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RESEARCH ARTICLE

Whittaker coefficients of geometric Eisenstein series

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Abstract
Geometric Langlands predicts an isomorphism between Whittaker coefficients of Eisenstein series and functions on the moduli space of $\check{N}$-local systems. We prove this formula by interpreting Whittaker coefficients of Eisenstein series as factorization homology and then invoking Beilinson and Drinfeld's formula for chiral homology of a chiral enveloping algebra.

Contents

1. Introduction

1.1. Notation and conventions

Let G be a simply connected complex reductive group with Langlands dual group $\check{G}$ defined over $k := \mathbf{C}$. Choose a maximal torus T and a Borel subgroup B with unipotent radical N. Let ρ be half the sum of the positive coroots. Let X/k be a smooth projective complex genus g curve. Choose a square root of the canonical bundle on X and form the anticanonical T-bundle $\omega^{-\rho}$.

We work in the framework of [16]. In particular, all functors are derived and categories are by default presentable stable DG-categories.

Let σ be a $\check{T}$-local system on X, and let $\mathrm{Loc}^{\sigma}_{\check{N}} := \mathrm{Loc}_{\check{B}} \times_{\mathrm{Loc}_{\check{T}}} \sigma$ be the derived moduli stack of $\check{B}$-local systems on X whose underlying $\check{T}$-local system is identified with σ; see (1.3). A $\check{T}$-local system is called regular if for every coroot, the associated rank 1 local system is nontrivial. If σ is regular, then $\mathrm{Loc}^{\sigma}_{\check{N}}$ is a classical affine scheme isomorphic to a vector space.

Let K be the Hecke σ-eigensheaf on Bun_T whose stalk at $\omega^{-\rho}$ twisted by a negative coweight valued divisor $\underline{\lambda} \cdot \underline{x} = \sum \lambda_i x_i$ is

$$K_{\omega^{-\rho}(-\underline{\lambda}\cdot\underline{x})} \simeq \Big(\bigotimes \sigma_{x_i}^{-\lambda_i}\Big)[d_T + d^{\lambda}]. \tag{1.1}$$

Above, σ^{λ}_x means the fiber at x of the rank 1 local system $\sigma^{\lambda} := \sigma \times_{\check{T}} k_{\lambda}$. Here, $d_T := \dim \mathrm{Bun}_T$ and $d^{\lambda} := \langle 2\check{\rho}, 4(g-1)\rho + \lambda\rangle$ is the shift appearing in Section 6.4.8 of [13].

The Whittaker or Poincaré series sheaf $\mathrm{Whit} := r_! \chi^* D \exp \simeq r_!(-\chi)^* \exp[-2]$ on Bun_G is the pullback then pushforward of the exponential sheaf along

$$\mathbf{A}^1 \xleftarrow{\chi} \mathrm{Bun}_{N^-}^{\omega^{-\rho}} \xrightarrow{r} \mathrm{Bun}_G;$$

see 5.4.1 of [10]. Here, D denotes Verdier duality. The function χ is defined for example in [9]. The character sheaf exp on $\mathbf{A}^1$ is normalized so that its costalks are in degree zero. If F is a $\mathbf{G}_m$-constructible sheaf on $\mathbf{A}^1$, then $\mathrm{Hom}(\exp, F)[2]$ and $\mathrm{Hom}(D\exp, F)$ both calculate t-exact vanishing cycles of F. The Whittaker sheaf does not have nilpotent singular support.

The automorphic and spectral Eisenstein series functors, $\mathrm{Eis}_! := p_! q^*$ and $\check{\mathrm{E}}\mathrm{is} := \check{p}_*^{\mathrm{IndCoh}} \check{q}^{\mathrm{IndCoh}\,*}$, are defined by pullback then pushforward along

$$\mathrm{Bun}_T \xleftarrow{q} \mathrm{Bun}_B \xrightarrow{p} \mathrm{Bun}_G \quad \text{and} \quad \mathrm{Loc}_{\check{T}} \xleftarrow{\check{q}} \mathrm{Loc}_{\check{B}} \xrightarrow{\check{p}} \mathrm{Loc}_{\check{G}}.$$

All of the above functors are left adjoints. For example, $\check{p}_*^{\mathrm{IndCoh}}$ is defined because $\check{p}$ is schematic, and a left adjoint because $\check{p}$ is proper.

1.2. Main theorem statement

Write $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ for the DG-category of ind-constructible sheaves on Bun_G with singular support [20] in the global nilpotent cone [14]. Let $\mathrm{Loc}_{\check{G}}$ be the restricted moduli space of $\check{G}$-local systems on X [3]. Write $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$ for the DG-category of ind-coherent sheaves with nilpotent singular support [2].

The geometric Langlands conjecture is supposed to be compatible with parabolic induction. Moreover, the Whittaker functional is expected to correspond under Langlands to global sections on $\mathrm{Loc}_{\check{G}}$ (up to a shift by $d_G := \dim \mathrm{Bun}_G$). Thus, commutativity of conjectural (since this paper was written, a proof was announced) diagram

$$\begin{tikzcd} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_T) \arrow[r, phantom, "\simeq"] \arrow[d, "\mathrm{Eis}_!((\omega^{-\rho}-)[d^{\lambda}])"'] & \mathrm{QCoh}(\mathrm{Loc}_{\check{T}}) \arrow[d, "\check{\mathrm{E}}\mathrm{is}(-)"] \\ \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \arrow[r, phantom, "\simeq"] \arrow[dr, "\mathrm{Hom}(\mathrm{Whit},-)[d_G]"'] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}}) \arrow[d, "\Gamma^{\mathrm{IndCoh}}(-)"] \\ & \mathrm{Vect} \end{tikzcd} \tag{1.2}$$

applied to the skyscraper k_{σ} predicts the following isomorphism.

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Main Theorem 1.1. *Let σ be a $\check{T}$-local system on X and let K be the Hecke eigensheaf on* Bun_T *defined in (1.1). Whittaker coefficients of Eisenstein series equals functions on moduli space of $\check{N}$-local systems:*

$$\mathrm{Hom}(\mathrm{Whit}, \mathrm{Eis}_! K)[d_G] \simeq \mathcal{O}(\mathrm{Loc}^{\sigma}_{\check{N}}).$$

The proof uses a combination of [24] and [6] to relate twisted cohomology of the Zastava space to the formal completion of $\mathrm{Loc}^{\sigma}_{\check{N}}$.

