Lemma 4.10, it suffices to prove that ψ' is injective.

Let $f \in R$ such that the image of $\beta_3 \circ \psi(f) = 0$. We may assume that f is of the form (4.16). We obtain

$$\tilde{\psi}(\alpha_1(f)) = \sum_{i=0}^{p} (a_i f_i(a_i, b_i\lambda_i, b_i\mu_i) + \tilde{f}(b_i\lambda_i, b_i\mu_i)). \tag{4.19}$$

As $\beta_3 \circ \psi(f) = 0$, there exist polynomials $w_i(a_i, b_i)$ and $z_i(b_i)$ in A with $z_i(b_i)$ depending only on b_i such that

$$\tilde{\psi}(\alpha_1(f)) = \sum_{i=0}^{p} (a_i w_i(a_i, b_i) + z_i(b_i)) h_i. \tag{4.20}$$

Since h_i is a polynomial of degree $p + 1$ in b_i, it is easy to see that every monomial in $\tilde{f}$ is of degree greater or equal than $p + 1$. We write

$$\tilde{f} = \sum_{k=0}^{p} x^{p+1-k} y^k \tilde{f}_k(x, y). \tag{4.21}$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Define $f' = \sum_{i=0}^{p} a_i w_i(a_i, \lambda_i^{-1} x) g_0 - \sum_{k=0}^{p} \tilde{f}_k(x, y) g_k$.

Claim: We have $\tilde{\psi}(\alpha_1(f')) = \tilde{\psi}(\alpha_1(f))$.

An easy calculation shows that

$$\tilde{\psi}(\alpha_1(f)) - \tilde{\psi}(\alpha_1(f')) = \sum_{i=0}^{p} h_i \Big[z_i(b_i) + \sum_{k=0}^{p} \lambda_i^{-k} \mu_i^k \tilde{f}_k(b_i \lambda_i, b_i \mu_i) \Big]. \tag{4.22}$$

We obtain from (4.19), (4.20), and (4.21) with $a_i = 0$ for all i that

$$-z_i(b_i) = \sum_{k=0}^{p} \lambda_i^{-k} \mu_i^k \tilde{f}_k(b_i \lambda_i, b_i \mu_i),$$

which by (4.22) proves the claim.

Since $\tilde{\psi}$ is injective by Lemma 4.13, we obtain that $\alpha_1(f') = \alpha_1(f)$. By construction $\alpha_3(f') = 0$, hence ψ' is injective. ■

Lemma 4.14 *The completion $\hat{\mathcal{O}}_{T_{\mathbf{M}},0}$ of the local ring of $T_{\mathbf{M}}$ at the origin* 0 *is isomorphic to $\mathbb{F}_{p^2}[[x, y]] / \prod_{i=0}^{p} (\lambda_i y - \mu_i x)$. In particular, $T_{\mathbf{M}}$ is of complete intersection.*

Proof We use the notations of (4.13). An elementary calculation shows that

$$\prod_{i=0}^{p} (\lambda_i y - \mu_i x)$$

is contained in the ideal $(g_k)_{0 \leq k \leq p}$ of R. Thus the morphism

$$\begin{aligned} \varphi^\circ \colon \mathbb{F}_{p^2}[[x, y]] / \prod_{i=0}^{p} (\lambda_i y - \mu_i x) &\to \hat{R}_{\mathbf{M},0}, \\ x &\mapsto x, \\ y &\mapsto y \end{aligned}$$

is well defined. The corresponding schemes are both of dimension 1 and have $(p+1)$ irreducible components. Let φ be the morphism of schemes corresponding to φ°. It is easy to see that the tangent map of φ is an isomorphism. Therefore, φ° is surjective. Furthermore, φ induced an isomorphism on irreducible components, hence φ° is injective. ■

From now on, we use the theory of displays as in [Zi2]. First of all, we recall the definition of a display. Let R be a ring of characteristic p. We denote by I_R the image of the Verschiebung τ' on $W(R)$. A $3n$-display over R is a tuple $\mathcal{P} = (P, Q, F, V^{-1})$ such that P is a finitely generated projective $W(R)$-module, Q is a submodule of P, and $F \colon P \to P$ and $V^{-1} \colon Q \to P$ are σ-linear maps such that the following conditions are satisfied [Zi2, Definition 1]:

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

(i) $I_R P \subset Q \subset P$ and P/Q is a direct summand of the $W(R)$-module $P/I_R P$.
(ii) $V^{-1}\colon Q \to P$ is a σ-linear epimorphism, *i.e.*, the map $W(R) \otimes_{F,W(R)} Q \to P$ with $w \otimes m \mapsto wm$ is surjective.
(iii) For $x \in P$ and $w \in W(R)$ we have $V^{-1}(\tau'(w)x) = wF(x)$.

A $3n$-display $\mathcal{P}$ is called a display if $\mathcal{P}$ satisfies the nilpotent condition of [Zi2, Definition 13].

Let M be a Dieudonné module over a perfect field k. Then

$$\mathcal{P}_M = (M, V(M), F, V^{-1})$$

is a $3n$-display. It is a display if and only if V is topologically nilpotent on M for the p-adic topology, *i.e.,* if the associated p-divisible group has no étale part.

There exists a functor from the category of displays over R to the category of formal p-divisible groups over R [Zi2, Theorem 81]. If R is of finite type over a field of characteristic p, this functor is an equivalence of categories [Zi2, Theorem 103]. For such a ring R, we will identify the elements of $\mathcal{N}_0(R)$ with the corresponding displays with additional structure.

For a $3n$-display $\mathcal{P} = (P, Q, F, V^{-1})$, denote by N the base change $N = P \otimes \mathbb{Q}$. Then N is a projective $W_{\mathbb{Q}}(R)$-module and F induces a σ-linear operator on N which we will again denote by F. The pair (N, F) is called the isodisplay of $\mathcal{P}$ [Zi2, Definition 61].

Now assume that R is torsion free as an abelian group and that P is of rank n. Then the morphism $W(R) \to W_{\mathbb{Q}}(R)$ is injective, hence P is a $W(R)$-submodule of N. Let P' be a projective $W(R)$-submodule of N of rank n and Q' be a $W(R)$-submodule of P' such that P/Q is a direct summand of the $W(R)$-module $P/I_R P$. Denote by V^{-1} the operator $p^{-1}F$ on N. If P' is F-invariant and $V^{-1}(Q') \subset P'$, the tuple $\mathcal{P}' = (P', Q', F, V^{-1})$ is a $3n$-display. If $\mathcal{P}$ is a display, the $3n$-display $\mathcal{P}'$ is a display. Let Qisg_R be the category of displays over R up to isogeny, *i.e.,* the objects are displays and the morphisms are $\mathrm{Hom}(\mathcal{P}, \mathcal{P}') \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the functor $\mathrm{Qisg}_R \to (\mathrm{Isodisplays})_R$ is fully faithful [Zi2, Proposition 66].

Let $\mathbf{M} \in \mathcal{N}_0(k)$ be a superspecial Dieudonné lattice. As $\mathbf{M}$ is τ-invariant, we obtain $V^2 = \sigma^{-2}\,\mathrm{id}_{\mathbf{M}}$, hence the corresponding $3n$-display is a display. Let $R_{\mathbf{M}}$ be the ring defined in (4.13). By base change we obtain a display $\mathcal{P}_{\mathbf{M}} = (P_{\mathbf{M}}, Q_{\mathbf{M}}, F, V^{-1})$ over $R_{\mathbf{M}}$ with

$$P_{\mathbf{M}} = \mathbf{M} \otimes_{W(\mathbb{F}_{p^2})} W(R_{\mathbf{M}}),$$

$$Q_{\mathbf{M}} = \langle e_1, f_2, e_3 \rangle_{W(R_{\mathbf{M}})} \oplus I_{R_{\mathbf{M}}} \langle f_1, e_2, f_3 \rangle_{W(R_{\mathbf{M}})} \subset P_{\mathbf{M}}$$

and induced F and V^{-1} [Zi2, Definition 20]. The display $\mathcal{P}_{\mathbf{M}}$ is equipped with an action of $\mathcal{O}_E$, *i.e.,* a $\mathbb{Z}/2\mathbb{Z}$-grading

$$P_{\mathbf{M}} = P_{\mathbf{M},0} \oplus P_{\mathbf{M},1} = \langle e_1, e_2, e_3 \rangle_{W(R_{\mathbf{M}})} \oplus \langle f_1, f_2, f_3 \rangle_{W(R_{\mathbf{M}})}$$

and an induced grading $Q_{\mathbf{M}} = Q_{\mathbf{M},0} \oplus Q_{\mathbf{M},1}$. The perfect alternating form $\langle \cdot\,, \cdot \rangle$ on $\mathbf{M}$ induces a perfect alternating form on the display $\mathcal{P}_{\mathbf{M}}$, *i.e.,* a perfect alternating

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

form $\langle\cdot,\cdot\rangle\colon P_{\mathbf{M}} \times P_{\mathbf{M}} \to W(R_{\mathbf{M}})$ such that $\tau'(\langle V^{-1}(x_1), V^{-1}(x_2)\rangle) = \langle x_1, x_2\rangle$ for all x_1, x_2 in $Q_{\mathbf{M}}$ [Zi2, Definition 18].

