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Limits of Hodge structures in several variables

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Abstract. We define the notion of a morphism of generalized semi-stable type, which is a generalization of the notion of a semistable degeneration over a curve. We partially generalize Steenbrink's results on the limit of Hodge structures to the case of such a morphism. As an application we prove the E_1-degeneration of the relative Hodge–De Rham spectral sequence for this case.

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Introduction

For a semistable degeneration $f\colon X \to \Delta$ over the unit disc Δ in $\mathbf{C}$, Steenbrink constructed, in [12], a cohomological mixed Hodge complex which gave us the limiting mixed Hodge structure of the variation of Hodge structure over the punctured disc Δ^* obtained from the morphism f. In this article we partially generalize his result to the case of higher dimensional parameter spaces. This is a partial answer to a problem stated by Steenbrink and Zucker [14, problem 7 in the Introduction]. In Section 6 we define the notion of a morphism of generalized semi-stable type (see Definition (6.2)). Roughly speaking, a morphism is said to be of generalized semi-stable type, if it is locally a product of smooth morphisms and semistable degenerations over 1-dimensional unit discs (see Lemma (6.5) precisely). Therefore such a morphism is of quasi-semistable type in the sense of F. Kato [9]. Illusie treated such morphisms in [8] for the algebraic case. Let $f\colon (X, D) \to (S, T)$ be a morphism of pairs which is proper and of generalized semi-stable type. Then the result of F. Kato [9] implies that the sheaf $R^q f_* \Omega^{\cdot}_{X/S}(\log D)$ is locally free of finite rank and commutes with base change for every integer q. Moreover it is trivial that this sheaf defines a variation of Hodge structure over $S \setminus T$. The main theorem in this article, Theorem (6.10), states that for every boundary point s on T there exists, under a certain Kähler condition, a $\mathbf{Q}$-mixed Hodge structure $(H_{\mathbf{Q}}, W, F)$ such that the $\mathbf{C}$-vector space $H_{\mathbf{C}}$ is isomorphic to the $\mathbf{C}$-vector space $R^q f_* \Omega^{\cdot}_{X/S}(\log D) \otimes \mathbf{C}(s)$ (where $\mathbf{C}(s)$ denotes the residue field at the point s) and, via this isomorphism, the filtration F on $H_{\mathbf{C}}$ is identified with the filtration on $R^q f_* \Omega^{\cdot}_{X/S}(\log D) \otimes \mathbf{C}(s)$ obtained from the stupid filtration on $\Omega^{\cdot}_{X/S}(\log D)$. In

Section 4 we construct a cohomological mixed Hodge complex which gives us such a candidate of the 'limiting' mixed Hodge structure by an elementary way following the idea of Steenbrink in [12], [13] and of Tu in [15]. Here we should mention L.-H. Tu's Ph.D. thesis [15], in which he studied the same problem and proposed a way of constructing such a cohomological complex. However, he has not finished the proof that his complex is actually a cohomological mixed Hodge complex, as far as the author knows. Our cohomological mixed Hodge complex constructed in Section 4 is the same as L.-H. Tu's at the **C**-structure level. But the construction of the underlying **Q**-structure is different from his but deeply influenced by the work of Steenbrink in [13]. In Section 1 we present some basic facts which we need later. In Section 2 we construct a complex of sheaves of **Q**-vector spaces, which will turn out to be the underlying **Q**-structure of our cohomological mixed Hodge complex in Section 4. Section 3 is concerned with the weight filtrations on sheaves of the logarithmic forms. In Section 5 we study the relation between the relative log De Rham complex for a morphism of generalized semi-stable type and the cohomological mixed Hodge complex constructed in Section 4.

There still remain, at least, two open problems related to the results in this article. The first is to prove that our mixed Hodge structure is the limit of Hodge structures in the sense of Schmid [11] and Cattani–Kaplan [1]. The second is the problem to generalize the results in this article to the case of log geometry in the sense of Fontaine–Illusie and K. Kato [10]. Log geometry does not appear explicitly in this article, but influences it deeply. So it is natural to study the generalization to the case of log geometry as in Steenbrink [13].

Notation

We use the following notation in this article.

(0.1) For a complex K and for an integer n we define a complex $K[n]$ by

$$K[n]^p = K^{p+n},$$

for every p with the differential defined by

$$d^p_{K[n]} = d^{n+p}_K \colon K[n]^p = K^{n+p} \to K^{n+p+1} = K[n]^{p+1}.$$

Notice that our definition is different from the usual one as in Hartshorne [6] on the sign of the differentials.

(0.2) Let k be a positive integer. A k-ple complex $K^\cdot$ means the collection of the data K^p indexed by the set $\mathbf{Z}^k$, that is, p runs through $\mathbf{Z}^k$, and the morphisms

$$d_i \colon K^p \to K^{p+e_i},$$

for $i = 1, \ldots, k$ for every p, where e_i denotes the ith unit vector of $\mathbf{Z}^k$, satisfying the conditions:

$$K^p = 0 \quad \text{unless}\, p \in (\mathbf{Z}_{\geqslant 0})^k$$

$$d_i^2 = 0$$

$$d_i \circ d_j + d_j \circ d_i = 0,$$

for every i and j. To a given k-ple complex $K^{\cdot}$, we associate a (single) complex sK, which is called the (single) complex associated to $K^{\cdot}$, by

$$sK^n = \bigoplus_{|p|=n} K^p$$

with the differential

$$d = \bigoplus_{i=1}^{k} d_i,$$

where $|p| = p_1 + \cdots + p_k$ for an element $p = (p_1, \ldots, p_k)$ of $\mathbf{Z}^k$. It is easy to see that the data $(sK; d)$ above actually give a complex, that is, the differential d above satisfies the condition $d^2 = 0$.

(0.3) For a positive integer N and for an integer m with $1 \leqslant m \leqslant N$ we define a set $\mathfrak{S}_m^N$ by

$$\mathfrak{S}_m^N = \{(\sigma_1, \ldots, \sigma_m) \in \mathbf{Z}^m | 1 \leqslant \sigma_1 < \cdots < \sigma_m \leqslant N\}.$$

For $m \geqslant 2$ and for an element $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathfrak{S}_m^N$, the symbol σ_i denotes the element $(\sigma_1, \ldots, \hat{\sigma}_i, \ldots, \sigma_m)$ of $\mathfrak{S}_{m-1}^N$ for every $i = 1, \ldots, m$, where the symbol $\hat{\cdot}$ means that we delete the integer under it.

(0.4) A reduced divisor Y on a complex manifold X is called a normal crossing divisor if for every point x on Y there exists a local coordinate system $(x_1, \ldots, x_n)$ around the point x such that Y is defined by the function $x_1 \cdots x_k$ for some k with $1 \leqslant k \leqslant n$. A reduced normal crossing divisor on a complex manifold is said to be a simple normal crossing divisor if every irreducible component is nonsingular.

(0.5) Let X be a complex manifold and $Y = \sum_{i=1}^N Y_i$ a reduced simple normal crossing divisor on X, where the Y_i's are the irreducible components of Y. For an element $\sigma = (\sigma_1, \ldots, \sigma_m)$ of $\mathfrak{S}_m^N$ we define a submanifold Y_σ of X by

$$Y_\sigma = Y_{\sigma_1} \cap \cdots \cap Y_{\sigma_m}.$$

Notice that Y_σ is of pure codimension m in X. Furthermore, for an integer m with $1 \leqslant m \leqslant N$ we set

$$Y^m = \coprod_{\sigma \in \mathfrak{S}_m^N} Y_\sigma,$$

where $\coprod$ denotes the disjoint union of complex manifolds. We use the convention that $Y^0 = X$ and $Y^m = \emptyset$ for $m > N$ or $m < 0$. We denote the canonical morphism from Y^m to X by

$$a_m: Y^m \to X$$

and, if there is no danger of confusion, omit the subscript m to a_m. Since a_m is a finite morphism of complex manifolds, the functor $(a_m)_*$ is an exact functor. For this reason we sometimes omit the symbol $(a_m)_*$. As for the pull-back of analytic objects on X to Y_σ (resp. Y^m), we sometimes use the symbols $\cap Y_\sigma$ (resp. $\cap Y^m$) or $|_{Y_\sigma}$ (resp. $|_{Y^m}$). For example, for an $\mathcal{O}_X$-module sheaf $\mathcal{F}$ on X the pull-back $a_m^*\mathcal{F}$ of $\mathcal{F}$ to Y^m is denoted by $\mathcal{F}|_{Y^m}$, for a subspace Z of X the subspace $a_m^{-1}Z$ of Y^m is denoted by $Z \cap Y^m$ and so on.

(0.6) For a point x of a complex analytic space X the residue field at the point x is denoted by $\mathbf{C}(x)$, that is, $\mathbf{C}(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$, which is isomorphic to $\mathbf{C}$. For a local section f of $\mathcal{O}_X$ around the point x, $f(x)$ denotes the class of the germ $f_x \in \mathcal{O}_{X,x}$ in the residue field $\mathbf{C}(x)$, which is often identified with a complex number by the isomorphism $\mathbf{C}(x) \simeq \mathbf{C}$ above. The complex number $f(x)$ above is called the value of f at the point x.

(0.7) Let X and S be complex manifolds, Y and T reduced simple normal crossing divisors on X and S respectively, and $f: X \to S$ a surjective morphism. (We permit the case $Y = 0$. In this case we need a trivial modification in the following.) Assume, in addition, that the pull-back $D = f^*T$ and the sum $D + Y$ are reduced simple normal crossing divisors on X too. Then we have a morphism

$$f^*\Omega_S^1(\log T) \to \Omega_X^1(\log(D+Y))$$

where $\Omega_X^1(\log(D+Y))$ and $\Omega_S^1(\log T)$ are the sheaves of the logarithmic 1-forms. We define an $\mathcal{O}_X$-module $\Omega_{X/S}^1(\log D)(\log Y)$ by

$$\Omega_{X/S}^1(\log D)(\log Y) = \operatorname{Coker}(f^*\Omega_S^1(\log T) \to \Omega_X^1(\log(D+Y)),$$

and $\mathcal{O}_X$-module sheaves $\Omega_{X/S}^p(\log D)(\log Y)$ by

$$\Omega_{X/S}^p(\log D)(\log Y) = \bigwedge^p \Omega_{X/S}^1(\log D)(\log Y),$$

for all p. (For the case that $Y = 0$, we use symbols $\Omega_{X/S}^1(\log D)$ and $\Omega_{X/S}^p(\log D)$ instead.) The derivation d on $\Omega_X^\cdot(\log(D+Y))$ induces a $f^{-1}\mathcal{O}_S$-derivation

$$d: \Omega_{X/S}^p(\log D)(\log Y) \to \Omega_{X/S}^{p+1}(\log D)(\log Y)$$

for every p and then $\Omega_{X/S}^\cdot(\log D)(\log Y)$ forms a complex of $f^{-1}\mathcal{O}_S$-modules.