Both sides of the main theorem are coweight graded vector spaces. On the automorphic side, let K^λ be the restriction to the degree $-\lambda - 2(g-1)\rho$ connected component Bun^{λ}_T. On the spectral side, the adjoint $\check{T}$-action on $\check{B}$ induces an action on $\mathrm{Loc}^{\sigma}_{\check{N}}$.

Remark 1.1. If we replace naive Eisenstein series by compactified Eisenstein series of [5], then geometric Langlands predicts that $\mathrm{Hom}(\mathrm{Whit}, \mathrm{Eis}_{!*} K')$ should equal global sections of a skyscraper sheaf at $\sigma \in \mathrm{Loc}_{\check{G}}$. This is verified by Gaitsgory in appendix B of [7].

1.3. Restricted, de Rham and Betti versions

Our results apply for all three versions of geometric Langlands [3]. On the automorphic side, $\mathrm{Eis}_! K$ is a constructible sheaf, equivalently regular holonomic D-module, with nilpotent singular support by [14]. On the spectral side, there are three versions of the moduli space of local systems, all having the same complex valued points. For a unipotent group,

$$\mathrm{Loc}^{\sigma,\mathrm{dR}}_{\check{N}} \simeq \mathrm{Loc}^{\sigma,\mathrm{restr}}_{\check{N}} \simeq \mathrm{Loc}^{\sigma,\mathrm{Betti}}_{\check{N}} \tag{1.3}$$

coincide by Proposition 4.3.3 and Section 4.8.1 of [3].

In the Betti setting, there is no exponential D-module. Because χ is $\mathbf{C}^\times$-equivariant for the 2ρ-action on $\mathrm{Bun}^{\omega^{-\rho}}_{N^-}$ and the weight 2 action on $\mathbf{A}^1$, the sheaf defined in 2.5.2 of [23] serves as a substitute.

1.4. Normalizations and shifts

First, we explain how the normalization (1.1) of the Hecke σ-eigensheaf K matches the normalization $\mathrm{Eis}_!((\omega^{-\rho}-)[d^\lambda])$ appearing in (1.2) (as in Section 4.1 of [11] or Section 6.4.8 [13]). Let $K' \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_T)$ correspond under class field theory to the skyscraper sheaf $k_\sigma \in \mathrm{QCoh}(\mathrm{Loc}_{\check{T}})$. The Hecke eigensheaf condition determines K' up to tensoring by a line. Whittaker normalization says that global sections of k_σ equals the costalk at the trivial T-bundle of $K'[d_T]$. Thus, K is only noncanonically isomorphic to a shift of K'. On the degree $-\lambda - 2(g-1)\rho$ connected component Bun^{λ}_T, there is a canonical identification $K \simeq \omega^{-\rho} K'[d^\lambda]$. We translated K' by $\omega^{-\rho}$ (having the effect of tensoring it by a certain line; see Section 4.1 of [11]).

Now we perform a consistence check. If σ is a regular, then Theorem 10.2 of [6] says that $\mathrm{Eis}_!(K^\lambda)[d_G - d^0_B]$ is perverse. The Whittaker functional $\mathrm{Hom}(\mathrm{Whit}, -)[d^0_B]$ is exact by [22] or [10], so the automorphic side of the main theorem is concentrated in degree 0. This is consistent with $\mathrm{Loc}^{\sigma}_{\check{N}}$ being a classical scheme if σ is regular. Here,

$$d^\lambda_B \coloneqq (g-1)\dim B + \langle 2\check{\rho}, \lambda + 2(g-1)\rho\rangle = \dim \mathrm{Bun}^\lambda_B \tag{1.4}$$

is the dimension of the degree $-\lambda - 2(g-1)\rho$ connected component.

1.5. Proof outline

It is convenient to take the coweight graded linear dual to avoid topological rings and because Lie algebra homology behaves better than Lie algebra cohomology. Here is the proof of our main theorem

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in one sentence:

$$\begin{aligned}\operatorname{Hom}(\mathrm{Whit},\mathrm{Eis}_! K)^*[-d_G] &\overset{(2.1)}{\simeq} \bigoplus_\lambda \operatorname{Hom}(\chi_Z^* D\exp, q_Z^! DK^\lambda)[d_T+d^0] \\ &\overset{(2.4)}{\simeq} \bigoplus_\lambda \Gamma(X^\lambda,\Upsilon_\sigma^\lambda) \overset{(2.7)}{\simeq} \Gamma(\mathrm{Ran}, C_\bullet(\check{\mathfrak{n}}_\sigma)) \overset{(2.10)}{\simeq} C_\bullet(\Gamma(X,\check{\mathfrak{n}}_\sigma)) \overset{(2.11)}{\simeq} \mathcal{O}(\mathrm{Loc}_{\check N}^\sigma)^*.\end{aligned} \tag{1.5}$$

In Section 2.1, we use [22] or [10] to exchange $\mathrm{Eis}_!$ for a right adjoint, then apply base change and a result of [1] to get a calculation on the Zastava space. In Section 2.2, we pushforward to the space of positive coweight valued divisors and, by Theorem 4.6.1 of [24], obtain a certain factorizable perverse sheaves Υ_σ^λ on X^λ.

In Section 2.3, we interpret Υ_σ in terms of the chiral enveloping algebra of $\check{\mathfrak{n}}_\sigma$ as in [6]. In Section 2.4, we explain, following [6], how the cohomology of Υ_σ equals factorization homology of $A := C_\bullet(\check{\mathfrak{n}}_\sigma)$. Beilinson and Drinfeld's formula says factorization homology of $C_\bullet(\check{\mathfrak{n}}_\sigma)$ is Lie algebra homology of $\Gamma(X,\check{\mathfrak{n}}_\sigma)$. In Section 2.5, we study moduli of $\check{\mathfrak{n}}$-local systems using deformation theory. Since $\Gamma(X,\check{\mathfrak{n}}_\sigma)$ is the shifted tangent complex of $\mathrm{Loc}_{\check N}^\sigma$, its Lie algebra homology is related the formal completion of $\mathrm{Loc}_{\check N}^\sigma$ at σ. Using that $\mathrm{Loc}_{\check N}^\sigma \simeq (\operatorname{Spec} R)/\check N$ is the quotient of an affine scheme by a unipotent group and using the contracting $\mathbf{G}_m$-action, we show that $C_\bullet(\Gamma(X,\check{\mathfrak{n}}_\sigma)) \simeq \mathcal{O}(\mathrm{Loc}_{\check N}^\sigma)^*$ is the graded linear dual ring of functions.