Let $N_{\mathbf{M}} = P_{\mathbf{M}} \otimes \mathbb{Q}$ and denote by $(N_{\mathbf{M}}, F)$ the isodisplay of $\mathcal{P}_{\mathbf{M}}$. As $R_{\mathbf{M}}$ is reduced (Proposition 4.12), the morphism $P_{\mathbf{M}} \hookrightarrow N_{\mathbf{M}}$ is injective and we obtain a $\mathbb{Z}/2\mathbb{Z}$-grading and a perfect alternating form on $N_{\mathbf{M}}$. We denote by V^{-1} the operator $p^{-1}F$ on $N_{\mathbf{M}}$.

We now construct a display $\mathcal{P}$ over $R_{\mathbf{M}}$ together with a quasi-isogeny $\rho\colon \mathcal{P} \to \mathcal{P}_{\mathbf{M}}$ such that $\mathcal{P}$ is equipped with all the data of $\mathcal{N}_0$. Then by base change this display will define a morphism $T_{\mathbf{M}} \to \mathcal{N}_0^{\mathrm{red}}$ (4.31).

Let $c_i = -\lambda_i \mu_i^{-1} a_i$. Consider the following elements of $N_{\mathbf{M}}$ analogously to (4.5):

$$\tilde{e}_1 = e_1 - [x]e_2 + \Big(\Big[\sum_{i=0}^{p} a_i^p \lambda_i^p\Big] - [x^{p+1}]\Big)\, p^{-1} e_1 + \Big(\Big[\sum_{i=0}^{p} a_i^p \mu_i^p\Big] - [xy^p]\Big)\, p^{-1} e_3, \tag{4.23}$$

$$\tilde{e}_2 = e_2 + [x^p]p^{-1}e_1 + [y^p]p^{-1}e_3, \tag{4.24}$$

$$\tilde{e}_3 = e_3 - [y]e_2 + \Big(\Big[\sum_{i=0}^{p} c_i^p \lambda_i^p\Big] - [x^p y]\Big)\, p^{-1} e_1 + \Big(\Big[\sum_{i=0}^{p} c_i^p \mu_i^p\Big] - [y^{p+1}]\Big)\, p^{-1} e_3, \tag{4.25}$$

$$\tilde{f}_1 = f_1 - [x^p]p^{-1}f_2 - \Big[\sum_{i=0}^{p} a_i^p \lambda_i^p\Big]\, p^{-1} f_1 - \Big[\sum_{i=0}^{p} a_i^p \lambda_i^{p-1}\mu_i\Big]\, p^{-1} f_3, \tag{4.26}$$

$$\tilde{f}_2 = f_2 + [x]f_1 + [y]f_3, \tag{4.27}$$

$$\tilde{f}_3 = f_3 - [y^p]p^{-1}f_2 - \Big[\sum_{i=0}^{p} c_i^p \lambda_i \mu_i^{p-1}\Big]\, p^{-1} f_1 - \Big[\sum_{i=0}^{p} c_i^p \mu_i^p\Big]\, p^{-1} f_3. \tag{4.28}$$

If we identify $R_{\mathbf{M},i}$ with R_i via η_i, as in (4.15), we obtain over $R_{\mathbf{M},i}$ the elements of (4.5) up to a Frobenius twist.

Let $P = P_0 \oplus P_1 \subset N_{\mathbf{M}}$ be the module given by

$$P_0 = \langle \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\rangle_{W(R_{\mathbf{M}})}, \qquad P_1 = \langle \tilde{f}_1, \tilde{f}_2, \tilde{f}_3\rangle_{W(R_{\mathbf{M}})}. \tag{4.29}$$

Denote by Q the submodule $Q = \langle \tilde{e}_1, \tilde{f}_2, \tilde{e}_3\rangle_{W(R_{\mathbf{M}})} \oplus I_{R_{\mathbf{M}}}\langle \tilde{f}_1, \tilde{e}_2, \tilde{f}_3\rangle_{W(R_{\mathbf{M}})}$ of P. We define a morphism

$$\begin{aligned} \rho'\colon \mathcal{P} &\to \mathcal{P}_{\mathbf{M}}, \\ \tilde{e}_i &\mapsto p\tilde{e}_i, \\ \tilde{f}_i &\mapsto p\tilde{f}_i. \end{aligned}$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Proposition 4.15 *The tuple $\mathcal{P} := (P, Q, F, V^{-1})$ is a subdisplay of the isodisplay $N_{\mathbf{M}}$. It is invariant under the action of $\mathcal{O}_E$ and satisfies the determinant condition of signature $(1, 2)$. The module P is free of rank 6 and the form $\langle\,\cdot\,,\,\cdot\,\rangle$ is given by the matrix*

$$t\begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix} \tag{4.30}$$

with respect to the above basis of P.

The morphism ρ' is a quasi-isogeny of height 6 of displays with $\mathcal{O}_E$-action such that for all $x_1, x_2 \in \mathcal{P}$ we have $\langle \rho'(x_1), \rho'(x_2)\rangle = p^2\langle x_1, x_2\rangle$.

Proof The display $\mathcal{P}$ is $\mathcal{O}_E$-invariant, because the $\mathbb{Z}/2\mathbb{Z}$-grading on $N_{\mathbf{M}}$ induces the $\mathbb{Z}/2\mathbb{Z}$-grading $P = P_0 \oplus P_1$ and a $\mathbb{Z}/2\mathbb{Z}$-grading $Q = Q_0 \oplus Q_1$. To show that $\mathcal{P}$ is a subdisplay of $N_{\mathbf{M}}$, we must prove that P is invariant under F and that $V^{-1}Q \subset P$. Consider

$$L = \langle \tilde{e}_1, \tilde{f}_2, \tilde{e}_3\rangle_{W(R_{\mathbf{M}})} \quad \text{and} \quad T = \langle \tilde{f}_1, \tilde{e}_2, \tilde{f}_3\rangle_{W(R_{\mathbf{M}})}.$$

We will show that $P = L \oplus T$ is a normal decomposition of P. It is sufficient to prove that $FT \subset P$ and $V^{-1}L \subset P$.

As an example, we will check that $V^{-1}(\tilde{e}_1) \in P$. The other inclusions can be proved by a similar calculation. By (4.23) we have

$$\begin{aligned} V^{-1}(\tilde{e}_1) = f_1 - [x^p]p^{-1}f_2 + \Big(\Big[\sum_{i=0}^{p} a_i^{p^2}\lambda_i\Big] - [x^{p(p+1)}]\Big)\,p^{-1}f_1 \\ + \Big(\Big[\sum_{i=0}^{p} a_i^{p^2}\mu_i\Big] - [x^p y^{p^2}]\Big)\,p^{-1}f_3. \end{aligned}$$

By (4.26) we obtain

$$\begin{aligned} V^{-1}(\tilde{e}_1) = \tilde{f}_1 + \Big(\Big[\sum_{i=0}^{p} a_i^{p}\lambda_i^{p}\Big] + \Big[\sum_{i=0}^{p} a_i^{p^2}\lambda_i\Big] - [x^{p(p+1)}]\Big)\,p^{-1}f_1 \\ + \Big(\Big[\sum_{i=0}^{p} a_i^{p}\lambda_i^{p-1}\mu_i\Big] + \Big[\sum_{i=0}^{p} a_i^{p^2}\mu_i\Big] - [x^p y^{p^2}]\Big)\,p^{-1}f_3. \end{aligned}$$

Now an easy calculation using the relations of $R_{\mathbf{M}}$ (4.13) shows that

$$\begin{aligned} V^{-1}(\tilde{e}_1) = \tilde{f}_1 + \Big(\Big[\sum_{i=0}^{p} a_i^{p}\lambda_i^{p}\Big] + \Big[\sum_{i=0}^{p} a_i^{p^2}\lambda_i\Big] - [x^{p(p+1)}]\Big)\,p^{-1}\tilde{f}_1 \\ + \Big(\Big[\sum_{i=0}^{p} a_i^{p}\lambda_i^{p-1}\mu_i\Big] + \Big[\sum_{i=0}^{p} a_i^{p^2}\mu_i\Big] - [x^p y^{p^2}]\Big)\,p^{-1}\tilde{f}_3. \end{aligned}$$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

By definition of $R_{\mathbf{M}}$ (4.13) we know that

$$\Big[\sum_{i=0}^{p} a_i\lambda_i\Big] + \Big[\sum_{i=0}^{p} a_i^p\lambda_i^p\Big] - [x^{p+1}] \in I_{R_{\mathbf{M}}},$$

$$\Big[\sum_{i=0}^{p} a_i\lambda_i^{1-p}\mu_i^p\Big] + \Big[\sum_{i=0}^{p} a_i^p\mu_i^p\Big] - [xy^p] \in I_{R_{\mathbf{M}}},$$

hence their images under σ are elements of $pW(R_{\mathbf{M}})$. Thus $V^{-1}(\tilde{e}_1) \in P$.