1. Preliminaries

(1.1) In this section we will prove several results which we need later for the construction of the **Q**-structure of our mixed Hodge structure.

(1.2) At first we recall the definition of the Koszul complex (see Illusie [7] or Steenbrink [13]).

DEFINITION (1.3) Let A be a ring and $\varphi\colon E \to F$ a morphism of A-modules. We fix a non-negative integer n. Then we define an A-module $\mathrm{Kos}_A^n(\varphi)^p$ for every integer p by

$$\mathrm{Kos}_A^n(\varphi)^p = \begin{cases} 0 & \text{if } \ p<0 \ \text{ or } \ p>n \\ \Gamma_{n-p}E \otimes_A \bigwedge^p F & 0 \leqslant p \leqslant n \end{cases}$$

where Γ_{n-p} denotes the $(n-p)$th graded piece of the divided power envelope of E over A. Moreover we define a morphism of A-modules $d\colon \mathrm{Kos}_A^n(\varphi)^p \to \mathrm{Kos}_A^n(\varphi)^{p+1}$ for every $p = 0, \ldots, n-1$ by

$$d((x_1^{[n_1]} \cdots x_k^{[n_k]}) \otimes y) = \sum_{j=1}^{k} (x_1^{[n_1]} \cdots x_j^{[n_j-1]} \cdots x_k^{[n_k]}) \otimes \varphi(x_j) \wedge y,$$

where $x_1, \ldots, x_k$ are elements of E, y of $\bigwedge^p F$, $n_1, \ldots, n_k$ are positive integers with $\sum_{j=1}^k n_j = n-p$. Then it is easy to see that $(\mathrm{Kos}_A^n(\varphi)^p, d)$ forms a complex of A-modules. We call it the Koszul complex associated to the morphism φ. We omit the subscript A, if there is no danger of confusion. We use the convention that $\mathrm{Kos}^n(\varphi)^p = 0$ for every p if n is a negative integer.

Remark (1.4) Let $\varphi\colon E \to F$ be a morphism of A-modules and B an A-algebra. Then we have a canonical isomorphism

$$\mathrm{Kos}_A^n(\varphi) \otimes_A B \simeq \mathrm{Kos}_B^n(\varphi \otimes_A B),$$

for every integer n, where $\varphi \otimes_A B$ denotes the morphism of B-modules from $E \otimes_A B$ to $F \otimes_A B$ obtained by the base extension $A \to B$. Therefore we have an isomorphism

$$H^p(\mathrm{Kos}_A^n(\varphi)) \otimes_A B \simeq H^p(\mathrm{Kos}_B^n(\varphi \otimes_A B))$$

for every integer p, if B is flat over A.

(1.5) Let A, E, F and φ be as above and n a fixed non-negative integer. Then the inclusion $\mathrm{Ker}(\varphi) \to E$ induces a morphism

$$\Gamma_{n-p}(\mathrm{Ker}(\varphi)) \otimes \bigwedge^p F \to \mathrm{Kos}^n(\varphi)^p$$

for $0 \leqslant p \leqslant n$. It is clear that the image of the morphism above is contained in the kernel of $d\colon \mathrm{Kos}^n(\varphi)^p \to \mathrm{Kos}^n(\varphi)^{p+1}$. Therefore we obtain a morphism

$$\Gamma_{n-p}(\mathrm{Ker}(\varphi)) \otimes \bigwedge^p F \to H^p(\mathrm{Kos}^n(\varphi)). \tag{1.5.1}$$

On the other hand, the canonical surjection $F \to \mathrm{Coker}(\varphi)$ induces a morphism

$$\Gamma_{n-p}(\mathrm{Ker}(\varphi)) \otimes \bigwedge^p F \to \Gamma_{n-p}(\mathrm{Ker}(\varphi)) \otimes \bigwedge^p \mathrm{Coker}(\varphi).$$

It is easy to see that the morphism (1.5.1) factors through the morphism above. Thus we obtain a morphism

$$\Gamma_{n-p}(\mathrm{Ker}(\varphi)) \otimes \bigwedge^p \mathrm{Coker}(\varphi) \to H^p(\mathrm{Kos}^n(\varphi)). \tag{1.5.2}$$

LEMMA (1.6) *In the situation as in* (1.5), *the morphism* (1.5.2) *is an isomorphism for every integer* p *if* E, F *and* $\mathrm{Coker}(\varphi)$ *are flat over* A.

Proof. See Illusie [7] and Steenbrink [13]. □

COROLLARY (1.7) *Assume that we have the following commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & E \arrow[r, "\varphi"] \arrow[d] & F \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] & E' \arrow[r, "\varphi'"] & F' \arrow[r] & C \arrow[r] & 0,
\end{tikzcd}$$

of flat A*-modules with exact lines. Then the canonical morphism*

$$\mathrm{Kos}^n(\varphi) \to \mathrm{Kos}^n(\varphi')$$

induced by the morphisms $E \to E'$ *and* $F \to F'$ *is a quasi-isomorphism.*

Proof. Trivial from the lemma above. □

(1.8) In addition to the assumption in Lemma (1.6), we assume that the base ring A contains the field of the rational numbers $\mathbf{Q}$ and that the kernel of φ is a free A-module of rank one, that is, $\mathrm{Ker}(\varphi) \simeq A$ and the cokernel of φ is a free A-module of finite rank, that is, $\mathrm{Coker}(\varphi) \simeq A^r$ for some non-negative integer r. We denote the free generator of $\mathrm{Ker}(\varphi)$ by e. For integers n and n' with $n \leqslant n'$ we have a morphism of complexes $\mathrm{Kos}^n(\varphi) \to \mathrm{Kos}^{n'}(\varphi)$ by

$$x \otimes y \mapsto e^{[n'-n]} x \otimes y$$

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at the pth step, where x is an element of $\Gamma_{n-p}E$ and y of $\bigwedge^p F$.

COROLLARY (1.9) *Under the assumption in* (1.8) *the morphism*

$$\mathrm{Kos}^n(\varphi) \to \mathrm{Kos}^{n'}(\varphi)$$

is a quasi-isomorphism if rank (Coker $(\varphi)) \leqslant n$.

Proof. Easy from Lemma (1.6). □

(1.10) In the situation as in (1.3), an increasing filtration W on $\mathrm{Kos}^n(\varphi)$ is defined by

$$\begin{aligned} W_m \mathrm{Kos}^n(\varphi)^p &= 0 \quad \text{for } m < 0 \\ W_m \mathrm{Kos}^n(\varphi)^p &= \text{image of } \Gamma_{n-p}E \otimes \wedge^m F \otimes \wedge^{p-m}\varphi(E) \text{ in } \mathrm{Kos}^n(\varphi) \\ &\quad \text{for } 0 \leqslant m \leqslant p \\ W_m \mathrm{Kos}^n(\varphi)^p &= \mathrm{Kos}^n(\varphi)^p \quad \text{for } m > p \end{aligned}$$

as in Steenbrink [13]. We easily see that $W_m \mathrm{Kos}^n(\varphi)$ forms a subcomplex of $\mathrm{Kos}^n(\varphi)$.

(1.11) From now on, we assume that the base ring A is a field. So we use the letter K instead of A. We will treat the cases $A = \mathbf{Q}, \mathbf{R}$ or $\mathbf{C}$ only in the following sections. We denote the cokernel of φ by C and the projection $F \to \mathrm{Coker}(\varphi) = C$ by $\pi\colon F \to C$. Moreover we assume that a direct sum decomposition

$$C = \bigoplus_{i=1}^{k} C_i \tag{1.11.1}$$

is fixed. Under this assumption, we define a K-subspace F_i of F by

$$F_i = \pi^{-1}\left(\bigoplus_{j \neq i} C_j\right), \tag{1.11.2}$$

for every $i = 1, \ldots, k$. Then the image of φ is contained in F_i for every i, therefore the morphism φ can be viewed as a morphism from E to F_i for every i. We denote this morphism by φ_i. Then we have a commutative diagram with exact lines

$$\begin{tikzcd} E \arrow[r, "\varphi_i"] \arrow[d, equal] & F_i \arrow[r] \arrow[d] & \bigoplus_{j \neq i} C_j \arrow[r] \arrow[d] & 0 \\ E \arrow[r, "\varphi"] & F \arrow[r] & C = \bigoplus_{j=i}^{k} C_j \arrow[r] & 0 \end{tikzcd}$$

for every $i = 1, \ldots, k$, where the vertical arrows $F_i \to F$ and $\bigoplus_{j \neq i} C_j \to C = \bigoplus_{j=1}^{k} C_j$ denote the inclusions.

(1.12) Under the assumption in (1.11), we define increasing filtrations on $\mathrm{Kos}^n(\varphi)$ as follows. A subspace $W(C_i)_m \mathrm{Kos}^n(\varphi)^p$ of $\mathrm{Kos}^n(\varphi)^p$ is defined by

$$\begin{aligned} W(C_i)_m \mathrm{Kos}^n(\varphi)^p &= 0 \quad \text{for } m < 0 \\ W(C_i)_m \mathrm{Kos}^n(\varphi)^p &= \text{image of } \Gamma_{n-p} E \otimes \wedge^m F \otimes \wedge^{p-m} F_i \text{ in } \mathrm{Kos}^n(\varphi)^p \\ &\qquad \text{for } 0 \leqslant m \leqslant p \\ W(C_i)_m \mathrm{Kos}^n(\varphi)^p &= \mathrm{Kos}^n(\varphi)^p \quad \text{for } m > p \end{aligned}$$

for every $i = 1, \ldots, k$. Then it is easy to see that $W(C_i)_m \mathrm{Kos}^n(\varphi)^p$ forms a subcomplex of $\mathrm{Kos}^n(\varphi)$. Thus we get increasing filtrations $W(C_i)$'s on $\mathrm{Kos}^n(\varphi)$.