The idea of using factorization homology to study the formal completion of $\mathrm{Loc}_{\check N}^\sigma$ is from [6] and [12]. Proposition 3.4.4 of [12] (whose proof is omitted) implies an isomorphism $\bigoplus \Gamma(X^\lambda, D\Upsilon_\sigma^\lambda) \simeq \mathcal{O}(\mathrm{Loc}_{\check N}^\sigma)$. For σ regular, Propositions 11.3 and 11.4 of [6] give an isomorphism between $\prod \Gamma(X^\lambda,\Upsilon_\sigma^\lambda)^*$ and the completed ring of functions $\mathcal{O}(\mathrm{Loc}_{\check N}^\sigma)^\wedge$. Sections 2.3 and 2.4 review some of their arguments and do not contain new content apart from filling in some details. Our main contribution is in Section 2.5, where we extend the results of [6] to the more interesting case of irregular σ, and we obtain a formula for the ring of functions on $\mathrm{Loc}_{\check N}^\sigma$ (not just its formal completion) using the contracting $\mathbf{G}_m$-action.

2. Proof of the main theorem

2.1. Base change to Zastava

In this section, we interpret Whittaker coefficients of Eisenstein series as twisted cohomology of the Zastava space Z.

The fiber product $Z' := \mathrm{Bun}_B \times_{\mathrm{Bun}_G} \mathrm{Bun}_{N^-}^{\omega^{-\rho}}$ has a stratification indexed by the Weyl group, determined by the generic relative position of two flags. Let $j : Z \hookrightarrow Z'$ be the open inclusion of the locus where the two flags are generically transverse, called the Zastava space.

$$\begin{tikzcd} & & Z' \arrow[dl] \arrow[dr] & & \\ & \mathrm{Bun}_B \arrow[dl, "q"'] \arrow[dr, "p"] & & \mathrm{Bun}_{N^-}^{\omega^{-\rho}} \arrow[dl, "r"'] \arrow[dr, "\chi"] & \\ \mathrm{Bun}_T & & \mathrm{Bun}_G & & \mathbf{A}^1 \end{tikzcd}$$

Consider the compositions

$$q_{Z'} : Z' \to \mathrm{Bun}_B \to \mathrm{Bun}_T \quad \text{and} \quad \chi_{Z'} : Z' \to \mathrm{Bun}_{N^-}^{\omega^{-\rho}} \to \mathbf{A}^1$$

and let $q_Z := q_{Z'} j$ and $\chi_Z := \chi_{Z'} j$ be their restrictions to Z.

Proposition 2.1. *There is an isomorphism*

$$\operatorname{Hom}(\mathrm{Whit},\mathrm{Eis}_! K^\lambda)^*[-d_G] \simeq \operatorname{Hom}(\chi_Z^* D\exp, q_Z^! DK^\lambda)[d_T+d^0]. \tag{2.1}$$

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Proof. We cannot directly apply adjunction to calculate Whittaker coefficients of Eisenstein series because $\mathrm{Eis}_!$ is a left not right adjoint. It is shown in [10] and [22] that the shifted Whittaker functional $\mathrm{Hom}(\mathrm{Whit}, -)[d_B^0]$ on nilpotent sheaves commutes with Verdier duality D. This allows us to exchange $\mathrm{Eis}_! := p_! q^*$ for $\mathrm{Eis}_* := p_* q^!$. Then apply adjunction and base change to reduce to a calculation on the fiber product Z'.

$$\mathrm{Hom}(\mathrm{Whit}, \mathrm{Eis}_! K^\lambda)^*[-2d_B^0] \simeq \mathrm{Hom}(\mathrm{Whit}, \mathrm{Eis}_* DK^\lambda) \simeq \mathrm{Hom}(\chi_{Z'}^* D\exp, q_{Z'}^! DK^\lambda)$$

Finally, by Equation (3.5) of [1], restriction to the open generically transverse locus Z does not change the calculation. More precisely, the map

$$\mathrm{Hom}(\chi_{Z'}^* D\exp, q_{Z'}^! DK^\lambda) \xrightarrow{\sim} \mathrm{Hom}(\chi_Z^* D\exp, q_Z^! DK^\lambda)$$

is an isomorphism. For the shifts, use (1.4) and $d_G + d_T + d^0 = 2d_B^0$. □

2.2. Pushforward to the configuration space

In this section, we recall how to factor the projection $q_Z : Z^\lambda \to \mathrm{Bun}_T^\lambda$ through the configuration space X^λ of positive coweight valued divisors of total degree λ. Hence, we obtain a description of the λ-graded piece of Proposition 2.1 as cohomology of a certain perverse sheaf Υ_σ^λ on X^λ.

Let $(F, F^-, E) \in Z^\lambda$ be a point in the λ connected component of Zastava space – that is, a G-bundle E equipped with generically transverse B, B^--reductions F, F^-, such that F has degree $-\lambda - 2(g-1)\rho$, and $F^- \times_{B^-} T$ is identified with $\omega^{-\rho}$. For each dominant weight $\check\mu$, the Plucker description gives maps

$$F^{\check\mu} \to E^{\check\mu} \to (F^-)^{\check\mu} \simeq \omega^{-\langle\check\mu,\rho\rangle}. \tag{2.2}$$

Here, $F^{\check\mu} := F \times_B \mathbf{C}_{\check\mu}$ is a line bundle and $E^{\check\mu} := E \times_G V_{\check\mu}$ is the vector bundle associated to the simple G-module of highest weight $\check\mu$.