The following statements follow by a straightforward calculation. The matrix of $\langle\,\cdot\,,\,\cdot\,\rangle$ on P with respect to the basis in (4.29) is given by the matrix

$$t\begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix},$$

hence the form is perfect on P. As $\det(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = 1$, the module P is free of rank 6. We have $\rho'(Q) \subset Q_{\mathbf{M}}$, hence ρ' is a quasi-isogeny. The height of ρ' is equal to 6.

To prove the determinant condition of signature $(1,2)$, note that $Q/I_{R_{\mathbf{M}}}P$ is isomorphic to the dual of the Lie algebra of the corresponding p-divisible group. By construction the dimension of $Q_0/I_{R_{\mathbf{M}}}P_0$ is equal to 2 and the dimension of $Q_1/I_{R_{\mathbf{M}}}P_1$ is equal to 1, hence the determinant condition is satisfied. ■

Let $\mathcal{P}_{\mathbb{M}}$ be the display over $R_{\mathbb{M}}$ of the p-divisible group $\mathbb{X}$ of Definition 1.1. Denote by $\rho_{\mathbf{M}}\colon \mathcal{P}_{\mathbf{M}} \to \mathcal{P}_{\mathbb{M}}$ the quasi-isogeny of height 0 induced by the two lattices $\mathbb{M}$ and $\mathbf{M}$ in N. Let $\rho = p^{-1}\rho_{\mathbf{M}} \circ \rho'\colon \mathcal{P} \to \mathcal{P}_{\mathbb{M}}$. Then by Proposition 4.15, the morphism ρ is a quasi-isogeny of height 0 of polarized displays. The pair $(\mathcal{P}, \rho)$ defines by base change a morphism

$$\Phi_{\mathbf{M}}\colon T_{\mathbf{M}} \to \mathcal{N}_0^{\mathrm{red}}. \tag{4.31}$$

Here we denote again by $T_{\mathbf{M}}$ the corresponding scheme over $\overline{\mathbb{F}}_p$. By construction of the display $\mathcal{P}$ of Proposition 4.15, the morphism $\Phi_{\mathbf{M}}$ depends on the choice of the basis $e_1, \dots, f_3$ of $\mathbf{M}$. We denote by $\Phi_{\mathbf{M},i}$ the restriction of $\Phi_{\mathbf{M}}$ to the irreducible component $T_{\mathbf{M},i}$ of $T_{\mathbf{M}}$.

Let ξ be a k-rational point of $T_{\mathbf{M},i}$ and let $(\underline{a}, x, y) \in k^{p+3}$ be the coordinates of ξ with $\underline{a} = (a_0, \dots, a_p)$. We have $a_j = 0$ for $j \neq i$ and $y = \lambda_i^{-1}\mu_i x$. We identify $T_{\mathbf{M},i}$ with T_i via the isomorphism η_i of (4.15), *i.e.*, the element ξ corresponds to (a_i, b_i) with $b_i = x\lambda_i^{-1}$. Consider the set $\mathcal{S}_{\mathbf{M},\sigma(i)}(k) \subset \mathcal{V}(\Lambda_{\sigma(i)})(k)$ as in (4.7) and consider the relative Frobenius $\mathrm{Fr}_{T_{\mathbf{M},i}}\colon T_{\mathbf{M},i} \to T_{\mathbf{M},\sigma(i)}$ (Remark 4.5).

Lemma 4.16 *On k-rational points $\Phi_{\mathbf{M},i}$ induces a bijection between $T_{\mathbf{M},i}$ and $\mathcal{S}_{\mathbf{M},\sigma(i)}(k)$,*

$$\begin{aligned} \Phi_{\mathbf{M},i}\colon T_{\mathbf{M},i}(k) &\to \mathcal{S}_{\mathbf{M},\sigma(i)}(k), \\ (a_i, b_i) &\mapsto M_{a_i^p, b_i^p}, \end{aligned} \tag{4.32}$$

i.e., *we have on k-rational points $\Phi_{\mathbf{M},i} = \Psi_{\mathbf{M},\sigma(i)}(k) \circ \mathrm{Fr}_{T_{\mathbf{M},i}}$. In particular, the morphism $\Phi_{\mathbf{M},i}$ is universally injective.*

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Proof Denote by $(\mathcal{P}_\xi, \rho_\xi)$ the image of ξ under $\Phi_{\mathbf{M}}$. If ξ is equal to zero, *i.e.*, $\underline{a} = \underline{0}$ and $x = y = 0$, then $(\mathcal{P}_\xi, \rho_\xi)$ is equal to the superspecial point $\mathbf{M}$.

If we compare the formulas for P_ξ in (4.23) to (4.28) with the formulas of $M_{a_i^p, b_i^p} \in \mathcal{S}_{\mathbf{M},\sigma(i)}(k)$ in (4.5), we find that $P_\xi = M_{a_i^p, b_i^p}$ as sublattices of the isocrystal N_k. Therefore, $\Phi_{\mathbf{M},i}$ is given by (4.32) and is equal to the composition $\Psi_{\mathbf{M},\sigma(i)} \circ \mathrm{Fr}_{T_{\mathbf{M},i}}$ (Proposition 4.8). As $\mathrm{Fr}_{T_{\mathbf{M},i}}$ and $\Psi_{\mathbf{M},\sigma(i)}$ are universally bijective on k-rational points, the morphism (4.32) is universally bijective. ∎

Proposition 4.17 *The morphism $\Phi_{\mathbf{M}}$ is universally injective, and the tangent morphism at each closed point is injective.*

Proof By Lemma 4.16 the morphisms $\Phi_{\mathbf{M},i}$ are universally injective for each i. The Frobenius induces a bijection on the set $\tilde{\mathcal{J}}$ of (4.3), hence $\Lambda_{\sigma(i)} \neq \Lambda_{\sigma(j)}$ for $i \neq j$. By Lemma 4.3 and Remark 4.1, we obtain $\mathcal{V}(\Lambda_{\sigma(i)})(k) \cap \mathcal{V}(\Lambda_{\sigma(j)})(k) = \{\mathbf{M}\}$. Therefore, the images of two irreducible components of $T_{\mathbf{M}}$ intersect only at the superspecial point $\mathbf{M}$. This proves that $\Phi_{\mathbf{M}}$ is universally injective.

Now we want to show that Ψ is injective on tangent spaces. Let ξ be a k-rational point of $T_{\mathbf{M}}$ and let $(\mathcal{P}_\xi, \rho_\xi)$ be its image under $\Phi_{\mathbf{M}}$. We know that $(\mathcal{P}_\xi, \rho_\xi)$ has no nontrivial automorphisms. Let α, β be two elements of the tangent space of $T_{\mathbf{M}}$ at ξ. We denote by $(\mathcal{P}_\alpha, \rho_\alpha)$ and $(\mathcal{P}_\beta, \rho_\beta)$ the images of α and β under $\Phi_{\mathbf{M}}$, respectively. We assume that $\Phi_{\mathbf{M}}(\alpha) = \Phi_{\mathbf{M}}(\beta)$. Then there exists an $\mathcal{O}_E$-linear isomorphism $\varphi\colon \mathcal{P}_\alpha \xrightarrow{\sim} \mathcal{P}_\beta$ of polarized displays such that $\rho_\alpha = \varphi \circ \rho_\beta$. We must show that $\alpha = \beta$.

By (4.29) the modules P_α and P_β are graded $W(k[\epsilon])$-modules. As φ is $\mathcal{O}_E$-invariant, it preserves the grading. We denote by $\tilde{e}_{\alpha,i}, \tilde{f}_{\alpha,i}$ the basis of $\mathcal{P}_\alpha$ as in (4.29) and similarly for $\mathcal{P}_\beta$. With respect to these bases, φ is given by a matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathrm{GL}_6(W(k[\epsilon])). \tag{4.33}$$

The polarizations on P_α and P_β with respect to the above bases are given by the matrix (4.30). As φ respects the polarizations, we obtain ${}^t A B^\sigma = {}^t B A^\sigma = I_3$. Thus $A = A^{\sigma^2}$ and $B = B^{\sigma^2}$. In particular, the matrices A and B do not depend on ϵ, *i.e.*, A and B are elements of $\mathrm{GL}_3(W(k))$. As $\mathcal{P}_\alpha \otimes_{W(k[\epsilon])} W(k) \cong \mathcal{P}_\xi$ and $\mathcal{P}_\beta \otimes_{W(k[\epsilon])} W(k) \cong \mathcal{P}_\xi$, the base change of the isomorphism φ induces an isomorphism $\overline{\varphi}$ of $\mathcal{P}_\xi$. Since A and B do not depend on ϵ, the morphism $\overline{\varphi}$ is given by the matrix (4.33). Since $(\mathcal{P}_\xi, \rho_\xi)$ has no nontrivial automorphisms, this shows that $A = B = I_3$.