(1.13) We set a subspace $\hat{F}$ by $\hat{F} = \pi^{-1}(C_k)$. Then we can view the morphism $\varphi: E \to F$ as a morphism from E to $\hat{F}$. We denote it by $\hat{\varphi}$. Then we have an exact sequence

$$E \xrightarrow{\hat{\varphi}} \hat{F} \to C_k \to 0.$$

Now we define another increasing filtration $\hat{W}$ on $\mathrm{Kos}^n(\varphi)$ by

$$\begin{aligned} \hat{W}_m \mathrm{Kos}^n(\varphi)^p &= 0 \quad \text{for } m < 0 \\ \hat{W}_m \mathrm{Kos}^n(\varphi)^p &= \text{image of } \Gamma_{n-p} E \otimes \wedge^m F \otimes \wedge^{p-m} \hat{F} \text{ in } \mathrm{Kos}^n(\varphi)^p \\ &\qquad \text{for } 0 \leqslant m \leqslant p \\ \hat{W}_m \mathrm{Kos}^n_A(\varphi)^p &= \mathrm{Kos}^n(\varphi)^p \quad \text{for } m > p \end{aligned}$$

as before.

(1.14) Once we fix a splitting $C \to F$ of $\pi: F \to C$, we have an identification

$$F \simeq \varphi(E) \oplus C = \varphi(E) \oplus \left(\bigoplus_{i=1}^{k} C_i \right).$$

Under the identification above the subspaces F_i's and $\hat{F}$ are identified with

$$\begin{aligned} F_i &\simeq \varphi(E) \oplus \left(\bigoplus_{j \neq i} C_j \right) \\ \hat{F} &\simeq \varphi(E) \oplus C_k. \end{aligned}$$

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Then we have the following identifications:

$$\mathrm{Kos}^n(\varphi)^p \simeq \Gamma_{n-p}E \otimes \left(\bigoplus_{\alpha+\beta_1+\cdots+\beta_k=p} \bigwedge^{\alpha} \varphi(E) \otimes \bigwedge^{\beta_1} C_1 \otimes \cdots \bigwedge^{\beta_k} C_k \right),$$

$$W_m \,\mathrm{Kos}^n(\varphi)^p \simeq \Gamma_{n-p}E \otimes \left(\bigoplus_{\substack{\alpha+\beta_1+\cdots+\beta_k=p \\ \beta_1+\cdots+\beta_k \leqslant m}} \bigwedge^{\alpha} \varphi(E) \otimes \bigwedge^{\beta_1} C_1 \otimes \cdots \bigwedge^{\beta_k} C_k \right),$$

$$W(C_i)_m \,\mathrm{Kos}^n(\varphi)^p \simeq \Gamma_{n-p}E \otimes \left(\bigoplus_{\substack{\alpha+\beta_1+\cdots+\beta_k=p \\ \beta_i \leqslant m}} \bigwedge^{\alpha} \varphi(E) \otimes \bigwedge^{\beta_1} C_1 \otimes \cdots \bigwedge^{\beta_k} C_k \right),$$

$$\hat{W}_m \,\mathrm{Kos}^n(\varphi)^p \simeq \Gamma_{n-p}E \otimes \left(\bigoplus_{\substack{\alpha+\beta_1+\cdots+\beta_k=p \\ \beta_1+\cdots+\beta_{k-1} \leqslant m}} \bigwedge^{\alpha} \varphi(E) \otimes \bigwedge^{\beta_1} C_1 \otimes \cdots \bigwedge^{\beta_k} C_k \right),$$

for all i and m with $0 \leqslant m \leqslant p$.

LEMMA (1.15) *We have*

$$\left(\sum_{i=1}^{k} W(C_i)_{q_i} \right) \cap W(C_k)_m = \sum_{i=1}^{k} (W(C_i)_{q_i} \cap W(C_k)_m)$$

on $\mathrm{Kos}^n(\varphi)^p$ *for every* $p, q_1, \ldots, q_k$ *and* m.

Proof. Easy by taking a splitting $C \to F$ as in (1.14). □

LEMMA (1.16) *We have*

$$\left(\sum_{i=1}^{k} W(C_i)_{q_i} + \hat{W}_q \right) \cap W(C_k)_m$$

$$= \sum_{i=1}^{k} (W(C_i)_{q_i} \cap W(C_k)_m) + \hat{W}_q \cap W(C_k)_m$$

on $\mathrm{Kos}^n(\varphi)^p$ *for every* $p, q, q_1, \ldots, q_k$ *and* m.

Proof. As above. □

LEMMA (1.17) *Under the assumption in* (1.11) *we have*

$$\mathrm{Kos}^{n-m}(\varphi_i)^{p-m} \otimes \bigwedge^{m} (F/F_i) \xrightarrow{\sim} \mathrm{Gr}_m^{W(C_i)} \mathrm{Kos}^n(\varphi)^p, \tag{1.17.1}$$

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for every $i = 1, \ldots, k$ *and for every* m, p. (*Recall the convention in Definition* (1.3) *for the case* $m > n$.) *More precisely, the isomorphism above is induced by the morphism*

$$\Gamma_{n-p} E \otimes \bigwedge^{p-m} F_i \otimes \bigwedge^{m} F \to \mathrm{Gr}_m^{W(C_i)} \mathrm{Kos}^n(\varphi)^p,$$

defined by

$$x \otimes y \otimes z \mapsto x \otimes (y \wedge z) = (-1)^{m(p-m)} x \otimes (z \wedge y) \bmod W(C_i)_{m+1},$$

where x *is an element of* $\Gamma_{n-p}E$, y *of* $\bigwedge^{p-m} F_i$ *and* z *of* $\bigwedge^m F$.

Proof. Fixing a splitting, we can identify $F \simeq F_i \oplus C_i$. Then it is easy to prove the conclusion. □

PROPOSITION (1.18) *For every integer* m, *the morphism* (1.17.1) *above induces an isomorphism of complexes*

$$\mathrm{Kos}^{n-m}(\varphi_i)[-m] \otimes \bigwedge^{m}(F/F_i) \xrightarrow{\sim} \mathrm{Gr}_m^{W(C_i)} \mathrm{Kos}^n(\varphi). \tag{1.18.1}$$

Proof. Trivial from the lemma above. □

LEMMA (1.19) *By the isomorphism* (1.18.1) *above, we have*

$$(W(C_j)_l \, \mathrm{Kos}^{n-m}(\varphi_i)[-m]) \otimes \bigwedge^{m}(F/F_i) \xrightarrow{\sim} W(C_j)_l \, \mathrm{Gr}_m^{W(C_i)} \mathrm{Kos}^n(\varphi),$$

for every i *and* j *with* $i \neq j$ *and for every integer* l, *where* $W(C_j)$ *on the right-hand side denotes the induced filtration on* $\mathrm{Gr}_m^{W(C_i)} \mathrm{Kos}^n(\varphi)$ *from* $W(C_j)$ *on* $\mathrm{Kos}^n(\varphi)$ *and* $W(C_j)$ *on the left-hand side is the filtration defined from the decomposition*

$$\mathrm{Coker}(\varphi_i) = \bigoplus_{j \neq i} C_j$$

in the same way as in (1.12).

Proof. Easy by fixing a splitting. □

LEMMA (1.20) *By the isomorphism* (1.18.1) *in Proposition* (1.18), *we have the identification*

$$(W_l \, \mathrm{Kos}^{n-m}(\varphi_k)[-m]) \otimes \bigwedge^{m}(F/F_i) \xrightarrow{\sim} \hat{W}_l \, \mathrm{Gr}_m^{W(C_k)} \mathrm{Kos}^n(\varphi),$$

where $\hat{W}$ *on the right-hand side is the induced filtration on* $\mathrm{Gr}_m^{W(C_k)} \mathrm{Kos}^n(\varphi)$ *from* $\hat{W}$ *on* $\mathrm{Kos}^n(\varphi)$ *and* W *on the left-hand side is the filtration on* $\mathrm{Kos}^{n-m}(\varphi_k)$ *defined in the same way as in* (1.10).

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Proof. Easy by fixing a splitting. □

(1.21) Once an element t of F is given, a morphism $\mathrm{Kos}^n(\varphi)^p \to \mathrm{Kos}^{n+1}(\varphi)^{p+1}$ is defined by

$$x \otimes y \mapsto x \otimes t \wedge y,$$

where x is an element of $\Gamma_{n-p}E, y$ of $\bigwedge^p F$, therefore $x \otimes y$ is an element of $\Gamma_{n-p}E \otimes \bigwedge^p F = \mathrm{Kos}^n(\varphi)^p$ and $x \otimes t \wedge y$ is an element of $\Gamma_{n-p}E \otimes \bigwedge^{p+1} F = \mathrm{Kos}^{n+1}(\varphi)^{p+1}$. We denote this morphism by $t\wedge$ if there is no danger of confusion. Then we can easily see that $(t\wedge)^2 = 0$. Moreover, we have

$$(t\wedge) \circ d + d \circ (t\wedge) = 0,$$

where the d's are the differentials of $\mathrm{Kos}^n(\varphi)$ and $\mathrm{Kos}^{n+1}(\varphi)$ at the appropriate places.

(1.22) We have

$$(t\wedge)(W_m \mathrm{Kos}^n(\varphi)^p) \subset W_{m+1}\mathrm{Kos}^{n+1}(\varphi)^{p+1} \tag{1.22.1}$$

$$(t\wedge)(W(C_i)_m \mathrm{Kos}^n(\varphi)^p \subset W(C_i)_{m+1}\mathrm{Kos}^{n+1}(\varphi)^{p+1} \tag{1.22.2}$$

$$(t\wedge)(\hat{W}_m \mathrm{Kos}^n(\varphi)^p) \subset \hat{W}_{m+1}\mathrm{Kos}^{n+1}(\varphi)^{p+1}, \tag{1.22.3}$$

for every m, every i and any $t \in F$. If t is contained in the subspace $\varphi(E)$, then we have

$$(t\wedge)(W_m \mathrm{Kos}^n(\varphi)^p) \subset W_m \mathrm{Kos}^{n+1}(\varphi)^{p+1}, \tag{1.22.4}$$

if $t \in F_i$, then

$$(t\wedge)(W(C_i)_m \mathrm{Kos}^n(\varphi)^p) \subset W(C_i)_m \mathrm{Kos}^{n+1}(\varphi)^{p+1}, \tag{1.22.5}$$

and if $t \in \hat{F}$, then

$$(t\wedge)(\hat{W}_m \mathrm{Kos}^n(\varphi)^p) \subset \hat{W}_m \mathrm{Kos}^{n+1}(\varphi)^{p+1}. \tag{1.22.6}$$

(1.23) Here we list up all the assumptions which we need later. Let K be a field, E and F be K-vector spaces and $\varphi: E \to F$ a K-linear map. We denote the cokernel of φ by C and the projection $F \to C$ by π. We assume the following:

(1.23.1) C is of finite dimension

(1.23.2) we are given a direct sum decomposition

$$C = \bigoplus_{i=1}^{k} C_i$$

(1.23.3) we are given an element t_i of F for every i, such that $\pi(t_i)$ is a non-zero element of C contained in the subspace C_i

We set $r_i = \dim_K C_i$ and $r = \dim_K C = \sum_{i=1}^k r_i$.