By the generic transversality condition, the composition (2.2) is nonzero map of line bundles, so λ is a non-negative coweight. For each point in the Zastava space, there is a unique positive coweight valued divisor $\underline{x} \cdot \underline{\lambda} \in X^\lambda$ such that (2.2) factors through an isomorphism $F^{\check\mu}(\langle \underline{x} \cdot \underline{\lambda}, \check\mu\rangle) \simeq \omega^{-\langle\check\mu,\rho\rangle}$. Since G is assumed simply connected, we can write $\lambda = \sum n_i\alpha_i$ as a sum of simple coroots and $X^\lambda = \prod X^{(n_i)}$ as a product of symmetric powers of the curve. Therefore, q_Z factors through a map π to the configuration space followed by the Abel-Jacobi map,

$$q_Z : Z^\lambda \xrightarrow{\pi} X^\lambda \xrightarrow{\mathrm{AJ}} \mathrm{Bun}_T^\lambda, \qquad (E, F, F^-) \mapsto \underline{x} \cdot \underline{\lambda} \mapsto \omega^{-\rho}(-\underline{x} \cdot \underline{\lambda}) \simeq F \times_B T.$$

Let λ be a coweight and $n := \langle\check\rho, \lambda\rangle$. Let $\check{\mathfrak{n}}_\sigma := \sigma \times_{\check T} \check{\mathfrak{n}}$, an $\check{\mathfrak{n}}$-local system on X. The Chevalley complex on the coweight graded Ran space gives a $\prod S_{n_i}$ equivariant perverse sheaf A_{X^n} on $\prod X^{n_i}$. Let $\mathrm{sym}^\lambda : X^n \to X^\lambda$ be the partial symmetrization map. There is a certain canonical summand $\Upsilon_\sigma^\lambda \subset (\mathrm{sym}_*^\lambda A_{X^n})^{\prod S_{n_i}}$ whose stalk at $\underline{x} \cdot \underline{\lambda} \in X^\lambda$ is

$$(\Upsilon_\sigma^\lambda)_{\underline{x}\cdot\underline{\lambda}} \simeq \bigotimes C_\bullet(\check{\mathfrak{n}}_\sigma)_{x_i}^{\lambda_i}; \tag{2.3}$$

see Section 3.1 of [6] and Section 4 of [24]. (The definition of Υ_σ^λ involves the Chevalley differential, but the associated graded of Υ_σ^λ with respect to the Cousin filtration is easier to describe; see Section 3.3 of [6].)

Remark 2.2. Since sym^λ is finite, $\mathrm{sym}_*^\lambda A_{X^n} \simeq \mathrm{sym}_!^\lambda A_{X^n}$ is perverse by Artin vanishing. Since $\prod S_{n_i}$-invariants is exact and commutes with taking (co)stalks, Υ_σ^λ is perverse.

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Proposition 2.3. *There is an isomorphism*

$$\mathrm{Hom}(\chi_Z^* D\exp, q_Z^! DK^\lambda)[d_T + d^0] \simeq \Gamma(X^\lambda, \Upsilon_\sigma^\lambda). \tag{2.4}$$

Proof. Pushing forward to the configuration space X^λ, the left of (2.4) becomes

$$\mathrm{Hom}(\pi_! \chi_Z^* D\exp, \mathrm{AJ}^! DK^\lambda)[d_T + d^0] \simeq \Gamma(\Upsilon^\lambda \otimes (\mathrm{AJ}^* K^\lambda)^*)[d_T + d^\lambda] \simeq \Gamma(X^\lambda, \Upsilon_\sigma^\lambda).$$

We used that the configuration space X^λ is smooth, so the dualizing sheaf is a rank 1 local system. And we used Theorem 4.6.1 of [24], which says that

$$D\pi_! \chi_Z^* D\exp \simeq \pi_* \chi_Z^! \exp \simeq \Upsilon^\lambda[d^\lambda - d^0].$$

Here, $d^\lambda - d^0 = \dim Z^\lambda$, and Υ^λ has stalks $\Upsilon^\lambda_{\underline{x}\cdot\underline{\lambda}} \simeq \bigotimes C_\bullet(\check{\mathfrak{n}})^{\lambda_i}$.

Under class field theory (1.1), the stalks of $\mathrm{AJ}^* K^\lambda$ are

$$(\mathrm{AJ}^* K^\lambda)_{\underline{\lambda}\cdot\underline{x}} \simeq \left(\bigotimes \sigma_{x_i}^{-\lambda_i}\right)[d_T + d^\lambda]$$

and its $*$-pullback to $\prod X^{n_i}$ is the $\prod S_{n_i}$ equivariant rank 1 local system $\boxtimes(\sigma^{-\alpha_i})^{\boxtimes n_i}$. By the projection formula, $\Upsilon^\lambda \otimes (\mathrm{AJ}^* K^\lambda)^* \simeq \Upsilon_\sigma^\lambda$. □

Combining Propositions 2.1 and 2.3 shows Whittaker coefficients of Eisenstein series is graded dual to global sections of Υ_σ on the configuration space.

2.3. The chiral enveloping algebra as a Chevalley complex

The local system $\check{\mathfrak{n}}_\sigma$ determines a Lie* algebra on the Ran space. Its Lie algebra homology $A := C_\bullet(\check{\mathfrak{n}}_\sigma)$ is a factorization algebra, related to Υ_σ by partial symmetrization (2.6).

A sheaf on the Ran space of X is a collection of sheaves A_{X^I} on each power of the curve X^I, together with compatibility isomorphisms for !-restrictions along partial diagonal maps; see Section 2.1 of [8] for the precise definition. Recall from Section 1.2.1 of [8] that the category of sheaves on the Ran space admits two tensor products with a map $\otimes^* \to \otimes^{\mathrm{ch}}$ between them.

Pushing forward along the main diagonal $\Delta : X \to \mathrm{Ran}$, we can regard $\Delta_* \check{\mathfrak{n}}_\sigma \in \mathrm{Shv}(\mathrm{Ran})$ as a Lie algebra for the $*$-tensor product. Restricting to X^2, the Lie* bracket $(\Delta_*\check{\mathfrak{n}}_\sigma \otimes^* \Delta_*\check{\mathfrak{n}}_\sigma)_{X^2} \simeq \check{\mathfrak{n}}_\sigma \boxtimes \check{\mathfrak{n}}_\sigma \to (\Delta_*\check{\mathfrak{n}}_\sigma)_{X^2} \simeq \Delta_*\check{\mathfrak{n}}_\sigma$ comes by adjunction from the Lie bracket.