Now let $(\underline{a}, x, y) \in k^{p+3}$ be the coordinates of ξ. The computation of the tangent space of $T_{\mathbf{M}}$ (Proposition 4.11) shows that $da_i = 0$ for $i = 0, \dots, p$. Thus the coordinates of α and β are given by $(\underline{a}, x_\alpha, y_\alpha)$ and $(\underline{a}, x_\beta, y_\beta)$ in $(k[\epsilon])^{p+3}$, with $x_\alpha \equiv x_\beta \equiv x \bmod(\epsilon)$ and $y_\alpha \equiv y_\beta \equiv y \bmod(\epsilon)$. Since φ is equal to the identity, we obtain that $\tilde{f}_{\alpha,2} = \tilde{f}_{\beta,2}$, hence $x_\alpha = x_\beta$ and $y_\alpha = y_\beta$. Therefore, $\alpha = \beta$ which proves the claim. ∎

5 The Global Structure of $\mathcal{N}^{\mathrm{red}}$ for GU$(1, 2)$

We use the notation of Section 4. In this section we will construct a scheme $\mathcal{T}$ by gluing together open subsets of the varieties $T_{\mathbf{M}}$ for every superspecial point $\mathbf{M} \in \mathcal{L}_0^{(1)}$. We will prove that $\mathcal{T}$ is isomorphic to the supersingular locus $\mathcal{N}_0^{\mathrm{red}}$.

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

As the intersection behaviour of the sets $\mathcal{V}(\Lambda)(k)$ is given by the simplicial complex $\mathcal{B}_0$ of Section 3, we will inductively glue together the varieties $T_{\mathbf{M}}$ using the complex $\mathcal{B}_0$. For $\Lambda \in \mathcal{L}_0^{(l)}$ we will, for simplicity, write Λ instead of $\{\Lambda\}$ for the corresponding vertex of Λ in $\mathcal{B}_0$.

As in Remark 1.8, let C be the $\mathbb{Q}_{p^2}$-vector space N_0^τ with perfect skew-hermitian form $\{\,\cdot\,,\,\cdot\,\}$. Denote by G the unitary group of $(C, \{\,\cdot\,,\,\cdot\,\})$ with respect to the extension $\mathbb{Q}_{p^2}/\mathbb{Q}_p$.

Lemma 5.1 *Let $\mathbf{M} \neq \mathbf{M}'$ be in $\mathcal{L}_0^{(1)}$ such that the corresponding vertices in $\mathcal{B}_0$ have a common neighbour Λ (Definition 3.3). Then there exists an element $g \in G(\mathbb{Q}_p)$ such that $g\mathbf{M} = \mathbf{M}'$ and such that Λ is invariant under g.*

Proof Let $\mathcal{G}$ be the unitary group of $(\Lambda, \{\,\cdot\,,\,\cdot\,\})$ with respect to $\mathbb{Z}_{p^2}/\mathbb{Z}_p$. Then G is isomorphic to the generic fibre of $\mathcal{G}$. Let $V = \Lambda/p\Lambda$ and let $\pi\colon \Lambda \to V$ be the natural projection. Then $\mathcal{G}_{\mathbb{F}_p}$ is equal to the unitary group of $(V, (\,\cdot\,,\,\cdot\,))$. Since $\mathcal{G}$ is smooth over $\operatorname{Spec} \mathbb{Z}_p$, the canonical map $\varphi\colon \mathcal{G}(\mathbb{Z}_p) \to \mathcal{G}(\mathbb{F}_p)$ is surjective. As Λ is a common neighbour of $\mathbf{M}$ and $\mathbf{M}'$, the lattices $\mathbf{M}$ and $\mathbf{M}'$ are contained in $\mathcal{V}(\Lambda)(\mathbb{F}_{p^2})$ (Proposition 3.4). They correspond to subspaces U, U' of V of dimension 2 satisfying $U^\perp \subset U$ and $(U')^\perp \subset U'$, respectively. As $(\,\cdot\,,\,\cdot\,)$ is a nondegenerate skew-hermitian form on V, there exists an element $\bar{g} \in \mathcal{G}(\mathbb{F}_p)$ with $\bar{g}(U) = U'$. Let g be a lift of $\bar{g}$ in $\mathcal{G}(\mathbb{Z}_p)$. As $\mathbf{M} = \pi^{-1}(U)$ and $\mathbf{M}' = \pi^{-1}(U')$, the automorphism g of Λ satisfies the claim. ■

Let $\mathbf{M}, \mathbf{M}', \Lambda$ and g be as in Lemma 5.1. We fix a basis $e_1, \dots, f_3$ of $\mathbf{M}$ as in Lemma 4.2. Then Λ is equal to Λ_i for some i with $0 \le i \le p$. We define

$$\begin{aligned} e'_j &= ge_j && \text{for } j = 1,2,3, \\ f'_j &= F^{-1}(ge_j) && \text{for } j = 1,3, \\ f'_2 &= F^{-1}(pg(e_2)). \end{aligned} \tag{5.1}$$

Let $e'_{\lambda_i,\mu_i} = p^{-1}([\lambda_i]e'_1 + [\mu_i]e'_3)$ and let $\Lambda'_i = \langle e'_1, e'_2, e'_3, e'_{\lambda_i,\mu_i}\rangle$.

Lemma 5.2 *The elements $e'_1, \dots, f'_3$ form a basis of $\mathbf{M}'$ which satisfies the conditions of Lemma 4.2. Furthermore, Λ is equal to Λ'_i.*

Proof This follows from Lemma 5.1. ■

For $\mathbf{M} \in \mathcal{L}_0^{(1)}$ consider the affine variety $T_{\mathbf{M}}$ and its closed subvarieties $T_{\mathbf{M},i}$ as in (4.13) and (4.14). Geometrically, we will glue to each $\mathbb{F}_{p^2}$-rational point of $T_{\mathbf{M}}$ a variety $T_{\mathbf{M}'}$ such that $T_{\mathbf{M},i}$ and $T_{\mathbf{M}',i}$ coincide on the open subsets of non-$\mathbb{F}_{p^2}$-rational points. We will prove that the scheme $\mathcal{T}$ obtained by iterating this process is isomorphic to $\mathcal{N}_0^{\mathrm{red}}$.

Let $\mathcal{T}_{\mathbf{M}} = T_{\mathbf{M}} \setminus \{\mathbb{F}_{p^2}\text{-rational points } \neq 0\}$. It is an open subvariety of $T_{\mathbf{M}}$. Denote by $\mathcal{T}_{\mathbf{M},i}$ the intersection of $T_{\mathbf{M},i}$ with $\mathcal{T}_{\mathbf{M}}$. Then $\mathcal{T}_{\mathbf{M},i}$ is a closed subvariety of $\mathcal{T}_{\mathbf{M}}$. Furthermore, let $\mathcal{T}^\circ_{\mathbf{M},i} = \mathcal{T}_{\mathbf{M},i} \setminus \{0\} = T_{\mathbf{M},i} \setminus \{\mathbb{F}_{p^2}\text{-rational points}\}$. We will always identify $T_{\mathbf{M},i}$ with T_i via (4.15).

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Remark 5.3 The variety $\mathcal{T}^{\circ}_{\mathbf{M},i}$ is equal to $T_i^{\circ} = \operatorname{Spec} R_i^{\circ}$ with

$$R_i^{\circ} = \mathbb{F}_{p^2}[a_i, b_i, (a_i^{p^2-1} - 1)^{-1}, (b_i^{p^2-1} - 1)^{-1}]/(a_i^p \lambda_i^p + a_i \lambda_i - b_i^{p+1} \lambda_i^{p+1}),$$

where we denote the indeterminates of R_i by a_i, b_i instead of a, b.

We will now inductively glue together the varieties $\mathcal{T}_{\mathbf{M}}$ with $\mathbf{M} \in \mathcal{L}_0^{(1)}$ along the open subsets $\mathcal{T}^{\circ}_{\mathbf{M},i}$. We choose a starting point $\widehat{\mathbf{M}} \in \mathcal{L}_0^{(1)}$. For each $\mathbf{M} \in \mathcal{L}_0^{(1)}$, we denote by $u_{\mathbf{M}}$ the distance of $\mathbf{M}$ to $\widehat{\mathbf{M}}$ (Remark 3.8). Let $\mathcal{I}$ be the set of pairs $(\mathbf{M}, \mathbf{M}')$ such that $\mathbf{M}, \mathbf{M}' \in \mathcal{L}_0^{(1)}$ and such that $\mathbf{M}$ and $\mathbf{M}'$ have a common neighbour Λ. We assume that $u_{\mathbf{M}} < u_{\mathbf{M}'}$.