(1.24) Under the assumption in (1.23) we define the subspaces F_i $(i = 1, \ldots, k)$ and $\hat{F}$ of F, the morphisms $\varphi_i: E \to F_i$ $(i = 1, \ldots, k)$ and the weight filtrations $W, W(C_i)(i = 1, \ldots, k)$ and $\hat{W}$ on the complex $\mathrm{Kos}^n(\varphi)$ as before. Notice that the morphism $\varphi_i: E \to F_i$, $\mathrm{Coker}(\varphi_i) = \bigoplus_{j \neq i} C_j$ and t_j for $j \neq i$ satisfy the conditions stated in (1.23) for every i. The element t_i is contained in F_j for every $j \neq i$ by the definition (1.11.2) of F_j, and t_k is contained in $\hat{F}$ by the definition of $\hat{F}$ in (1.13). Therefore we have

$$(t_i\wedge)(W(C_j)_m \operatorname{Kos}^n(\varphi)^p) \subset W(C_j)_m \operatorname{Kos}^{n+1}(\varphi)^{p+1},$$

for every $j \neq i$ and for every m as in (1.22.5) and

$$(t_k)(\hat{W}_m \operatorname{Kos}^n(\varphi)^p) \subset \hat{W}_m \operatorname{Kos}^{n+1}(\varphi)^{p+1},$$

for every m as in (1.22.6).

(1.25) From now on, we use multi-index notation. We denote the ith unit vector in $\mathbf{Z}^k$ by e_i. For an element $q = (q_1, \ldots, q_k)$ in $\mathbf{Z}^k$, we set $|q| = \sum_{i=1}^k q_i$. We fix a non-negative integer n. Under the assumptions in (1.23) we define a K-vector space $A(\varphi; n)^{p,q}$ for an integer p and for an element q of $\mathbf{Z}^k$ by

$$A(\varphi; n)^{p,q} = \operatorname{Kos}^{n+|q|+k}(\varphi)^{p+|q|+k} / \sum_{i=1}^{k} W(C_i)_{q_i} \operatorname{Kos}^{n+|q|+k}(\varphi)^{p+|q|+k}$$

for $p \in \mathbf{Z}_{\geqslant 0}$ and $q \in (\mathbf{Z}_{\geqslant 0})^k$, and

$$A(\varphi; n)^{p,q} = 0$$

if $p < 0$ or $q \in \mathbf{Z}^k \setminus (\mathbf{Z}_{\geqslant 0})^k$. Moreover, for the case that n is negative we define $A(\varphi; n)^{p,q} = 0$. We have filtrations on $A(\varphi; n)^{p,q}$ induced from the filtrations W, $W(C_i)$ and $\hat{W}$ on $\operatorname{Kos}^{n+|q|+k}(\varphi)^{p+|q|+k}$. We denote these filtrations by the same letters $W, W(C_i)$ and $\hat{W}$ by abuse of language. The differential d of $\operatorname{Kos}^{n+|q|+k}(\varphi)$ induces a morphism

$$d_0: A(\varphi; n)^{p,q} \to A(\varphi; n)^{p+1,q}.$$

Moreover, the morphism

$$t_i \wedge : \operatorname{Kos}^{n+|q|+k}(\varphi)^{p+|q|+k} \to \operatorname{Kos}^{n+|q|+k+1}(\varphi)^{p+|q|+k+1}$$

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induces a morphism

$$d_i\colon A(\varphi;n)^{p,q} \to A(\varphi;n)^{p,q+e_i},$$

for every i because of (1.24). It is easy to see that we have the following equalities:

$$(d_i)^2 = 0 \text{ for every } i$$

$$d_i \circ d_j + d_j \circ d_i = 0 \text{ for every } i \text{ and } j \text{ with } i \neq j,$$

which means that $(A(\varphi;n)^{p,q})$ forms a $(k+1)$-ple complex of K-vector spaces. The single complex associated to $(A(\varphi;n)^{p,q})$ is denoted by $sA(\varphi;n)$. As for the filtrations, we have

$$\begin{cases} d_0(W_m A(\varphi;n)^{p,q} \subset W_m A(\varphi;n)^{p+1,q} \\ d_i(W_m A(\varphi;n)^{p,q}) \subset W_{m+1} A(\varphi;n)^{p,q+e_i} \\ d_0(W(C_i)_m A(\varphi;n)^{p,q}) \subset W(C_i)_m A(\varphi;n)^{p+1,q} \\ d_i(W(C_i)_m A(\varphi;n)^{p,q}) \subset W(C_i)_{m+1} A(\varphi;n)^{p,q+e_i} \\ d_i(W(C_j)_m A(\varphi;n)^{p,q}) \subset W(C_j)_m A(\varphi;n)^{p,q+e_i} \quad \text{for } i \neq j \\ d_0(\hat{W}_m A(\varphi;n)^{p,q} \subset \hat{W}_m A(\varphi;n)^{p+1,q} \\ d_i(\hat{W}_m A(\varphi;n)^{p,q}) \subset \hat{W}_{m+1} A(\varphi;n)^{p,q+e_i} \\ d_k(\hat{W}_m A(\varphi;n)^{p,q} \subset \hat{W}_m A(\varphi;n)^{p,q+e_k}. \end{cases}$$

(1.26) We define new filtrations on $sA(\varphi;n)$ as follows. We set

$$L_m sA(\varphi;n)^s = \bigoplus_{p+|q|=s} W_{m+2|q|+k} A(\varphi;n)^{p,q}$$

$$L(C_j)_m sA(\varphi;n)^s = \bigoplus_{p+|q|=s} W(C_j)_{m+2q_j+1} A(\varphi;n)^{p,q} \quad \text{for } j = 1, \dots, k$$

$$\hat{L}_m sA(\varphi;n)^s = \bigoplus_{p+|q|=s} \hat{W}_{m+2|\hat{q}|+k-1} A(\varphi;n)^{p,q},$$

for every m and s, where $\hat{q}$ denotes the element $(q_1, \dots, q_{k-1})$ of $\mathbf{Z}^{k-1}$ for a given element $q = (q_1, \dots, q_k)$ of $\mathbf{Z}^k$. Then it is easy to see that $L(C_k)_m$,L_m and $\hat{L}_m$ define filtrations on the complex $sA(\varphi;n)$ by using (1.25.1).

LEMMA (1.27) *We have*

$$L_m sA(\varphi;n)^s = \sum_{\alpha+\beta=m} \hat{L}_\alpha sA(\varphi;n)^s \cap L(C_k)_\beta sA(\varphi;n)^s$$

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for every m *and* s.

Proof. Because of the equalities

$$L_m sA(\varphi;n)^s = \bigoplus_{p+|q|=s} W_{m+2|q|+k} A(\varphi;n)^{p,q}$$

and

$$\sum_{\alpha+\beta=m} \hat{L}_\alpha sA(\varphi;n)^s \cap L(C_k)_\beta sA(\varphi;n)^s$$

$$= \bigoplus_{p+|q|=s} \sum_{\alpha+\beta=m} \hat{W}_{\alpha+2|\hat{q}|+k-1} A(\varphi;n)^{p,q} \cap W(C_k)_{\beta+2q_k+1} A(\varphi;n)^{p,q}$$

the following lemma implies the conclusion. □

LEMMA (1.28) *For every* $p \in \mathbf{Z}_{\geqslant 0}$ *and every* $q \in (\mathbf{Z}_{\geqslant 0})^k$, *we have*

$$W_m A(\varphi;n)^{p,q} = \sum_{\alpha+\beta=m} \hat{W}_\alpha A(\varphi;n)^{p,q} \cap W(C_k)_\beta A(\varphi;n)^{p,q},$$

for every m.

Proof. Easy by taking a splitting of $\pi\colon F \to C$. □

COROLLARY (1.29) *In the situation above the natural morphism*

$$\bigoplus_{\alpha+\beta=m} \mathrm{Gr}_\alpha^{\hat{L}} \mathrm{Gr}_\beta^{L(C_k)} sA(\varphi;n) \xrightarrow{\sim} \mathrm{Gr}_m^L sA(\varphi;n),$$

is an isomorphism.

Proof. Easy from Lemma (1.27). □

(1.30) In order to compute the left-hand side of the isomorphism above, we need the following results.

DEFINITION (1.31) Let $\mathcal{C}$ be an abelian category, V an object of $\mathcal{C}$ and E, F and W subobjects of V with the condition $F \subset E$. Then the subobject $W(E/F)$ of E/F is defined as the image of the natural morphism $W \cap E \to E/F$.

LEMMA (1.32) (Zassenhaus' Lemma) *Let* $\mathcal{C}$ *be an abelian category,* V *an object of* $\mathcal{C}$, *and* E, F *and* W *subobjects of* V *with the condition* $F \subset E$. *Then the natural surjection*

$$E/F \to E(V/W)/F(V/W)$$

induces an isomorphism

$$(E/F)/W(E/F) \xrightarrow{\sim} E(V/W)/F(V/W).$$

Proof. Easy. □

LEMMA (1.33) *In the situation above, assume that another subobject $\hat{W}$ is given. If we have*

$$(W+\hat{W})\cap E = W\cap E + \hat{W}\cap E$$

then

$$\hat{W}(E/F)((E/F)/W(E/F)) \xrightarrow{\sim} \hat{W}(V/W)(E(V/W)/F(V/W))$$

via the identification in the lemma above, where the left-hand side is the subobject of $(E/F)/W(E/F)$ induced by the subobject $\hat{W}(E/F)$ of E/F and the right hand side is the subobject of $E(V/W)/F(V/W)$ induced by the subobject $\hat{W}(V/W)$ of V/W.