Let $A := C_\bullet(\check{\mathfrak{n}}_\sigma) \in \mathrm{Shv}(\mathrm{Ran})$ be Lie algebra homology of $\Delta_*\check{\mathfrak{n}}_\sigma$ with respect to the $*$-tensor product, viewed by the forgetful functor as a cocommutative coalgebra with respect to the ch-tensor product. Proposition 6.1.2 of [8] says that A corresponds to the chiral enveloping algebra of $\Delta_*\check{\mathfrak{n}}_\sigma$ under the equivalence between factorization and chiral algebras.

The Chevalley complex $A = \bigoplus A^\lambda$ is coweight graded because $\mathrm{Sym}(\check{\mathfrak{n}}_\sigma[1])$ is coweight graded and because the Chevalley differential preserves the grading. Choose a coweight λ and let $n := \langle \check{\rho}, \lambda\rangle$. The sheaf $A^\lambda_{X^n}$ on X^n is S_n-equivariant and perverse. Symmetrize it along $\mathrm{sym} : X^n \to X^{(n)}$ to get a perverse sheaf $(\mathrm{sym}_* A^\lambda_{X^n})^{S_n}$ on the nth symmetric power. (In other words, we pushed forward $A^\lambda_{X^n}$ from the stack quotient X^n/S_n to the coarse quotient $X^{(n)}$.)

Now we describe a certain canonical summand $A^\lambda_{X^{(n)}} \subset (\mathrm{sym}_* A^\lambda_{X^n})^{S_n}$ defined in Section 3 of [6]. Let $X_i^{(n)} \subset X^{(n)}$ be the space of effective degree n divisors supported at exactly i points. The !-restriction of $(\mathrm{sym}_* A^\lambda_{X^n})^{S_n}$ to $X_i^{(n)}$ is a local system whose stalk at a divisor $\underline{n}\cdot\underline{x} \in X_i^{(n)}$ is given by

$$(\mathrm{sym}_* A^\lambda_{X_i^n})^{S_n}_{\underline{n}\cdot\underline{x}} \simeq \bigoplus_{\lambda = \sum \lambda_j} \bigotimes C_\bullet(\mathfrak{n}_\sigma)^{\lambda_j}_{x_j}.$$

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The !-restriction of $A^{\lambda}_{X^{(n)}}$ to $X_i^{(n)} \subset X^{(n)}$ is the summand whose stalks are

$$(A^{\lambda}_{X_i^{(n)}})_{\underline{n}\cdot\underline{x}} \simeq \bigoplus_{\substack{\lambda=\sum \lambda_j,\\ \langle\check{\rho},\lambda_j\rangle=n_j}} \bigotimes C_\bullet(\check{\mathfrak{n}}_\sigma)^{\lambda_j}_{x_j}. \tag{2.5}$$

By Section 11.6 of [6], the pushforward of $\Upsilon^{\lambda}_{\sigma}$ – see (2.3) – along the partial symmetrization map ${}^{\lambda}\operatorname{sym} : X^{\lambda} \to X^{(n)}$ is

$$^{\lambda}\mathrm{sym}_* \Upsilon^{\lambda}_{\sigma} \simeq A^{\lambda}_{X^{(n)}}. \tag{2.6}$$

2.4. Factorization homology

In this section, we review, following [6], how factorization homology of $A^{\lambda} := C_\bullet(\check{\mathfrak{n}}_\sigma)^{\lambda}$ can be computed as cohomology on the symmetric power $X^{(n)}$, where $n := \langle\check{\rho}, \lambda\rangle$.

Let FSet be the category whose objects are finite nonempty sets and whose morphisms are surjective maps. For each surjection $J \twoheadrightarrow I$, there is a partial diagonal map $\Delta : X^I \to X^J$. A sheaf on the Ran space comes with isomorphisms $A_{X^I} \simeq \Delta^! A_{X^J}$, so adjunction gives maps $\Delta_* A_{X^I} \to A_{X^J}$. Factorization homology is defined in Section 6.3.3 of [8] or Section 4.2.2 of [4] as the colimit over these maps

$$\Gamma(\mathrm{Ran}, A) \simeq \operatorname*{colim}_{\mathrm{FSet}^{\mathrm{op}}} \Gamma(A_{X^I}).$$

The following proposition is stated in 11.6 of [6], and below, we fill in the proof using the Cousin filtration and ideas from Section 4.2 of [4].

Proposition 2.4. *The cohomology of Υ_σ – see (2.3) – is the factorization homology of the Chevalley complex,*

$$\bigoplus_{\lambda} \Gamma(X^{\lambda}, \Upsilon^{\lambda}_{\sigma}) \simeq \Gamma(\mathrm{Ran}, A). \tag{2.7}$$

Proof. Equation (2.6) relates Υ_σ to the symmetrization of A. Thus, it suffices to show that

$$\Gamma(X^{\lambda}, \Upsilon^{\lambda}_{\sigma}) \simeq \Gamma(A^{\lambda}_{X^{(n)}}) \to \Gamma(A^{\lambda}_{X^n}) \to \Gamma(\mathrm{Ran}, A^{\lambda}) \tag{2.8}$$

is an isomorphism for $n := \langle\check{\rho}, \lambda\rangle$. Indeed, we will prove that (2.8) is compatible with the Cousin filtration and that it induces an isomorphism on the associated graded pieces.

Consider the filtration on (2.8) whose $\leq i$th filtered piece consists of sections supported on the partial diagonals of dimensions $\leq i$. The ith graded piece is

$$\Gamma(A^{\lambda}_{X_i^{(n)}}) \to \Gamma(A^{\lambda}_{X_i^n}) \to \operatorname*{colim}_{\mathrm{FSet}^{\mathrm{op}}} \Gamma(A^{\lambda}_{X_i^I}) \simeq \mathrm{gr}_i \Gamma(\mathrm{Ran}, A^{\lambda}). \tag{2.9}$$

Here, $A^{\lambda}_{X_i^{(n)}}$ is the !-restriction of $A^{\lambda}_{X^{(n)}}$ to the space $X_i^{(n)} \subset X^{(n)}$ of effective degree n divisors supported at exactly i points. Similarly, $A^{\lambda}_{X_i^I}$ is the !-restriction of $A^{\lambda}_{X^I}$ to the space $X_i^I \subset X^I$ of I-tuples supported at exactly i points.