For the rest of this section we fix the following elements. For each $(\mathbf{M}, \mathbf{M}') \in \mathcal{I}$, we choose an element $g \in G(\mathbb{Q}_p)$ such that $g(\mathbf{M}) = \mathbf{M}'$ and $g(\Lambda) = \Lambda$, where Λ is the common neighbour of $\mathbf{M}$ and $\mathbf{M}'$. We choose a basis of $\widehat{\mathbf{M}}$ as in Lemma 4.2. For each $\mathbf{M} \in \mathcal{L}_0^{(1)}$ we choose inductively a basis $e_1, \dots, f_3$ of $\mathbf{M}$ satisfying the conditions of Lemma 4.2 such that for each $(\mathbf{M}, \mathbf{M}') \in \mathcal{I}$ the chosen basis of $\mathbf{M}'$ is given by (5.1). This is possible, as $\mathcal{B}_0$ is a tree by Proposition 3.7.

For $\mathbf{M} \in \mathcal{L}_0^{(1)}$ consider the morphism $\Phi_{\mathbf{M}} \colon T_{\mathbf{M}} \to \mathcal{N}_0^{\mathrm{red}}$ of (4.31) with respect to the fixed basis $e_1, \dots, f_3$ of $\mathbf{M}$. We denote by the same symbol $\Phi_{\mathbf{M}}$ its restriction to $\mathcal{T}_{\mathbf{M}}$. Let $\Phi_{\mathbf{M},i}$ be the restriction of $\Phi_{\mathbf{M}}$ to $\mathcal{T}_{\mathbf{M},i}$.

Let $(\mathbf{M}, \mathbf{M}') \in \mathcal{I}$ and let the common vertex Λ of $\mathbf{M}$ and $\mathbf{M}'$ be of the form $\Lambda = \Lambda_i$ with respect to the chosen bases of $\mathbf{M}$ and $\mathbf{M}'$.

Lemma 5.4 *There exists an isomorphism $\theta_g \colon T^{\circ}_{\mathbf{M},i} \xrightarrow{\sim} T^{\circ}_{\mathbf{M}',i}$ such that the restriction of the map $\Psi_{\mathbf{M},i}(k)$ of* (4.7) *to $T^{\circ}_{\mathbf{M},i}(k)$ is equal to $\Psi_{\mathbf{M}',i}(k) \circ \theta_g$.*

Proof Denote by $Y^{\circ}_{\Lambda_i}$ the open subvariety of Y_{Λ_i},

$$Y^{\circ}_{\Lambda_i} = Y_{\Lambda_i} \setminus \{\mathbb{F}_{p^2}\text{-rational points}\}.$$

Then the morphism (4.6) induces by Remark 4.7(i) an isomorphism $\delta_{\mathbf{M},i} \colon T_i^{\circ} \xrightarrow{\sim} Y^{\circ}_{\Lambda_i}$ and similarly for $\delta_{\mathbf{M}',i}$. Note that $\delta_{\mathbf{M},i}$ and $\delta_{\mathbf{M}',i}$ depend on the chosen bases of $\mathbf{M}$ and $\mathbf{M}'$ respectively. We use the notation of the proof of Lemma 5.1. The element $(\overline{g})^{-1} \in \mathcal{G}(\mathbb{F}_p)$ induces an isomorphism on Y_{Λ_i} that preserves the set of $\mathbb{F}_{p^2}$-rational points. There exists an automorphism θ_g of T_i° such that the following diagram of isomorphisms commutes

$$\begin{tikzcd}
Y^{\circ}_{\Lambda_i} \arrow[r, "(\overline{g})^{-1}"] & Y^{\circ}_{\Lambda_i} \arrow[r, "\overline{g}"] & Y^{\circ}_{\Lambda_i} \\
T_i^{\circ} \arrow[u, "\delta_{\mathbf{M},i}"] \arrow[r, "\theta_g"] & T_i^{\circ} \arrow[u, "\delta_{\mathbf{M},i}"'] \arrow[ur, "\delta_{\mathbf{M}',i}"'] &
\end{tikzcd}$$

The above triangle commutes by definition of g and the choice of the basis of $\mathbf{M}'$. Thus $\delta_{\mathbf{M},i} = \delta_{\mathbf{M}',i} \circ \theta_g$. As $T^{\circ}_{\mathbf{M},i}$ and $T^{\circ}_{\mathbf{M}',i}$ are equal to T_i° by Remark 5.3, we obtain an isomorphism $\theta_g \colon T^{\circ}_{\mathbf{M},i} \xrightarrow{\sim} T^{\circ}_{\mathbf{M}',i}$. The claim follows from diagram (4.7). ■

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Proposition 5.5 *Let $\theta_g^{(p)}$ be the Frobenius pullback of the isomorphism θ_g of Lemma* 5.4. *Then the diagram*

$$
\begin{tikzcd}
T^\circ_{\mathbf{M},\sigma(i)} \arrow[dd, "\theta_g^{(p)}", "\sim"'] \arrow[dr, "\Phi_{\mathbf{M},\sigma(i)}"] & \\
& \mathcal{N}_0^{\mathrm{red}} \\
T^\circ_{\mathbf{M}',\sigma(i)} \arrow[ur, "\Phi_{\mathbf{M}',\sigma(i)}"'] &
\end{tikzcd}
$$

commutes.

Proof It is sufficient to prove the claim on k-rational points. Lemma 4.16 shows that on k-rational points $\Phi_{\mathbf{M},\sigma(i)}$ is equal to $\Psi_{\mathbf{M},i}(k) \circ \mathrm{Fr}_{T^\circ_{\mathbf{M},\sigma(i)}}$ Similarly, we obtain $\Phi_{\mathbf{M}',\sigma(i)} = \Psi_{\mathbf{M}',i}(k) \circ \mathrm{Fr}_{T^\circ_{\mathbf{M}',\sigma(i)}}$. As $\theta_g^{(p)}$ is defined over $\mathbb{F}_{p^2}$, we obtain

$$\mathrm{Fr}_{T^\circ_{\mathbf{M}',\sigma(i)}} \circ \theta_g^{(p)} = \theta_g \circ \mathrm{Fr}_{T^\circ_{\mathbf{M},\sigma(i)}} .$$

Thus $\Phi_{\mathbf{M}',i} \circ \theta_g^{(p)} = \Psi_{\mathbf{M}',i}(k) \circ \theta_g \circ \mathrm{Fr}_{T^\circ_{\mathbf{M},\sigma(i)}}$, and the claim follows from Lemma 5.4. ■

Proposition 5.6 *There exists a reduced scheme $\mathcal{T}$ locally of finite type over $\mathbb{F}_{p^2}$ of dimension* 1 *and a morphism $\Phi \colon \mathcal{T} \to \mathcal{N}_0^{\mathrm{red}}$ that satisfies the following conditions:*

(i) *For each $\mathbf{M} \in \mathcal{L}_0^{(1)}$, the scheme $\mathcal{T}_{\mathbf{M}}$ can be identified with an open subscheme of $\mathcal{T}$ such that the restriction of Φ to $\mathcal{T}_{\mathbf{M}}$ is equal to $\Phi_{\mathbf{M}}$.*

(ii) *The open subschemes $\mathcal{T}_{\mathbf{M}}$ with $\mathbf{M} \in \mathcal{L}_0^{(1)}$ form an open covering of $\mathcal{T}$.*

(iii) *Let $\mathbf{M}, \mathbf{M}' \in \mathcal{L}_0^{(1)}$. The open subschemes $\mathcal{T}_{\mathbf{M}}$ and $\mathcal{T}_{\mathbf{M}'}$ of $\mathcal{T}$ intersect if and only if $\mathbf{M}$ and $\mathbf{M}'$ have a common neighbour in $\mathcal{B}_0$. In this case, there exists an integer i such that the intersection $\mathcal{T}_{\mathbf{M}} \cap \mathcal{T}_{\mathbf{M}'}$ is equal to the open subschemes $\mathcal{T}^\circ_{\mathbf{M},i}$ and $\mathcal{T}^\circ_{\mathbf{M}',i}$ of $\mathcal{T}_{\mathbf{M}}$ and $\mathcal{T}_{\mathbf{M}'}$ respectively.*

Proof We glue the schemes $\mathcal{T}_{\mathbf{M}}$ together along the open subsets $\mathcal{T}^\circ_{\mathbf{M},i}$ by induction over the distance $u_{\mathbf{M}}$ of $\mathbf{M}$ from $\widehat{\mathbf{M}}$. Let α be an even nonnegative integer. Assume that the schemes $\mathcal{T}_{\mathbf{M}}$ with $\mathbf{M}$ of distance $u_{\mathbf{M}} \leq \alpha$ have been glued together to a scheme $\mathcal{T}_\alpha$ with morphism Φ_α satisfying condition (i) and the condition corresponding to (iii) of the proposition. Let $\mathbf{M}' \in \mathcal{L}_0^{(1)}$ be of distance $\alpha + 2$. As $\mathcal{B}_0$ is a tree, there exists exactly one element $\mathbf{M} \in \mathcal{L}_0^{(1)}$ of distance α such that $\mathbf{M}$ and $\mathbf{M}'$ have a common neighbour Λ in $\mathcal{B}_0$. Let $\Lambda = \Lambda_i$ with respect to the chosen bases of $\mathbf{M}$ and $\mathbf{M}'$. We glue together $\mathcal{T}_\alpha$ and $\mathcal{T}_{\mathbf{M}'}$ along the open subschemes $\mathcal{T}^\circ_{\mathbf{M},\sigma(i)}$ and $\mathcal{T}^\circ_{\mathbf{M}',\sigma(i)}$, respectively, via the isomorphism $\theta_g^{(p)}$ of Proposition 5.5. The same proposition shows that the morphisms Φ_α and $\Phi_{\mathbf{M}}$ glue together to a morphism.