Proof. Easy. □

LEMMA (1.34) *Let $\mathcal{C}$ be an abelian category, V be an object of $\mathcal{C}$ and E, F and $W_1,\dots,W_k$ subobjects of V with the condition $F\subset E$. If we have*

$$(W_1+\cdots+W_k)\cap E = W_1\cap E+\cdots+W_k\cap E,$$

then the canonical projection

$$E/F \to E(V/(W_1+\cdots+W_k))/F(V/(W_1+\cdots+W_k)),$$

induces an isomorphism

$$\begin{aligned}&(E/F)/(W_1(E/F)+\cdots+W_k(E/F))\\ &\quad\xrightarrow{\sim} E(V/(W_1+\cdots+W_k))/F(V/(W_1+\cdots+W_k)).\end{aligned}$$

Proof. Easy by Lemma (1.32). □

LEMMA (1.35) *We fix integers n and p with $p\leqslant n$. For an integer m with $m\geqslant r_i=\operatorname{rank} C_i$ we have*

$$W(C_i)_m \operatorname{Kos}^n(\varphi)^p = \operatorname{Kos}^n(\varphi)^p.$$

Therefore if $m>r_k$ then $\operatorname{Gr}_m^{W(C_k)} A(\varphi;n)^{p,q}=0$ for every $n\in\mathbf{Z}_{\geqslant 0}$, every $p\in\mathbf{Z}_{\geqslant 0}$ and every $q\in(\mathbf{Z}_{\geqslant 0})^k$.

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Proof. Easy by taking a splitting of $\pi\colon F \to C$. $\square$

LEMMA (1.36) *We fix non-negative integers n and p with $p \leqslant n$ and an element $q = (q_1, \ldots, q_k)$ in $(\mathbf{Z}_{\geqslant 0})^k$. For an integer m with $m > p + q_k$ we have*

$$W(C_k)_m \operatorname{Kos}^{n+|q|+k}(\varphi)^{p+|q|+k} + \sum_{i=1}^{k} W(C_i)_{q_i} \operatorname{Kos}^{n+|q|+k}(\varphi)^{p+|q|+k}$$

$$= \operatorname{Kos}^{n+|q|+k}(\varphi)^{p+|q|+k}.$$

Therefore if $m > p + q_k + 1$ then

$$W(C_k)_{m-1} A(\varphi; n)^{p,q} = A(\varphi; n)^{p,q}$$

and

$$\operatorname{Gr}_m^{W(C_k)} A(\varphi; n)^{p,q} = 0.$$

Proof. As above. $\square$

COROLLARY (1.37) $\operatorname{Gr}_m^{W(C_k)} A(\varphi; n)^{p,q} = 0$ *unless* $q_k < m \leqslant \min(r_k, p + q_k + 1)$.

Proof. Lemma (1.35) and Lemma (1.36) imply the second inequality. The first inequality is trivial. $\square$

LEMMA (1.38) *For non-negative integers n and p, for an element $q = (q_1, \ldots, q_k)$ of $(\mathbf{Z}_{\geqslant 0})^k$ and for an integer m with $q_k < m$, the morphism* (1.17.1) *in Lemma* (1.17) *induces an isomorphism*

$$A(\varphi_k; n + q_k + 1 - m)^{p+q_k+1-m,\hat{q}} \otimes \bigwedge^{m}(F/F_k) \to \operatorname{Gr}_m^{W(C_k)} A(\varphi; n)^{p,q}$$

where $\hat{q}$ denotes the element $(q_1, \ldots, q_{k-1})$ of $\mathbf{Z}^{k-1}$. Moreover, we have the following identification for the filtrations

$$W_l A(\varphi_k; n + q_k + 1 - m)^{p+q_k+1-m,\hat{q}} \otimes \bigwedge^{m}(F/F_k) \simeq \hat{W}_l \operatorname{Gr}_m^{W(C_k)} A(\varphi; n)^{p,q},$$

via the isomorphism above, where $\hat{W}$ on the right-hand side is the filtration induced by $\hat{W}$ on $A(\varphi; n)^{p,q}$.

Proof. If the integer m does not satisfy the condition $m \leqslant \min(r_k, p + q_k + 1)$, we obtain the conclusion by Corollary (1.37). Because of Lemma (1.15) and Lemma (1.34) we have

$$\operatorname{Gr}_m^{W(C_k)} A(\varphi; n)^{p,q}$$
$$\simeq \operatorname{Gr}_m^{W(C_k)} \operatorname{Kos}^{n+|q|+k}(\varphi)^{p+|q|+k} \Big/ \left(\sum_{i=1}^{k} W(C_i)_{q_i} \operatorname{Gr}_m^{W(C_k)} \operatorname{Kos}^{n+|q|+k}(\varphi)^{p+|q|+k} \right)$$

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and then obtain the first result by Lemma (1.17) and Lemma (1.19). The latter half follows from Lemma (1.16), Lemma (1.20) and Lemma (1.33). □

PROPOSITION (1.39) *For a fixed non-negative integer* n, *the morphism* (1.18.1) *for* $i = k$ *in Lemma* (1.18) *induces an isomorphism*

$$\bigoplus_{l \geqslant \max(0,-\beta)} sA(\varphi_k; n-\beta-l)[-\beta-2l] \otimes \bigwedge^{\beta+2l+1}(F/F_k) \to \mathrm{Gr}_\beta^{L(C_k)} sA(\varphi;n)$$

and an isomorphism

$$\bigoplus_{l \geqslant \max(0,-\beta)} L_\alpha sA(\varphi_k; n-\beta-l)[-\beta-2l] \otimes \bigwedge^{\beta+2l+1}(F/F_k)$$
$$\to \hat{L}_\alpha \, \mathrm{Gr}_\beta^{L(C_k)} sA(\varphi;n),$$

where $\hat{L}$ *on the right-hand side is the filtration induced by* $\hat{L}$ *on* $sA(\varphi;n)$.

Proof. Because

$$\mathrm{Gr}_\beta^{L(C_k)} sA(\varphi;n)^s = \bigoplus_{p+|q|=s} \mathrm{Gr}^{W(C_k)}_{\beta+2q_k+1} A(\varphi;n)^{p,q},$$

we get

$$\mathrm{Gr}_\beta^{L(C_k)} sA(\varphi;n)^s \simeq \bigoplus_{\substack{l \geqslant \max(0,-\beta) \\ p+|\hat{q}|=s-l}} A(\varphi_k; n-\beta-l)^{p-\beta-l,\hat{q}} \otimes \bigwedge^{\beta+2l+1}(F/F_k)$$

by Lemma (1.38) and by putting $l = q_k$. □

THEOREM (1.40) *For a fixed non-negative integer* n, *we have an isomorphism*

$$\mathrm{Gr}_m^L sA(\varphi;n) \simeq \bigoplus_{\substack{\alpha+\beta=m \\ l \geqslant \max(0,-\beta)}} \mathrm{Gr}_\alpha^L sA(\varphi_k; n-\beta-l)[-\beta-2l] \otimes \bigwedge^{\beta+2l+1}(F/F_k)$$

induced from the isomorphism (1.17.1) *in Lemma* (1.17).

Proof. Corollary (1.29) and Proposition (1.39) imply the result. □

(1.41) Assume that we are given the following data:

(1.41.1) a commutative diagram of K-vector spaces with exact lines

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d, equal] & E' \arrow[r, "\varphi'"] \arrow[d] & F' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G \arrow[r] & E \arrow[r, "\varphi"] & F \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

(1.41.2) a direct sum decomposition

$$C = \bigoplus_{i=1}^{k} C_i$$

(1.41.3) elements t_i' of F' for $i = 1, \dots, k$

such that the data $\varphi'\colon E' \to F'$, $\mathrm{Coker}(\varphi) = C = \oplus_{i=1}^{k} C_i$ and $t_i' \in F'$ satisfies the conditions in (1.23). Then, setting t_i the images of t_i' by the morphism $F' \to F$, the data $\varphi\colon E \to F$, $\mathrm{Coker}(\varphi) = C = \oplus_{i=1}^{k} C_i$ and $t_i \in F$ satisfies the conditions in (1.23) too. Therefore we obtain two complexes $sA(\varphi'; n)$ and $sA(\varphi; n)$ and the morphism

$$sA(\varphi'; n) \to sA(\varphi; n),$$

induced by the morphisms $E' \to E$ and $F' \to F$ for a fixed non-negative integer n.

PROPOSITION (1.42) *In the situation above, the morphism above*

$$sA(\varphi'; n) \to sA(\varphi; n)$$

is a filtered quasi-isomorphism with respect to the filtrations L on both sides.

Proof. We have to prove that the morphism

$$\mathrm{Gr}_m^L sA(\varphi'; n) \to \mathrm{Gr}_m^L sA(\varphi; n),$$

induced by the morphism in the statement is a quasi-isomorphism. We have isomorphisms

$$\mathrm{Gr}_m^L sA(\varphi'; n) \simeq \bigoplus_{\substack{\alpha+\beta=m \\ l \geqslant \max(0,-\beta)}} \mathrm{Gr}_\alpha^L sA(\varphi_k'; n-\beta-l)[-\beta-2l] \otimes \bigwedge^{\beta+2l+1} (F'/F_k')$$

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and

$$\mathrm{Gr}_m^L sA(\varphi;n) \simeq \bigoplus_{\substack{\alpha+\beta=m \\ l\geqslant \max(0,-\beta)}} \mathrm{Gr}_\alpha^L sA(\varphi_k; n-\beta-l)[-\beta-2l] \otimes \bigwedge^{\beta+2l+1}(F/F_k)$$

by Theorem (1.40). Under the identification above, the morphism $\mathrm{Gr}_m^L sA(\varphi';n) \to \mathrm{Gr}_m^L sA(\varphi;n)$ in question is identified with the morphism which is a direct sum of the tensor product of the morphism defined in the same way as $\mathrm{Gr}_m^L sA(\varphi';n) \to \mathrm{Gr}_m^L sA(\varphi;n)$ for

$$\begin{tikzcd}
E' \arrow[r, "\varphi'_k"] \arrow[d] & F'_k \arrow[r] \arrow[d] & \bigoplus_{i=1}^{k-1} C_i \arrow[r] \arrow[d, equal] & 0 \\
E \arrow[r, "\varphi_k"] & F_k \arrow[r] & \bigoplus_{i=1}^{k-1} C_i \arrow[r] & 0
\end{tikzcd}$$

and the isomorphism

$$\bigwedge^{\beta+2l+1}(F'/F'_k) \to \bigwedge^{\beta+2l+1}(F/F_k)$$

induced from the morphism $F' \to F$. Therefore we reduce the problem to the case of $k = 1$ by induction on k. In the case of $k = 1$, the double complex $A(\varphi;n)^{p,q}$ is given by

$$A(\varphi;n)^{p,q} = \mathrm{Kos}^{n+q+1}(\varphi)^{p+q+1}/W_q\,\mathrm{Kos}^{n+q+1}(\varphi)^{p+q+1}$$

and then we have

$$\mathrm{Gr}_m^L sA(\varphi;n) \simeq \bigoplus_{l\geqslant \max(0,-m)} \mathrm{Kos}^{n-m-l}(\overline{\varphi})[-m-2l] \otimes \bigwedge^{m+2l+1} C$$

for every m, where $\overline{\varphi}$ is the morphism $E \to \varphi(E)$ induced from φ. Because we have the same result on $sA(\varphi';n)$ and because the isomorphism above is functorial, we complete the proof by Corollary (1.7). $\square$

