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A COUNTEREXAMPLE IN FINITE FIXED POINT THEORY

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This note answers a question raised by Lee Mohler in 1970, by exhibiting a finite topological space X which is the union of closed subspaces Y, Z, such that Y, Z, and $Y \cap Z$, but not X, have the fixed point property. The example is a triangulation Δ of S^3, the points of X being the simplices of Δ and the closed sets the subcomplexes of Δ. Corresponding examples in geometric simplicial complexes have been known since 1967 [1]. But the present question concerns a different property. It is easy to prove (and probably long known) that a finite complex X, considered as a finite space, has the fixed point property for continuous maps if and only if the geometric realization $|X|$ has the fixed point property for simplicial maps.

(**Proof.** Simplicial maps are continuous on X, and the barycenter of a fixed simplex is a fixed point in $|X|$. Conversely, given continuous $f: X \to X$, associate to each vertex v a vertex $g(v)$ of $f(v)$. The vertices of each simplex s will go to the vertices of a face of $f(s)$, so g is a simplicial map. Thus g must fix a point of $|X|$. If t is the carrier of such a point then t is a face of $f(t)$, and $f(t)$ of $f^2(t)$, and so on; some $f^n(t)$ is fixed.)

The crux of the example is $Y \cap Z$, which is a triangulation of a spherical shell $S^2 \times I$. Let U be an octahedral surface. Note, U has a fixed point free simplicial involution σ, and each vertex of U is on four edges. Form V from U by adding a vertex v in one of the triangular faces t and subdividing t into three triangles. Now (i) $|V|$ has the fixed point property for simplicial maps f. For, first, a simplicial automorphism must fix the vertex v of order 3. If f is not an automorphism, it is not surjective. In the 2-sphere $|V|$, this means f is null-homotopic and has a fixed point.

Let $U^{(1)}$ be the first barycentric subdivision of U. It has a fixed point free simplicial automorphism $\sigma^{(1)}$. Define a triangulation of $S^2 \times I$ which is V on $S^2 \times \{0\}$ and $U^{(1)}$ on $S^2 \times \{1\}$ as follows. Begin with the cell complex $U \times I$. Subdivide $U \times \{0\}$ to form V, and $U \times \{1\}$ to form $U^{(1)}$. For each of the twelve edges e of U, the boundary of $e \times I$ has 5 edges. Subdivide $e \times I$ into five triangles: a cone over the boundary. Do the same with the 8 triangular prisms $t \times I$ (t a triangle in U); subdivide each as a cone over its boundary. The

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resulting complex is $Y \cap Z$, and (ii) any simplicial map $f:|Y \cap Z| \rightarrow |Y \cap Z|$ has a fixed point. For V is the only subcomplex of $Y \cap Z$ that has only seven vertices and is not contractible in $Y \cap Z$. Either $f(|V|) = |V|$, and v is fixed, or the restriction of f to $|V|$ is null-homotopic. In the latter case, f is null-homotopic since $|V|$ is a deformation retract of $|Y \cap Z|$. If f had no fixed point, it would extend to a fixed point free map of a 3-ball.

Y is a 3-ball constructed from $Y \cap Z$ by adding a cone over V. The whole complex X is constructed from the union of two copies of Y by identifying their boundaries, two copies of $U^{(1)}$, by means of $\sigma^{(1)}$. Thus X has a fixed point free simplicial involution. But finally, Z consists of X minus one open cone over V; like Y, it is a ball and has the fixed point property.

REFERENCE

1. W. Lopez, *An example in the fixed point theory of polyhedra*, Bull. Amer. Math. Soc. **73** (1967), 922–924.

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