As $\mathcal{B}_0$ is a tree and the glueing is defined inductively, no cocycle condition has to be checked. ■

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

For $\Lambda \in \mathcal{L}_i^{(3)}$ let $\mathbf{M} \in \mathcal{L}_0^{(1)}$ be a neighbour of Λ in $\mathcal{B}_0$. Let $\Lambda = \Lambda_i$ with respect to the chosen basis of $\mathbf{M}$. Denote by $\mathcal{T}_\Lambda$ the closure of the open subvariety $\mathcal{T}_{\mathbf{M},i}$ in $\mathcal{T}$.

Proposition 5.7 *The variety $\mathcal{T}_\Lambda$ only depends on Λ and is isomorphic to Y_Λ. The scheme $\mathcal{T}$ is connected and the varieties $\mathcal{T}_\Lambda$ with $\Lambda \in \mathcal{L}_0^{(3)}$ are its irreducible components.*

Proof As $\mathcal{B}_0$ is a tree, the scheme $\mathcal{T}$ is connected. The claim follows from the construction of $\mathcal{T}$. ■

Denote by Φ_Λ the restriction of Φ to the irreducible component $\mathcal{T}_\Lambda$. As $\mathcal{T}_\Lambda$ is isomorphic to Y_Λ by Proposition 5.7, it is projective. By construction and Lemma 4.16, the morphism Φ_Λ is universally injective and the image of Φ_Λ on k-rational points is equal to $\mathcal{V}(\Lambda)(k)$. The moduli space $\mathcal{N}_0^{\mathrm{red}}$ is separated Remark 1.2, hence Φ_Λ is finite. We denote by $\mathcal{V}(\Lambda)$ the scheme theoretic image of Φ_Λ. To show that Φ_Λ induces an isomorphism onto $\mathcal{V}(\Lambda)$, we need the following lemma.

Lemma 5.8 *Let $f\colon X \to X'$ be a morphism of reduced schemes of finite type over an algebraically closed field k. We assume that f is finite, universally bijective and that the tangent morphism is injective at every closed point. Then f is an isomorphism.*

Proof Obviously, the morphism f is a homeomorphism. We will show that

$$f_x^\#\colon \mathcal{O}_{X',f(x)} \to \mathcal{O}_{X,x}$$

is surjective for every closed point x of X. Then f will be a closed immersion and, as X' is reduced, f will be an isomorphism.

We may assume that $X = \operatorname{Spec} R$ and $X' = \operatorname{Spec} R'$ are affine. Denote by $\mathfrak{m}$ a maximal ideal of R and by $\mathfrak{m}' = f(\mathfrak{m})$ its image in R'. The morphism

$$f_{R'_{\mathfrak{m}'}}\colon \operatorname{Spec}(R \otimes_{R'} R'_{\mathfrak{m}'}) \to \operatorname{Spec} R'_{\mathfrak{m}'}$$

is finite, hence $R \otimes_{R'} R'_{\mathfrak{m}'}$ is a semi-local ring. As $f_{R'_{\mathfrak{m}'}}$ is universally bijective, $R \otimes_{R'} R'_{\mathfrak{m}'}$ is a local ring and we obtain $R_\mathfrak{m} = R \otimes_{R'} R'_{\mathfrak{m}'}$. Thus $R_\mathfrak{m}$ is a finite $R'_{\mathfrak{m}'}$-module.

Furthermore, the tangent morphism at $\mathfrak{m}$ is injective, hence the morphism

$$\mathfrak{m}'/(\mathfrak{m}')^2 \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$$

is surjective. We obtain a surjective morphism $\hat{R}'_{\mathfrak{m}'} \twoheadrightarrow \hat{R}_\mathfrak{m}$. Since $R_\mathfrak{m}$ is a finite $R'_{\mathfrak{m}'}$-module, $\hat{R}_\mathfrak{m} = R_\mathfrak{m} \otimes_{R'_{\mathfrak{m}'}} \hat{R}'_{\mathfrak{m}'}$. Therefore, the morphism $f_\mathfrak{m}^\#\colon R'_{\mathfrak{m}'} \to R_\mathfrak{m}$ is surjective. ■

Theorem 5.9 *The morphism Φ is an isomorphism which induces an isomorphism of $\mathcal{T}_\Lambda$ onto $\mathcal{V}(\Lambda)$ for every $\Lambda \in \mathcal{L}_0^{(3)}$.*

Proof By Proposition 4.17 and Proposition 5.6, the tangent morphism of Φ_Λ is injective at every closed point. Therefore, the morphism Φ_Λ induces an isomorphism of $\mathcal{T}_\Lambda$ onto $\mathcal{V}(\Lambda)$ by Lemma 5.8.

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

Now we will prove that Φ is an isomorphism. By construction of Φ, the $\mathbb{F}_{p^2}$-rational points of $\mathcal{T}$ correspond to the superspecial points of $\mathcal{N}_0^{\mathrm{red}}$ and the intersection behaviour of the varieties $\mathcal{V}(\Lambda)$ is equal to the intersection behaviour of the irreducible components $\mathcal{T}_\Lambda$ of $\mathcal{T}$. Thus Φ is universally bijective and the varieties $\mathcal{V}(\Lambda)$ are the irreducible components of $\mathcal{N}_0^{\mathrm{red}}$. We obtain that Φ is an isomorphism locally at every point of $\mathcal{T}$ which is not $\mathbb{F}_{p^2}$-rational.

Let x be an $\mathbb{F}_{p^2}$-rational point of $\mathcal{T}$, and let $\mathbf{M} = \Phi(x)$ be the corresponding superspecial point. The variety $\mathcal{T}_{\mathbf{M}}$ is an open neighbourhood of x in $\mathcal{T}$. Denote by $\overline{\mathcal{T}}_{\mathbf{M}}$ the closure of $\mathcal{T}_{\mathbf{M}}$ in $\mathcal{T}$, and denote by $Z_{\mathbf{M}}$ the image of $\overline{\mathcal{T}}_{\mathbf{M}}$ under Φ, *i.e.*, the union of the varieties $\mathcal{V}(\Lambda)$ that contain $\mathbf{M}$. The induced morphism $\Phi|_{\overline{\mathcal{T}}_{\mathbf{M}}} \colon \overline{\mathcal{T}}_{\mathbf{M}} \to Z_{\mathbf{M}}$ is bijective and injective on tangent spaces, by Propositions 4.17 and 5.6. The morphism $\Phi|_{\overline{\mathcal{T}}_{\mathbf{M}}}$ is universally closed as its restriction to every irreducible component is finite, hence $\Phi|_{\overline{\mathcal{T}}_{\mathbf{M}}}$ is finite. By Lemma 5.8 it is an isomorphism, which proves the claim. ■

Let i be an even integer and let $\Lambda \in \mathcal{L}_i^{(3)}$. The isomorphism $\Psi_i \colon \mathcal{N}_i \xrightarrow{\sim} \mathcal{N}_0$ of (1.15) maps $\mathcal{V}(\Lambda)(k)$ to a set $\mathcal{V}(\Lambda')(k)$ for a lattice $\Lambda' \in \mathcal{L}_0^{(3)}$ (Remark 2.3(iv)). We denote by $\mathcal{V}(\Lambda)$ the preimage of the variety $\mathcal{V}(\Lambda')$ by Ψ_i.

Theorem 5.10 *The schemes $\mathcal{N}_i^{\mathrm{red}}$ with $i \in \mathbb{Z}$ even are the connected components of $\mathcal{N}^{\mathrm{red}}$ which are all isomorphic to each other. The varieties $\mathcal{V}(\Lambda)$ with $\Lambda \in \mathcal{L}_i^{(3)}$ are the irreducible components of $\mathcal{N}_i^{\mathrm{red}}$.*

The singular points of $\mathcal{N}^{\mathrm{red}}$ are the superspecial points. Each $\mathcal{V}(\Lambda)$ contains p^3+1 superspecial points and each superspecial point is the intersection of $p+1$ irreducible components $\mathcal{V}(\Lambda)$. Two irreducible components intersect transversally in at most one superspecial point, and the intersection graph of $\mathcal{N}_i^{\mathrm{red}}$ is a tree.

Each variety $\mathcal{V}(\Lambda)$ is isomorphic to the Fermat curve $\mathcal{C}$ (Remark 4.7). The superspecial points of $\mathcal{V}(\Lambda)$ correspond to the $\mathbb{F}_{p^2}$-rational points of $\mathcal{C}$.

The scheme $\mathcal{N}^{\mathrm{red}}$ is equidimensional of dimension 1 and of complete intersection.