PROPOSITION (1.43) *In addition to the assumption in* (1.23), *we assume that the kernel of* φ *is of dimension one. Once we fix a non-zero element* e *of* $\mathrm{Ker}(\varphi)$ *we have the morphism*

$$\mathrm{Kos}^n(\varphi) \to \mathrm{Kos}^{n'}(\varphi),$$

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defined in (1.8) *for two integers* n *and* n' *with* $n \leqslant n'$*. If the integer* n *satisfies the condition* $n \geqslant r = \dim_K C$*, then the morphism*

$$sA(\varphi; n) \to sA(\varphi; n'),$$

induced by the morphism above is a filtered quasi-isomorphism with respect to the filtrations L *on both sides.*

Proof. Similarly to the proposition above, we can reduce the problem to the case of $k = 1$. (Because the conditions $l \geqslant \max(0, -\beta)$, $n - \beta - l < r - r_k$ and $n \geqslant r$ implies the inequality $\beta + 2l + 1 > r_k$, and then we have

$$\bigwedge^{\beta+2l+1} F/F_k = 0,$$

since $r_k = \dim_K(F/F_k)$. Therefore we have

$$sA(\varphi_k; n - \beta - l)[-\beta - 2l] \otimes \bigwedge^{\beta+2l+1} (F/F_k) = 0,$$

if $n - \beta - l < r - r_k = \dim_K \operatorname{Coker}(\varphi_k)$ for $l \geqslant \max(0, -\beta)$. Therefore we can regard the direct sum decomposition of $\mathrm{Gr}_m^L sA(\varphi; n)$ and $\mathrm{Gr}_m^L sA(\varphi; n')$ in the proof of the proposition above as the direct sums in which the index l runs through the integers with the conditions $l \geqslant \max(0, -\beta)$ and $n - \beta - l \geqslant r - r_k = \dim_K \operatorname{Coker}(\varphi_k)$. Thus the induction on k works.) For the case of $k = 1$, we have

$$\mathrm{Gr}_m^L sA(\varphi; n) \simeq \bigoplus_{l \geqslant \max(0, -m)} \mathrm{Kos}^{n-m-l}(\overline{\varphi})[-m - 2l] \otimes \bigwedge^{m+2l+1} C$$

and

$$\mathrm{Gr}_m^L sA(\varphi; n') \simeq \bigoplus_{l \geqslant \max(0, -m)} \mathrm{Kos}^{n'-m-l}(\overline{\varphi})[-m - 2l] \otimes \bigwedge^{m+2l+1} C$$

where $\overline{\varphi}$ is the morphism $E \to \varphi(E)$ induced by φ. So we conclude the result by Corollary (1.9). $\square$

2. A complex of sheaves of Q-vector spaces

(2.1) In this section we construct a complex of sheaves of **Q**-vector spaces, which will turn out to be the underlying **Q**-structure of our cohomological complex in Section 4, from the data $(X; D_1, \ldots, D_k; t_1, \ldots, t_k)$ where X be a connected complex manifold, $D_1, \ldots, D_k$ reduced simple normal crossing divisors with defining functions $t_1, \ldots, t_k$.

(2.2) Let X be a topological space, K a field and $\varphi\colon E \to F$ a morphism of sheaves of K-vector spaces on X. Then we can define a complex of sheaves on X in the same way as in the last section, that is,

$$\mathrm{Kos}^n(\varphi)^p = \Gamma_{n-p} E \otimes_K \bigwedge^p F$$

at the pth step with $0 \leqslant p \leqslant n$. We define a filtration W on $\mathrm{Kos}^n(\varphi)$ similarly as in the last section. We denote the cokernel of φ by C as before. Once a direct sum decomposition

$$C = \bigoplus_{i=1}^{k} C_i \tag{2.2.1}$$

is fixed we obtain the filtrations $W(C_i)$ on $\mathrm{Kos}^n(\varphi)$ as in the last section. If we fix a global section t of the sheaf F, in addition, we have a morphism

$$t\wedge\colon \mathrm{Kos}^n(\varphi)^p \to \mathrm{Kos}^{n+1}(\varphi)^{p+1},$$

as in (1.21).

(2.3) Assume, in addition, that we have a continuous map $f\colon Y \to X$. Because the inverse image functor f^{-1} commutes with taking the tensor product and taking the wedge product, we have

$$f^{-1}\mathrm{Kos}^n(\varphi) = \mathrm{Kos}^n(f^{-1}\varphi) \tag{2.3.1}$$

where $f^{-1}\varphi\colon f^{-1}E \to f^{-1}F$ is the topological pull-back of the morphism φ. For the case that we have a direct sum decomposition (2.2.1), the cokernel of $f^{-1}\varphi$ admits a direct sum decomposition

$$\mathrm{Coker}(f^{-1}\varphi) = f^{-1}C = \bigoplus_{i=1}^{k} f^{-1}C_i$$

because the functor f^{-1} is exact. Via the identification (2.3.1), the filtrations $f^{-1}W$ and $f^{-1}W(C_i)$ $(i = 1, \ldots, k)$ on the left-hand side are identified with the filtrations W and $W(f^{-1}C_i)$ on the right-hand side. We can easily see it by using the exactness of the functor f^{-1}.

(2.4) Let t be a global section of the sheaf F as in (2.2). Since we have the canonical morphism $f^{-1}\colon \Gamma(X, F) \to \Gamma(Y, f^{-1}F)$, we obtain an element $f^{-1}t$ of $\Gamma(Y, f^{-1}F)$ for an element t of $\Gamma(X, F)$. Then we have a morphism

$$t\wedge\colon \mathrm{Kos}^n(\varphi)^p \to \mathrm{Kos}^{n+1}(\varphi)^{p+1}$$

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on X and a morphism

$$(f^{-1}t)\wedge\colon \mathrm{Kos}^n(f^{-1}\varphi)\to \mathrm{Kos}^n(f^{-1}\varphi)$$

on Y. Via the identification (2.3) the last morphism is identified with the inverse image of the former one, that is,

$$(f^{-1}t)\wedge = f^{-1}(t\wedge).$$

(2.5) Now we assume that we are given the following data.

(2.5.1) A morphism of sheaves of K-vector spaces $\varphi\colon E\to F$ on X. We denote the cokernel of φ by C and the projection $F\to C$ by π.

(2.5.2) A direct sum decomposition

$$C=\bigoplus_{i=1}^{k} C_i$$

of the cokernel of φ.

(2.5.3) Elements t_i of $\Gamma(X,F)$ for $i=1,\dots,k$ such that the germ $\pi(t_i)_x$ is contained in the subspace $(C_i)_x$ of the stalk C_x for every i at any point x of X, and is not equal to zero if the stalk $(C_i)_x$ is not zero.

For a fixed non-negative integer n we define a $(k+1)$-ple complex $A_X(\varphi;n)^{p,q}$, where p is an integer and q is an element of $\mathbf{Z}^k$, in the same way as in the last section, that is,

$$A_X(\varphi;n)^{p,q}=\mathrm{Kos}^{n+|q|+k}(\varphi)^{p+|q|+k}/\sum_{i=1}^{k}W(C_i)_{q_i}\mathrm{Kos}^{n+|q|+k}(\varphi)^{p+|q|+k}$$

for $p\in\mathbf{Z}_{\geqslant 0}$ and $q=(q_1,\dots,q_k)\in(\mathbf{Z}_{\geqslant 0})^k$. The associated single complex of the $(k+1)$-ple complex $A_X(\varphi;n)^{p,q}$ is denoted by $sA_X(\varphi;n)$ as before. We define three filtrations L, $L(C_k)$ and $\hat{L}$ on $sA_X(\varphi;n)$ as in (1.26).

(2.6) If we are given a continuous map $f\colon Y\to X$ in addition, the data $f^{-1}\varphi\colon f^{-1}E\to f^{-1}F$, $\mathrm{Coker}(f^{-1}\varphi)=f^{-1}C=\oplus_{i=1}^{k}f^{-1}C_i$ and $f^{-1}t_i\in\Gamma(Y,f^{-1}F)$ satisfy the conditions (2.5.1)–(2.5.3) because $(f^{-1}C)_y=C_{f(y)}$ for every point y of Y. Then we have a $(k+1)$-ple complex $A_Y(f^{-1}\varphi;n)^{p,q}$ on Y and the single complex $sA_Y(f^{-1}\varphi;n)$ associated to it on Y. Then we have

$$f^{-1}A_X(\varphi;n)^{p,q}=A_Y(f^{-1}\varphi;n)^{p,q}$$

for every p and q, and

$$f^{-1}sA_X(\varphi;n)=sA_Y(f^{-1}\varphi;n)$$

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by (2.3), (2.4) and the exactness of the functor f^{-1}.