The symmetric group S_i acts freely on the space $X_i^i \subset X^i$ of distinct i-tuples of points. By Section 4.2.3 of [4], the ith graded piece of the factorization homology of A^{λ} is $\mathrm{gr}_i\, \Gamma(\mathrm{Ran}, A^{\lambda}) \simeq \Gamma(A^{\lambda}_{X_i^i})_{S_i}$.

The connected components of $X_i^{(n)}$ are indexed by partitions $\underline{n} = n_1 + \ldots n_i$. Also, the local system $A^{\lambda}_{X_i^i}$ splits as a direct sum indexed by such partitions; see 6.4.9 of [8]. Restricting (2.9) to the connected

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component $X_i^{(\underline{n})} \subset X_i^{(n)}$ indexed by a certain partition,

$$\Gamma(A^\lambda_{X_i^{(\underline{n})}}) \to \mathrm{gr}_i \Gamma(\mathrm{Ran}, A^\lambda) \simeq \Gamma(A^\lambda_{X_i^i})_{S_i}$$

is an isomorphism onto the corresponding summand of $\Gamma(A^\lambda_{X_i^i})_{S_i}$ by (2.5). Summing over partitions shows that the ith graded piece of (2.8) is an isomorphism. □

Since the factorization algebra $A := C_\bullet(\check{\mathfrak{n}}_\sigma)$ corresponds to the chiral enveloping algebra $U(\check{\mathfrak{n}}_\sigma)$, Beilinson and Drinfeld's formula for chiral homology of an enveloping algebra – see Theorem 4.8.1.1 of [4] or 6.4.4 of [8] – says

$$\Gamma(\mathrm{Ran}, A) \simeq C_\bullet(\Gamma(X, \check{\mathfrak{n}}_\sigma)). \tag{2.10}$$

2.5. *Deformation theory*

In this section, we show that

$$C_\bullet(\Gamma(X, \check{\mathfrak{n}}_\sigma)) \simeq \mathcal{O}(\mathrm{Loc}^\sigma_{\check{N}})^*, \tag{2.11}$$

Lie algebra homology of the shifted tangent complex equals the graded dual ring of functions on $\mathrm{Loc}^\sigma_{\check{N}}$. Deformation theory says that $C_\bullet(\Gamma(X, \check{\mathfrak{n}}_\sigma)) \simeq \Gamma^{\mathrm{IndCoh}}(\omega_{(\mathrm{Loc}^\sigma_{\check{N}})^\wedge})$ is global sections of the dualizing sheaf on the formal completion at σ. Using the structure of $\mathrm{Loc}^\sigma_{\check{N}}$ described below, we recover the graded dual ring of functions on $\mathrm{Loc}^\sigma_{\check{N}}$, not just its completion, from $\Gamma^{\mathrm{IndCoh}}(\omega_{(\mathrm{Loc}^\sigma_{\check{N}})^\wedge})$.

First, we show that $\mathrm{Loc}^\sigma_{\check{N}} \simeq \mathrm{Loc}^{\sigma,x}_{\check{N}}/\check{N}$ is the quotient by a unipotent group of an affine derived scheme with a contracting $\mathbf{G}_m$-action. Let $\mathrm{Loc}^x_{\check{B}} := \check{B}^{2g} \times_{\check{B}} 1$ (respectively, $\mathrm{Loc}^x_{\check{T}} := \check{T}^{2g} \times_{\check{T}} 1$) be the Betti moduli of $\check{B}$ (respectively, $\check{T}$) local systems trivialized at a point x. Let $\mathrm{Loc}^{\sigma,x}_{\check{N}} := \mathrm{Loc}^x_{\check{B}} \times_{\mathrm{Loc}^x_{\check{T}}} \sigma$ be the moduli of $\check{B}$-local systems with underlying $\check{T}$-local system identified with σ, plus a $\check{T}$-reduction at x.

Since $\check{T}$ is abelian, it acts by automorphisms on $\sigma \in \mathrm{Loc}_{\check{T}}$ so there is a canonical lift $\sigma \in \mathrm{Loc}^x_{\check{T}}$. We also sometimes regard σ as a point in $\mathrm{Loc}^{\sigma,x}_{\check{N}}$ via the inclusion $\check{T} \subset \check{B}$.

Let $\check{B}$ act on $\mathrm{Loc}^x_{\check{B}}$ by changing the trivialization at x, equivalently by the adjoint action on $\check{B}^{2g} \times_{\check{B}} 1$. Restricting the adjoint action along $\check{\rho}$ gives a $\mathbf{G}_m$-action that contracts $\check{B}$ to $\check{T}$. Thus, we expect a $\mathbf{G}_m$-action that contracts $\mathrm{Loc}^x_{\check{B}}$ to $\mathrm{Loc}^x_{\check{T}}$, as is made precise below.

Proposition 2.5. *The moduli space* $\mathrm{Loc}^{\sigma,x}_{\check{N}} \simeq \mathrm{Spec}\, R$ *is a finite type affine scheme with a* $\check{B}$*-action. Restricting the action along* $\check{\rho}$ *gives a non-negative grading* $R = \bigoplus_{n\geq 0} R_n$ *such that* $\sigma \simeq \mathrm{Spec}(R/R_{>0})$ *is cut out by the ideal of strictly positively graded functions.*

Proof. We argue in the Betti setting, but the restricted and de Rham versions also follow by (1.3). First, rewrite

$$\mathrm{Loc}^{\sigma,x}_{\check{N}} \simeq \mathrm{Loc}^x_{\check{B}} \times_{\mathrm{Loc}^x_{\check{T}}} \sigma \simeq \check{B}^{2g} \times_{\check{T}^{2g} \times_{\check{T}} \check{B}} \sigma \simeq (\check{B}^{2g} \times_{\check{T}^{2g}} \sigma) \times_{\check{B} \times_{\check{T}} 1} 1 \simeq \mathrm{Spec}(R' \otimes_S k). \tag{2.12}$$