Proof We first consider the case $i = 0$. The incidence relation of the varieties $\mathcal{V}(\Lambda)$ follows from Proposition 3.7. The geometric statements follow from Theorem 5.9, Proposition 5.7, and Lemma 4.14.

Now consider the general case. The varieties $\mathcal{V}(\Lambda)$ with $\Lambda \in \mathcal{L}_i^{(l)}$ correspond to the varieties $\mathcal{V}(\Lambda)$ with $\Lambda \in \mathcal{L}_0^{(l)}$ under the isomorphism Ψ_i. The claim follows from the case $i = 0$. ■

6 The Structure of the Supersingular Locus of $\mathcal{M}$ for $\mathrm{GU}(1,2)$

Let $\mathcal{M}$ be the moduli space of abelian varieties defined in the introduction. In this section, we carry over our results on the moduli space $\mathcal{N}$ to $\mathcal{M}$ in the case of $\mathrm{GU}(1,2)$.

We use the notation of the introduction. In particular, we now denote by E an imaginary quadratic extension of $\mathbb{Q}$ such that p is inert in E and denote by E_p the completion of E with respect to the p-adic topology. Consider the supersingular locus $\mathcal{M}^{\mathrm{ss}}$ of the special fibre $\mathcal{M}_{\mathbb{F}_{p^2}}$ of $\mathcal{M}$. It is a closed subscheme of $\mathcal{M}_{\mathbb{F}_{p^2}}$ which contains an $\overline{\mathbb{F}}_p$-rational point [BW, Lemma 5.2]. We view $\mathcal{M}^{\mathrm{ss}}$ as a scheme over $\overline{\mathbb{F}}_p$. We say that a point of $\mathcal{M}^{\mathrm{ss}}$ is superspecial if the underlying abelian variety is superspecial. Let $(A \otimes_{\mathbb{Z}}$

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

$\mathbb{Z}_{(p)}, \iota \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \overline{\lambda}, \overline{\eta})$ be an $\overline{\mathbb{F}}_p$-valued point of $\mathcal{M}^{ss}$ with corresponding p-divisible group $(\mathbb{X}, \iota, \lambda)$. Let $\mathcal{N}$ be the moduli space of quasi-isogenies of Definition 1.1 with respect to $(\mathbb{X}, \iota, \lambda)$.

Let J be the group of similitudes of the isocrystal N of $\mathbb{X}$ with additional structure as in (1.13). We denote by $I(\mathbb{Q})$ the group of quasi-isogenies in $\operatorname{End}_{\mathcal{O}_E}(A) \otimes \mathbb{Q}$ which respect the homogeneous polarization $\overline{\lambda}$. It is a subgroup of J. Using the level structure of A, one can define an injective morphism of $I(\mathbb{Q})$ into $G(\mathbb{A}_f^p)$ [RZ, 6.15]. By [RZ, Theorem 6.30], the set $I(\mathbb{Q})\backslash G(\mathbb{A}_f^p)/C^p$ is finite. Denote by $g_1, \dots, g_m \in G(\mathbb{A}_f^p)$ representatives of the different elements of $I(\mathbb{Q})\backslash G(\mathbb{A}_f^p)/C^p$. For every integer j with $1 \leq j \leq m$, let Γ_j be the group $\Gamma_j = I(\mathbb{Q}) \cap g_j^{-1} C^p g_j$. We view Γ_j as a subgroup of J.

We recall the uniformization theorem of Rapoport and Zink in case of GU$(1, 2)$. We will formulate this theorem only for the underlying schemes, not for the formal schemes.

There exists an isomorphism of schemes over $\operatorname{Spec} \overline{\mathbb{F}}_p$:

$$I(\mathbb{Q})\backslash(\mathcal{N}^{\mathrm{red}} \times G(\mathbb{A}_f^p)/C^p) \xrightarrow{\sim} \mathcal{M}^{ss}. \tag{6.1}$$

The left-hand side is isomorphic to the disjoint union of the quotients $\Gamma_j\backslash\mathcal{N}^{\mathrm{red}}$ for $1 \leq j \leq m$. Each group $\Gamma_j \subset J$ is discrete and cocompact modulo center. If C^p is small enough, Γ_j is torsion free.

Indeed, the proof of [RZ, Theorem 6.30] shows that (6.1) is an isomorphism if G satisfies the Hasse principle, *i.e.,* if the kernel of the Hasse map with respect to G,

$$\mathrm{H}^1(\mathbb{Q}, G) \to \prod_{v \text{ place of } \mathbb{Q}} \mathrm{H}^1(\mathbb{Q}_v, G), \tag{6.2}$$

is trivial. By [K2, §7], the kernel of (6.2) is equal to the kernel of the Hasse map with respect to the group $\operatorname{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$. But $\mathrm{H}^1(\mathbb{Q}, \operatorname{Res}_{E/\mathbb{Q}}(\mathbb{G}_m))$ is trivial, thus $\operatorname{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$, and hence G, satisfy the Hasse principle.

The decomposition into the schemes $\Gamma_j\backslash\mathcal{N}^{\mathrm{red}}$ follows from the proof of [RZ, Theorem 6.23]. The properties of the subgroups Γ_j are proved as well.

Let J^0 be the subgroup of J as in Remark 1.16(ii), and denote by Γ_j^0 the intersection of Γ_j with J^0. Consider the morphism $\Psi\colon \coprod_{j=1}^m \mathcal{N}^{\mathrm{red}} \to \mathcal{M}^{ss}$ induced by (6.1).

Theorem 6.1 *Let C^p be small enough. The morphism Ψ is surjective and an isomorphism locally at each point. The restriction of Ψ to each irreducible component $\mathcal{V}(\Lambda)$ of $\mathcal{N}^{\mathrm{red}}$ is a closed immersion, and the images of two irreducible components of $\mathcal{N}^{\mathrm{red}}$ in $\mathcal{M}^{ss}$ intersect in at most one superspecial point.*

Proof We must show that $\Psi_j\colon \mathcal{N}^{\mathrm{red}} \to \Gamma_j\backslash\mathcal{N}^{\mathrm{red}}$ is an isomorphism locally at each point for every integer j with $1 \leq j \leq m$. We use the notation of Remark 1.16. Let $g \in J$ and let $\alpha = v_p(c(g))$. By Remark 1.16(iii) the action of g defines an

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

isomorphism of $\mathcal{N}_i$ with $\mathcal{N}_{i+\alpha}$ for every integer i. Note that α is even (Lemma 1.17) and that $\mathcal{N}_i$ is empty if i is odd (Lemma 1.7). The schemes $\mathcal{N}_i^{\mathrm{red}}$ with i even are the connected components of $\mathcal{N}^{\mathrm{red}}$ (Theorem 5.10), which are all isomorphic to each other (Proposition 1.18). Thus we obtain

$$\Gamma_j \backslash \mathcal{N}^{\mathrm{red}} \xrightarrow{\sim} \coprod_{(\Gamma_j J^0)\backslash J} \Gamma_j^0 \backslash \mathcal{N}_0^{\mathrm{red}}. \tag{6.3}$$

In particular, the index of $(\Gamma_j J^0)$ in J is finite as $\mathcal{M}^{\mathrm{ss}}$ is of finite type over Spec $\overline{\mathbb{F}}_p$.

We now want to understand the action of Γ_j^0 on $\mathcal{N}_0^{\mathrm{red}}$. As in Remark 1.16(i), we view J as the group of similitudes of $(C, \{\cdot\, , \cdot\})$. For $l = 1, 3$ the group Γ_j^0 acts on the set $\mathcal{L}_0^{(l)}$ (Definition 2.2). We will show that the action of Γ_j^0 on $\mathcal{L}_0^{(l)}$ has no fixed points.

Indeed, for $\Lambda \in \mathcal{L}_0^{(l)}$ denote by Stab(Λ) the stabilizer of Λ in J^0. This is a compact open subgroup of J^0, hence $\Gamma_j^0 \cap \mathrm{Stab}(\Lambda)$ is finite. If C^p is small enough, Γ_j^0 has no torsion. Thus the intersection of Γ_j^0 with Stab(Λ) is trivial.

We have proved that each element of Γ_j^0 maps every irreducible component $\mathcal{V}(\Lambda)$ of $\mathcal{N}_0^{\mathrm{red}}$ with $\Lambda \in \mathcal{L}_0^{(3)}$ onto a different irreducible component. Furthermore, it fixes no superspecial points. As two irreducible components of $\mathcal{N}_0^{\mathrm{red}}$ intersect in at most one superspecial point, the action of Γ_j^0 on $\mathcal{N}_0^{\mathrm{red}}$ has no fixed points. Thus the morphism $\mathcal{N}_0^{\mathrm{red}} \to \Gamma_j^0 \backslash \mathcal{N}_0^{\mathrm{red}}$ is an isomorphism locally at every point.