(2.7) Let X be a complex manifold and D a reduced simple normal crossing divisor on X. We define a monoid sheaf $M_X(D)$ on X by

$$M_X(D) = j_*\mathcal{O}^*_{X\setminus D} \cap \mathcal{O}_X,$$

where j is the open immersion $X \setminus D \hookrightarrow X$ and the intersection is taken in the sheaf $j_*\mathcal{O}_{X\setminus D}$. The sheaf $M_X(D)$ is considered as a monoid sheaf by the multiplication in $\mathcal{O}_X$. Then $\mathcal{O}^*_X$ is a monoid subsheaf of $M_X(D)$. The abelian sheaf associated to the monoid sheaf $M_X(D)$ is denoted by $M_X(D)^{gp}$. Then we have the following exact sequence

$$0 \to \mathcal{O}^*_X \to M_X(D)^{gp} \to (a_1)_*\mathbf{Z}_{D^1} \to 0,$$

where D^1 is the disjoint union of all the irreducible components of the divisor D, and $\mathbf{Z}_{D^1}$ is the constant sheaf on D^1 with value $\mathbf{Z}$. Composing the exponential map $f \mapsto e^f$ of $\mathcal{O}_X$ to $\mathcal{O}^*_X$ and the inclusion $\mathcal{O}^*_X \hookrightarrow M_X(D)^{gp}$, we obtain a morphism of abelian sheaves $e\colon \mathcal{O}_X \to M_X(D)^{gp}$. It is trivial that the kernel of this morphism is the sheaf $2\pi\sqrt{-1}\mathbf{Z}$. We set $E_X = \mathcal{O}_X \otimes_{\mathbf{Z}} \mathbf{Q} (\simeq \mathcal{O}_X)$ and $F_X(D) = M_X(D)^{gp} \otimes_{\mathbf{Z}} \mathbf{Q}$. Then we have a morphism $E_X \to F_X(D)$ by tensoring $\mathbf{Q}$ to the morphism $e\colon \mathcal{O}_X \to M_X(D)^{gp}$ above. We denote it by $\varphi_X(D)$. Moreover, we denote the cokernel of $\varphi_X(D)$ by $C_X(D)$ and the projection $F_X(D) \to C_X(D)$ by $\pi_X(D)$. Then we have an exact sequence

$$0 \to 2\pi\sqrt{-1}\mathbf{Q} \to E_X \xrightarrow{\varphi_X(D)} F_X(D) \xrightarrow{\pi_X(D)} C_X(D) \to 0 \tag{2.7.1}$$

and a natural isomorphism $C_X(D) \to (a_1)_*\mathbf{Q}_{D^1}$. For a fixed non-negative integer n we denote the Koszul complex $\mathrm{Kos}^n(\varphi_X(D))$ simply by $\mathrm{Kos}^n_X(D)$. The filtration W on $\mathrm{Kos}^n_X(D)$ defined as in (2.2) is denoted by $W_X(D)$.

(2.8) Let X be a complex manifold and $D_1, \ldots, D_k$ reduced simple normal crossing divisors on X such that $D_1 + \cdots + D_k$ is a reduced simple normal crossing divisor too. Then, setting $D = \sum_{i=1}^k D_i$, we apply the construction above to D. In this case we have a direct sum decomposition

$$C_X(D) = \bigoplus_{i=1}^k C_X(D_i). \tag{2.8.1}$$

Then we have the filtration $W(C_X(D_i))$ on the Koszul complex $\mathrm{Kos}^n_X(D)$, which is denoted by $W_X(D_i)$ for every $i = 1, \ldots, k$. Notice that we have

$$\bigoplus_{j \neq i} C_X(D_j) = C_X(D - D_i)$$

$$F_X(D)_i = \pi_X(D)^{-1}\left(\bigoplus_{j\neq i} C_X(D_j)\right) = F_X(D - D_i)$$

for every $i = 1, \dots, k$ and that the projection $F_X(D)_i \to \bigoplus_{j\neq i} C_X(D_j)$ induced by the morphism $\pi_X(D)$ coincides with the morphism $\pi_X(D-D_i)\colon F_X(D-D_i) \to C_X(D - D_i)$.

(2.9) Moreover, we assume that we are given a global defining function t_i of the divisor D_i for every i. Then the t_i's are global sections of the sheaf $M_X(D)$. We denote the images of t_i's in $\Gamma(X, M_X(D)^{gp})$ or $\Gamma(X, F_X(D))$ by the same letters t_i if there is no danger of confusion. Then the data

$$\varphi_X(D)\colon E_X \to F_X(D)$$

$$\mathrm{Coker}(\varphi_X(D)) = C = \bigoplus_{i=1}^{k} C_X(D_i)$$

$$t_i \in \Gamma(X, F_X(D)) \quad \text{for } i = 1, \dots, k$$

satisfy the conditions (2.5.1)–(2.5.3). Therefore we have a complex of sheaves of **Q**-vector spaces $A_X(\varphi_X(D); n)^{p,q}$ and $sA_X(\varphi_X(D); n)$ on X as in the last section. We denote them by $A_X(D_1, \dots, D_k; n)^{p,q}$ and $sA_X(D_1, \dots, D_k; n)$ respectively. We denote the filtrations L, $L(C_X(D_k))$ and $\hat{L}$ on $sA_X(D_1, \dots, D_k; n)$ by $L_X(D)$, $L_X(D_k)$ and $\hat{L}_X$ respectively.

(2.10) In addition, we assume that we are given a reduced simple normal crossing divisor Y on X such that the divisor $D + Y$ is a reduced simple normal crossing divisor on X again. Now we fix a non-negative integer m, and denote the canonical morphism $a_m\colon Y^m \to X$ simply by a (see (0.5)). On the complex manifold Y^m we have reduced simple normal crossing divisors $D_1 \cap Y^m, \dots, D_k \cap Y^m$ such that the sum $D \cap Y^m = D_1 \cap Y^m + \cdots + D_k \cap Y^m$ is again a reduced simple normal crossing divisor on Y^m. Moreover, if we have the global defining functions t_i's of D_i's, then the a^*t_i's are the global defining functions of $D_i \cap Y^m$. Thus we have $\mathrm{Kos}^n_{Y^m}(D \cap Y^m)$, $A_{Y^m}(D_1 \cap Y^m, \dots, D_k \cap Y^m; n)^{p,q}$ and $sA_{Y^m}(D_1 \cap Y^m, \dots, D_k \cap Y^m; n)$ on Y^m.

(2.11) It is clear that the morphism $a^{-1}\mathcal{O}_X \to \mathcal{O}_{Y^m}$ induces a morphism $a^{-1}M_X(D) \to M_{Y^m}(D \cap Y^m)$ of monoid sheaves on Y^m. It is easy to see that we have a commutative diagram

$$\begin{tikzcd}
a^{-1}\mathcal{O}_X \arrow[r] \arrow[d] & a^{-1}M_X(D)^{gp} \arrow[d] \\
\mathcal{O}_{Y^m} \arrow[r] & M_{Y^m}(D \cap Y^m)^{gp}
\end{tikzcd}$$

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and that the two horizontal arrows have the same kernel and cokernel. Tensoring **Q** with the above diagram we have

$$\begin{tikzcd} a^{-1}E_X \arrow[r, "a^{-1}\varphi_X(D)"] \arrow[d] & a^{-1}F_X(D) \arrow[d] \\ E_{Y^m} \arrow[r, "\varphi_{Y^m}(D\cap Y^m)"] & F_{Y^m}(D\cap Y^m) \end{tikzcd}$$

and the two horizontal arrows $a^{-1}\varphi_X(D)$ and $\varphi_{Y^m}(D\cap Y^m)$ have the same kernel $2\pi\sqrt{-1}\mathbf{Q}_{Y^m}$ and the same cokernel $a^{-1}C_X(D) = C_{Y^m}(D\cap Y^m)$. Therefore we obtain the morphism

$$a^{-1}\operatorname{Kos}^n_X(D) = \operatorname{Kos}^n(a^{-1}\varphi_X(D)) \to \operatorname{Kos}^n_{Y^m}(D\cap Y^m), \tag{2.11.1}$$

from the commutative diagram above for every non-negative integer n. It is easy to see that these morphisms satisfy the conditions in (1.41). Therefore the morphism

$$a^{-1}sA_X(D_1,\dots,D_k;n) \to sA_{Y^m}(D_1\cap Y^m,\dots,D_k\cap Y^m;n)$$

induced from the morphism (2.11.1) is a filtered quasi-isomorphism with respect to the filtration $a^{-1}L_X(D)$ on the left-hand side and the filtration $L_{Y^m}(D\cap Y^m)$ on the right hand side by applying Proposition (1.42) and by (2.6). Because a is a finite morphism, we have a filtered quasi-isomorphism

$$a_*(a^{-1}sA_X(D_1,\dots,D_k;n)) \to a_*(sA_{Y^m}(D_1\cap Y^m,\dots,D_k\cap Y^m;n)),$$

with respect to the filtration $a_*(a^{-1}L_X(D))$ on the left and $a_*L_{Y^m}(D\cap Y^m)$ on the right. On the other hand, we easily see that the canonical morphism

$$sA_X(D_1,\dots,D_k;n) \to a_*(a^{-1}sA_X(D_1,\dots,D_k;n))$$

induces an isomorphism

$$sA_X(D_1,\dots,D_k;n)\otimes\bigwedge^m C_X(Y) \to a_*(a^{-1}sA_X(D_1,\dots,D_k;n))$$

and that the filtration $L_X(D)\otimes\bigwedge^m C_X(Y)$ on the left-hand side corresponds to the filtration $a_*(a^{-1}L_X(D))$ on the right via the isomorphism above. Therefore we have a filtered quasi-isomorphism

$$sA_X(D_1,\dots,D_k;n)\otimes\bigwedge^m C_X(Y) \to a_*(sA_{Y^m}(D_1\cap Y^m,\dots,D_k\cap Y^m;n)),$$

with respect to the filtration $L_X(D) \otimes \bigwedge^m C_X(Y)$ on the left and $a_* L_{Y^m}(D \cap Y^m)$ on the right.

(2.12) In addition to the assumption in (2.11) above, assume that we have a global defining function u of Y on X. Then a $(k+2)$-ple complex $A_X(D_1, \dots, D_k, Y; n)^{p,q}$ is defined for a non-negative integer n. We also have the complex $sA_X(D_1, \dots, D_k, Y; n)$ associated to the $(k+2)$-ple complex $A_X(D_1, \dots, D_k, Y; n)^{p,q}$.

(2.13) We abbreviate $A_X(D_1, \dots, D_k, Y; n)^{p,q}$ and $sA_X(D_1, \dots, D_k, Y; n)$ as $A_X(n)$ and $sA_X(n)$ respectively for a while. We use the similar abbreviation for $A_{Y^m}(D_1 \cap Y^m, \dots, D_k \cap Y^m; n)$ and $sA_{Y^m}(D_1 \cap Y^m, \dots, D_k \cap Y^m; n)$. By combining the results above and Theorem (1.40) we obtain the following.

PROPOSITION (2.14) *We have a quasi-isomorphism*

$$\mathrm{Gr}_m^{L_X} sA_X(n) \to \bigoplus_{\substack{\alpha+\beta=m \\ l \geqslant \max(0,-\beta)}} \mathrm{Gr}_\alpha^{L_{Y^{\beta+2l+1}}} a_*(sA_{Y^{\beta+2l+1}}(n-\beta-l)[-\beta-2l]),$$

where a denotes the morphism $a_{\beta+2l+1}$ for every l and β.