The contracting $\check{\rho}$-action induces non-negative gradings on the classical rings $R' := \mathcal{O}(\check{B}^{2g} \times_{\check{T}^{2g}} \sigma)$ and $S := \mathcal{O}(\check{B} \times_{\check{T}} 1) \simeq \mathcal{O}(\check{N})$. Since $\check{N}$ is smooth, the augmentation module $k \simeq S/S_{>0}$ admits a finite graded resolution by free S-modules, with all but one term shifted into strictly positive $\check{\rho}$-gradings. Therefore, $R \simeq R' \otimes_S k$ is a finite type non-negatively graded ring and $\sigma \simeq \mathrm{Spec}\, R/R_{>0}$. □

Now we review some derived deformation theory. Let $Y^\wedge$ be the formal completion of a derived stack Y at a point σ. The shifted tangent bundle $T_\sigma Y[-1]$ is a DG Lie algebra whose enveloping algebra is endomorphisms of the skyscraper at σ. By Chapter 7 of [17] or Remark 2.4.2 of [21], there is an equivalence

$$\mathrm{Mod}(T_\sigma Y[-1]) \simeq \mathrm{IndCoh}(Y^\wedge)$$

between Lie algebra modules for the shifted tangent complex and indcoherent sheaves on the formal completion. Let $p : Y^\wedge \to \mathrm{pt}$ be the map to a point. By Chapter 7, Section 5.2 of [17], the trivial $T_\sigma Y[-1]$-module corresponds to the dualizing sheaf $\omega_{Y^\wedge} \simeq p^! k \in \mathrm{IndCoh}(Y^\wedge)$. Moreover, Lie algebra homology corresponds to global sections

$$C_\bullet(T_\sigma Y[-1]) \simeq \Gamma^{\mathrm{IndCoh}}(\omega_{Y^\wedge}). \tag{2.13}$$

Suppose $Y^\wedge \simeq \mathrm{Spec}\, R$ is the spectrum of an Artinian local ring R. By properness, $p^!$ is right adjoint to p_*^{IndCoh}. Therefore, the dualizing complex $\omega_{Y^\wedge} \simeq R^*$ is the linear dual of R viewed as an R-module.

Suppose $Y^\wedge \simeq \mathrm{Spf}\, R^\wedge \simeq \mathrm{colim}\, Y_n$ where $Y_n \simeq \mathrm{Spec}\, R/\mathfrak{m}^n$ and let $i_n : Y_n \to Y^\wedge$. Since $Y_n \to Y_{n+1}$ is proper, $\mathrm{IndCoh}(Y^\wedge)$ is the colimit under $*$-pushforward of $\mathrm{IndCoh}(Y_n)$; see Chapter 1, Proposition 2.5.7 of [16]. The dualizing sheaf can be written as a colimit, $\omega_{Y^\wedge} \simeq \mathrm{colim}\, i_{n*}^{\mathrm{IndCoh}} \omega_{Y_n}$; see Chapter 7, Corollary 5.3.3 of [17]. Since $\Gamma^{\mathrm{IndCoh}}(Y^\wedge, -)$ is continuous, it follows that

$$\Gamma^{\mathrm{IndCoh}}(\omega_{Y^\wedge}) \simeq \mathrm{colim}((R/\mathfrak{m}^n)^*) \simeq (R^\wedge)^* \tag{2.14}$$

is the *topological* dual of the completed local ring $R^\wedge$. In this case, Equation (2.13) is Corollary 5.2 of [18].

Proposition 2.6. *Let $R = \bigoplus_{n\geq 0} R_n$ be a non-negatively graded finite type derived ring with $R_0 \simeq k$. Let $R^\wedge$ be the formal completion with respect to the ideal of positively graded functions. Then the graded dual $R^* := \bigoplus R_n^*$ equals the topological dual of the completion $(R^\wedge)^*$.*

Proof. First, suppose R is classical and choose homogeneous generators $f_1, \dots f_r \in R$. Let d be the maximum of their degrees, so $R_{\geq dn} \subset (f_1, \dots f_r)^n \subset R_{\geq n}$. Therefore, the graded dual R^* (linear functionals that vanish on some $R_{\geq n}$) equals the topologogical dual $(R^\wedge)^*$ (linear functionals that vanish on some $(f_1, \dots f_r)^n$).

Now suppose that R is derived. The finite type assumption means that after taking cohomology, $H^\bullet(R)$ is a finitely generated module over $H^0(R)$, a finitely generated graded classical ring. Choose a finite collection of homogeneous elements $f_1, \dots f_r \in R$ whose images generate $H^0(R)$.

The formal completion is the topological ring

$$R^\wedge \simeq R \otimes_{k[f_1,\dots f_r]} k[[f_1, \dots f_r]] \simeq \lim_n R \otimes_{k[f_1,\dots f_r]} (k[f_1, \dots f_r]/k[f_1, \dots f_r]_{>n}).$$

For the first equality, see Section 6.7 of [15]. The second equality uses that fiber products commute with filtered colimits and that $k[[f_1, \dots f_r]] \simeq \lim(k[f_1, \dots f_r]/k[f_1, \dots f_r]_{>n})$. (The formal completion of a classical positively graded polynomial algebra can be computed using the grading filtration.)

Since $k[f_1, \dots f_r]$ is smooth, $R \otimes_{k[f_1,\dots f_r]} k[f_1, \dots f_r]/k[f_1, \dots f_r]_{>n}$ has finite dimensional cohomology and therefore is concentrated in bounded degrees. Hence, for m sufficiently large, the quotient map factors through

$$R \to R/R_{>m} \to R \otimes_{k[f_1,\dots f_r]} k[f_1, \dots f_r]/k[f_1, \dots f_r]_{>n} \to R/R_{>n}.$$

Therefore, the formal completion of R can be computed using the grading filtration

$$R^\wedge \simeq \lim_n R \otimes_{k[f_1,\dots f_r]} (k[f_1, \dots f_r]/k[f_1, \dots f_r]_{>n}) \simeq \lim_n R/R_{>n}.$$

Taking the topological dual proves $(R^\wedge)^* \simeq \mathrm{colim}((R/R_{>n})^*) \simeq \bigoplus R_n^* \simeq R^*$. □

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The following proposition shows (2.11), completing the final step of (1.5) and the proof of the main theorem.