The action of Γ_j^0 on $\mathcal{L}_0^{(1)}$ and $\mathcal{L}_0^{(3)}$ induces an action of Γ_j^0 on the simplicial complex $\mathcal{B}_0$ of Definition 3.1. As $\mathcal{B}_0$ describes the incidence relation of the irreducible components and the superspecial points of $\mathcal{N}_0^{\mathrm{red}}$, we can choose C^p small enough such that for every $g \in \Gamma_j^0$ and every $\Lambda \in \mathcal{L}_0$ the distance $u(\Lambda, g\Lambda)$ (Remark 3.8) is greater than or equal to 6. In this case, the restriction of the morphism $\mathcal{N}_0^{\mathrm{red}} \to \Gamma_j^0 \backslash \mathcal{N}_0^{\mathrm{red}}$ to every irreducible component is a closed immersion and the images of two irreducible components of $\mathcal{N}_0^{\mathrm{red}}$ in $\Gamma_j^0 \backslash \mathcal{N}_0^{\mathrm{red}}$ intersect in at most one point. This proves the theorem. ■

Corollary 6.2 *Let C^p be small enough. The supersingular locus $\mathcal{M}^{\mathrm{ss}}$ is locally isomorphic to $\mathcal{N}^{\mathrm{red}}$. It is equidimensional of dimension 1 and of complete intersection.*

The singular points of $\mathcal{M}^{\mathrm{ss}}$ are the superspecial points. Each superspecial point is the pairwise transversal intersection of $p+1$ irreducible components. The irreducible components are isomorphic to the Fermat curve $\mathcal{C}$ (Remark 4.7) and contain p^3+1 superspecial points. Two irreducible components intersect in at most one superspecial point.

Proof The claim follows from Theorem 6.1 and Theorem 5.10. ■

Proposition 6.3 *Let $C_{J,p}$ and $C'_{J,p}$ be maximal compact subgroups of J such that $C_{J,p}$ is hyperspecial and $C'_{J,p}$ is not hyperspecial,* i.e., *$C_{J,p}$ is the stabilizer in J of a lattice $\Lambda \in \mathcal{L}_0^{(3)}$ and $C'_{J,p}$ is the stabilizer in J of a lattice $M \in \mathcal{L}_0^{(1)}$. If $C^{J,p}$ is small enough, we*

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

have

$$\begin{aligned}
\#\{\textit{irreducible components of } \mathcal{M}^{ss}\} &= \#(I(\mathbb{Q})\backslash(J/C_{J,p} \times G(\mathbb{A}_f^p)/C^p)),\\
\#\{\textit{superspecial points of } \mathcal{M}^{ss}\} &= \#(I(\mathbb{Q})\backslash(J/C'_{J,p} \times G(\mathbb{A}_f^p)/C^p)),\\
\#\{\textit{connected components of } \mathcal{M}^{ss}\} &= \#(I(\mathbb{Q})\backslash(J^0\backslash J \times G(\mathbb{A}_f^p)/C^p))\\
&= \#(I(\mathbb{Q})\backslash(\mathbb{Z} \times G(\mathbb{A}_f^p)/C^p)).
\end{aligned}$$

All numbers are finite.

Proof All numbers are finite as $\mathcal{M}^{ss}$ is of finite type over Spec $\overline{\mathbb{F}}_p$.

The number of connected components follows from (6.1) and the decomposition (6.3), since $\mathcal{N}_0^{red}$ is connected.

To compute the number of irreducible components and superspecial points, we have to count these objects in $\Gamma_j^0\backslash\mathcal{N}_0^{red}$. The number of irreducible components is equal to the number of orbits of the action of Γ_j^0 on $\mathcal{L}_0^{(3)}$. The group J^0 acts transitively on $\mathcal{L}_0^{(3)}$ by Lemma 1.14, hence the number of irreducible components of $\Gamma_j^0\backslash\mathcal{N}_0^{red}$ is equal to $\#(\Gamma_j^0\backslash J^0/C_{J,p})$. An easy calculation shows the above formula.

A similar argument proves the claim in case of the superspecial points. ■

Remark 6.4 The numbers in Proposition 6.3 can be expressed in terms of class numbers.

Acknowledgements I want to thank everyone who helped me while writing this paper. Special thanks go to M. Rapoport for his advice and his interest in my work. I want to express my gratitude to U. Görtz and E. Viehmann for helpful discussions on this subject. I am indebted to T. Wedhorn for enduring all my questions. Furthermore, I thank S. S. Kudla, T. Zink and the referee for useful remarks on an earlier version of this paper.

References

[BG] E. Bachmat and E. Z. Goren, *On the non-ordinary locus in Hilbert-Blumenthal surfaces.* Math. Ann. **313**(1999), no. 3, 475–506. doi:10.1007/s002080050270

[BW] O. Bültel and T. Wedhorn, *Congruence relations for Shimura varieties associated to some unitary groups.* J. Inst. Math. Jussieu **5**(2006), 229–261.

[DL] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*. Ann. of Math. **103**(1976), no. 1, 103–161. doi:10.2307/1971021

[Gn] E. Z. Goren, *Hilbert modular varieties in positive characteristic.* In: The Arithmetic and Geometry of Algebraic Cycles. CRM Proc Lecture Notes 24, American Mathematical Society, Providence, RI, 2000, pp. 283–303.

[GO] E. Z. Goren and F. Oort, *Stratifications of Hilbert modular varieties*. J. Algebraic Geom. **9**(2000), no. 1, 111–154.

[Ka] C. Kaiser, *Ein getwistetes fundamentales Lemma für die* GSp_4. Bonner Mathematische Schriften 303. Universität Bonn, Mathematisches Institut, 1997.

[KO1] T. Katsura, F. Oort, *Supersingular abelian varieties of dimension two or three and class numbers*. In: Algebraic Geometry. Adv. Stud. Pure Math. 10, North-Holland, Amsterdam, 1987, pp. 253–281.

[KO2] ———, *Families of supersingular abelian surfaces.* Compositio Math. **62**(1987), no. 2, 107–167.

[Kb] N. Koblitz, *p-adic variation of the zeta-function over families of varieties defined over finite fields.* Compositio Math. **31**(1975), no. 2, 119–218.

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press

[K1] R. E. Kottwitz. *Stable trace formula: cuspidal tempered terms.* Duke Math. J. **51**(1984), no. 3, 611–650. doi:10.1215/S0012-7094-84-05129-9

[K2] ———, *Points on some Shimura varieties over finite fields.* J. Amer. Math. Soc. **5**(1992), 373–444. doi:10.2307/2152772

[KR1] S. S. Kudla and M. Rapoport, *Arithmetic Hirzebruch-Zagier cycles.* J. Reine Angew. Math. **515**(1999), 155–244.

[KR2] ———, *Cycles on Siegel threefolds and derivatives of Eisenstein series.* Ann. Sci. École Norm. Sup. **33**(2000), no. 5, 695–756.

[LO] K.-Z. Li and F. Oort, *Moduli of supersingular abelian varieties.* Lecture Notes in Mathematics 1680, Springer-Verlag, Berlin, 1998.

[Lu1] G. Lusztig, *Coxeter orbits and eigenspaces of Frobenius.* Invent. Math. **38**(1976), no. 2, 101–159. doi:10.1007/BF01408569

[Lu2] ———, *On the finiteness of the number of unipotent classes.* Invent. Math. **34** (1976), no. 3, 201–213. doi:10.1007/BF01403067

[Ri] M. Richartz: *Klassifikation von selbstdualen Dieudonnégittern in einem dreidimensionalen polarisierten supersingulären Isokristall.* Bonner Mathematische Schriften 311, Universität Bonn, 1998.

[RZ] M. Rapoport and T. Zink, *Period spaces for p-divisible groups.* Annals of Mathematics Studies 141, Princeton University Press, Princeton, NJ, 1996.

[St] H. Stamm, *On the reduction of the Hilbert-Blumenthal-moduli scheme with $\Gamma_0(p)$-level structure.* Forum Math. **9**(1997), no. 4, 405–455. doi:10.1515/form.1997.9.405

[Ti] J. Tits, *Reductive groups over local fields.* In: Automorphic Forms, Representations and L-Functions. Proc. Sympos. Pure Math. 33, American Mathematical Society, Providence, RI, 1979, pp. 29–69.

[Yu] C.-F. Yu, *On the supersingular locus in Hilbert-Blumenthal 4-folds.* J. Algebraic Geom. **12**(2003), no.4, 653–698.

[Zi1] T. Zink, *Cartiertheorie kommutativer formaler Gruppen.* Teubner-Texte zur Mathematik 68. B. G, Teubner Verlagsgessellschaft, Leipzig, 1984.

[Zi2] ———, *The display of a formal p-divisible group.* In: Cohomologies p-adiques et applications arithmétiques, I. Astérisque No. 278 (2002), 127–248.

Mathematisches Institut, Universität Bonn, Beringstr. 1, 53115 Bonn, Germany

(Current address) *c/o Torsten Wedhorn, Institut für Mathematik, Universität Paderborn, Warburger Str. 100, 33098 Paderborn, Germany*
e-mail: ivollaard@gmx.de

https://doi.org/10.4153/CJM-2010-031-2 Published online by Cambridge University Press