Proposition 2.7. *Lie algebra homology of the shifted tangent complex of $Y := \mathrm{Loc}^{\sigma}_{\check{N}}$ equals the graded dual of the ring of functions,*

$$C_\bullet(T_\sigma Y[-1]) \simeq \mathcal{O}(Y)^*.$$

Proof. Write $\mathrm{Loc}^{\sigma,x}_{\check{N}} \simeq \mathrm{Spec}\, R$ as in Proposition 2.5. Let $\check{N}$ act by changing the $\check{T}$-reduction at x. Since $\check{T}$ normalizes $\check{N}$, the quotient $Y \simeq (\mathrm{Spec}\, R)/\check{N}$ retains the $\check{\rho}$-action. The formal completion of Y at σ is the inf-scheme $Y^\wedge \simeq \mathrm{Spf}(R^\wedge)/\exp(\check{\mathfrak{n}})$, the quotient by the formal group $\exp(\check{\mathfrak{n}})$.

Deformation theory says

$$C_\bullet(T_\sigma Y[-1]) \simeq \Gamma^{\mathrm{IndCoh}}(\omega_{Y^\wedge}) \simeq ((R^\wedge)^*)_{\check{\mathfrak{n}}}.$$

The first equality is equation (2.13). For the second equality, we pushed forward the dualizing sheaf $\omega_{Y^\wedge}$ in two steps,

$$Y^\wedge \to \mathrm{pt}/\exp(\check{\mathfrak{n}}) \to \mathrm{pt}.$$

The pushforward of $\omega_{Y^\wedge}$ to $\mathrm{pt}/\exp(\check{\mathfrak{n}})$ is an $\check{\mathfrak{n}}$-module. By proper base change and (2.14), the underlying vector space is $\Gamma^{\mathrm{IndCoh}}(\omega_{\mathrm{Spf}\, R^\wedge}) \simeq (R^\wedge)^*$ and the $\check{\mathfrak{n}}$-module structure comes from the $\check{N}$-action. Further pushing forward along $\mathrm{pt}/\exp(\check{\mathfrak{n}}) \to \mathrm{pt}$ corresponds to taking $\check{\mathfrak{n}}$-coinvariants, so $\Gamma^{\mathrm{IndCoh}}(\omega_{Y^\wedge}) \simeq ((R^\wedge)^*)_{\check{\mathfrak{n}}}$.

Now we show that $\check{\mathfrak{n}}$-coinvariants of the topological dual of $R^\wedge$ equals the graded dual ring of functions on Y,

$$((R^\wedge)^*)_{\check{\mathfrak{n}}} \simeq \mathrm{colim}(((R/R_{>n})^*)_{\check{\mathfrak{n}}}) \simeq \mathrm{colim}(((R/R_{>n})^{\check{\mathfrak{n}}})^*) \simeq \mathrm{colim}(((R^{\check{\mathfrak{n}}}/(R^{\check{\mathfrak{n}}})_{>n})^*) \simeq (R^{\check{N}})^*.$$

The ideal $R_{>n}$ is an $\check{\mathfrak{n}}$-module because the $\check{\mathfrak{n}}$-action increases $\check{\rho}$-weights. For the first equality, Proposition 2.6 says that $(R^\wedge)^* \simeq \mathrm{colim}((R/R_{>n})^*)$, and coinvariants commutes with colimits. For the second equality, $((R/R_{>n})^*)_{\check{\mathfrak{n}}} \simeq ((R/R_{>n})^{\check{\mathfrak{n}}})^*$ because $R/R_{>n}$ has finite dimensional cohomology. For the third equality, the image of $(R_{>n})^{\check{\mathfrak{n}}} \to R^{\check{\mathfrak{n}}}$ is concentrated in degrees $> n$ so we get a map $(R/R_{>n})^{\check{\mathfrak{n}}} \to R^{\check{\mathfrak{n}}}/(R^{\check{\mathfrak{n}}})_{>n}$. Moreover, since $(R/R_{>n})^{\check{\mathfrak{n}}}$ is concentrated in bounded degrees, for m sufficiently large, the quotient map factors through

$$R^{\check{\mathfrak{n}}}/(R^{\check{\mathfrak{n}}})_{>m} \to (R/R_{>n})^{\check{\mathfrak{n}}} \to R^{\check{\mathfrak{n}}}/(R^{\check{\mathfrak{n}}})_{>n}.$$

For the fourth equality, we used the van Est isomorphism; see Theorem 5.1 of [19]. Since $\check{N}$ is unipotent, Lie algebra cohomology $R^{\check{\mathfrak{n}}}$ coincides with group cohomology $R^{\check{N}}$. □

Example 2.8. Let $G = \mathrm{SL}(2)$ and let σ be a $\check{T}$-local system, viewed as a rank 1 local system using the positive coroot. Then σ is regular if and only if it is nontrivial.

If σ is regular, then $\mathrm{Loc}^{\sigma}_{\check{N}} \simeq H^1(X,\sigma)$ is a classical affine scheme because the other cohomologies vanish. The shifted tangent complex $T_\sigma \mathrm{Loc}^{\sigma}_{\check{N}}[-1] \simeq H^1(X,\sigma)[-1]$ is an abelian Lie algebra with enveloping algebra $U := \mathrm{Sym}(H^1(X,\sigma)[-1])$. Lie algebra homology of the shifted tangent complex is

$$k \otimes_U k \simeq \mathrm{Sym} H^1(X,\sigma) \simeq \mathcal{O}(\mathrm{Loc}^{\sigma}_{\check{N}})^*.$$

If σ is trivial, then $C_\bullet(T_\sigma \mathrm{Loc}_{\check{N}}[-1]) \simeq \mathrm{Sym}(H^2(X)[-1] \oplus H^1(X) \oplus H^0(X)[1])$ is the graded dual ring of functions on $\mathrm{Loc}_{\check{N}} \simeq H^2(X)[-1] \times H^1(X) \times \mathrm{pt}/H^0(X)$.

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