3. Differential forms with logarithmic poles

(3.1) In this section we collect the results on differential forms with logarithmic poles which we need later. Let X be a complex manifold and Y a reduced simple normal crossing divisor on X. We denote by $\Omega_X^p(\log Y)$ the sheaf of p-forms with logarithmic poles along Y. The weight filtration defined in Deligne [2] is denoted by $W_X(Y)$.

DEFINITION (3.2) Let X be a complex manifold and Y and Z reduced simple normal crossing divisors on X such that $Y + Z$ is again a reduced simple normal crossing divisor on X. (We permit $Y = 0$ or $Z = 0$.) Then we define an increasing filtration $W_X(Y)$ on $\Omega_X^p(\log(Y + Z))$ by

$$\begin{aligned} &W_X(Y)_m \Omega_X^p(\log(Y+Z)) \\ &\quad = \text{the image of } \ \Omega_X^{p-m}(\log Z) \otimes \Omega_X^m(\log(Y+Z)) \ \text{ in } \ \Omega_X^p(\log(Y+Z)) \end{aligned}$$

and call it the weight filtration with respect to Y. It is easy to see that the subsheaves $(W_X(Y)_m \Omega_X^p(\log(Y + Z)))$ form a subcomplex of $\Omega_X^{\cdot}(\log(Y + Z))$.

(3.3) Let X be a complex manifold and Y and D reduced simple normal crossing divisors on X such that $D + Y$ is a reduced simple normal crossing divisor too. For an element σ of $\mathfrak{S}_m^N$, where N is the number of the irreducible components of Y and m is an integer with $1 \leqslant m \leqslant N$, we have a morphism of sheaves

$$\Omega_{Y_\sigma}^p(\log D \cap Y_\sigma) \to \mathrm{Gr}_m^{W_X(Y)} \Omega_X^{p+m}(\log(D + Y)),$$

defined in El Zein [4] for every p. Taking the direct sum of the morphisms above, we have an isomorphism

$$(a_m)_*\Omega^p_{Y^m}(\log D\cap Y^m)\to \mathrm{Gr}^{W_X(Y)}_m\Omega^{p+m}_X(\log(D+Y))$$

(for the proof, see El Zein [4] or Deligne [2]). The inverse of this isomorphism is called the residue isomorphism with respect to Y. For an integer m with $1\leqslant m\leqslant p$,

$$\mathrm{Res}^Y_m\colon \mathrm{Gr}^{W_X(Y)}_m\Omega^p_X(\log(D+Y))\to (a_m)_*\Omega^{p-m}_{Y^m}(\log D\cap Y^m),$$

denotes the residue isomorphism with respect to Y. For an element $\sigma\in\mathfrak{S}^N_m$,

$$\mathrm{Res}^Y_\sigma\colon \mathrm{Gr}^{W_X(Y)}_m\Omega^p_X(\log(D+Y))\to \Omega^{p-m}_{Y_\sigma}(\log D\cap Y_\sigma)$$

denotes the corresponding direct summand of the isomorphism Res^Y_m. For the case of $m=0$, we have an isomorphism

$$\mathrm{Gr}^{W_X(Y)}_0\Omega^p_X(\log(D+Y))\to \Omega^p_X(\log D)$$

also called the residue isomorphism with respect to Y and denoted by Res^Y_0. We can easily see that the weight filtration $W_{Y^m}(D\cap Y^m)_l\Omega^p_{Y^m}(\log D\cap Y^m)$ is identified with the filtration $W_X(D)_l\,\mathrm{Gr}^{W_X(Y)}_m\Omega^{p+m}_X(\log(D+Y))$, which is induced by $W_X(D)_l$ on $\Omega^{p+m}_X(\log(D+Y))$. In the case that the simple normal crossing divisor D is written as the sum of two simple normal crossing divisors, that is, $D=D_1+D_2$ by simple normal crossing divisors D_1 and D_2, it is easy to see, as before, that the weight filtration $W_{Y^m}(D_i\cap Y^m)_l\Omega^p_{Y^m}(\log D\cap Y^m)$ is identified via the residue isomorphism above, with the filtration $W_X(D_i)_l\,\mathrm{Gr}^{W_X(Y)}_m\Omega^{p+m}_X(\log(D+Y))$ which is induced from the filtration $W_X(D_i)_l$ on $\Omega^{p+m}_X(\log(D_1+D_2+Y))$ for $i=1,2$.

(3.4) Now we assume that we are given a global defining function u of Y on X. Then the wedge product with du/u defines a morphism

$$\frac{du}{u}\wedge\colon \Omega^p_X(\log(D+Y))\to\Omega^{p+1}_X(\log(D+Y))$$

for every p, that is,

$$\left(\frac{du}{u}\wedge\right)(\omega)=\frac{du}{u}\wedge\omega$$

for a local section ω of $\Omega^p_X(\log(D+Y))$. Then we easily see

$$\left(\frac{du}{u}\wedge\right)(W_X(Y)_m\Omega^p_X(\log(D+Y)))\subset W_X(Y)_{m+1}\Omega^{p+1}_X(\log(D+Y))$$

$$\left(\frac{du}{u}\wedge\right)(W_X(D)_m\Omega_X^p(\log(D+Y))) \subset W_X(D)_m\Omega_X^{p+1}(\log(D+Y))$$

for every p and m. In the case that $D = D_1 + D_2$ with reduced simple normal crossing divisors D_1 and D_2, we obtain the same result as the second one for D_i for $i = 1, 2$ instead of D. Moreover, we have the equality

$$d \circ \left(\frac{du}{u}\wedge\right) + \left(\frac{du}{u}\wedge\right) \circ d = 0$$

where d denotes the derivation d on the complex of the sheaves of the logarithmic forms $\Omega_X^{\cdot}(\log(D+Y))$.

(3.5) Now we study properties of the stalk of the sheaf of logarithmic differential forms at a given point x. So, we may assume the following: Let X be the polydisk in $\mathbf{C}^{N+N'}$ and a point x the origin. We denote the coordinate functions by $z_1, \ldots, z_N, z_{N+1}, \ldots, z_{N+N'}$. Let Y_i be a divisor on X defined by the function z_i for $i = 1, \ldots, N$ and $Y = \sum_{i=1}^N Y_i$. Moreover, let D be a reduced simple normal crossing divisor on X defined by the function $z_{i_1} \cdots z_{i_l}$ with $N + 1 \leqslant i_1 < \cdots < i_l \leqslant N + N'$. In the situation above, for an element $\sigma = (\sigma_1, \ldots, \sigma_m)$ of $\mathfrak{S}_m^N$, the closed subspace Y_σ of X is defined by the functions $z_{\sigma_1}, \ldots, z_{\sigma_m}$. Therefore, X is isomorphic to the product of Y_σ and the m-dimensional polydisk, and then we have a projection

$$\pi_\sigma: X \to Y_\sigma.$$

So we have a restriction morphism

$$\iota_\sigma^*: \Omega_X^p(\log D)_x \to \Omega_{Y_\sigma}^p(\log D \cap Y_\sigma)_x$$

induced by the inclusion $\iota_\sigma: Y_\sigma \hookrightarrow X$ and a morphism

$$\pi_\sigma^*: \Omega_{Y_\sigma}^p(\log D \cap Y_\sigma)_x \to \Omega_X^p(\log D)_x$$

induced by the projection π_σ. Trivially the composite $(\iota_\sigma^*) \circ (\pi_\sigma^*)$ is equal to the identity. It is also trivial that we have

$$(\pi_\sigma^*)W_{Y_\sigma}(D \cap Y_\sigma)_l\Omega_{Y_\sigma}^p(\log D \cap Y_\sigma)_x \subset W_X(D)_l\Omega_X^p(\log D)_x.$$

In the case that D is written in the form $D = D_1 + D_2$ with simple normal crossing divisors D_1 and D_2, we also have

$$(\pi_\sigma^*)W_{Y_\sigma}(D_i \cap Y_\sigma)_l\Omega_{Y_\sigma}^p(\log D \cap Y_\sigma)_x \subset W_X(D_i)_l\Omega_X^p(\log D)_x.$$

for $i = 1, 2$.

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(3.6) Let p be a positive integer, m an integer with $1 \leqslant m \leqslant p$ and $\sigma = (\sigma_1, \ldots, \sigma_m)$ an element of $\mathfrak{S}^N_m$. We define a morphism

$$s^Y_\sigma \colon \mathrm{Gr}^{W_X(Y)}_m \Omega^p_X(\log(D+Y))_x \to \Omega^p_X(\log(D+Y))_x$$

as follows. We have the residue morphism

$$\mathrm{Res}^Y_\sigma \colon \mathrm{Gr}^{W_X(Y)}_m \Omega^p_X(\log(D+Y))_x \to \Omega^{p-m}_{Y_\sigma}(\log D \cap Y_\sigma)_x$$

and the morphism

$$\pi^*_\sigma \colon \Omega^{p-m}_{Y_\sigma}(\log D \cap Y_\sigma)_x \to \Omega^{p-m}_X(\log D)_x.$$

Then for an element ϖ of $\mathrm{Gr}^{W_X(Y)}_m \Omega^p_X(\log(D+Y))_x$ we get an element $\pi^*_\sigma \circ \mathrm{Res}^Y_\sigma(\varpi)$ of $\Omega^{p-m}_X(\log D)_x$. Thus, we define an element $s^Y_\sigma(\varpi)$ of $\Omega^p_X(\log(D+Y))_x$ for an element ϖ of $\mathrm{Gr}^{W_X(Y)}_m \Omega^p_X(\log(D+Y))_x$ by

$$s^Y_\sigma(\varpi) = (\pi^*_\sigma \circ \mathrm{Res}^Y_\sigma(\varpi)) \wedge \frac{dz_{\sigma_1}}{z_{\sigma_1}} \wedge \cdots \wedge \frac{dz_{\sigma_m}}{z_{\sigma_m}}.$$

Moreover, we set

$$s^Y_m = \sum_{\sigma \in \mathfrak{S}^N_m} s^Y_\sigma \colon \mathrm{Gr}^{W_X(Y)}_m \Omega^p_X(\log(D+Y))_x \to \Omega^p_X(\log(D+Y))_x$$

then it is trivial that the image of s^Y_m is contained in $W_X(Y)_m \Omega^p_X(\log(D+Y))_x$ and that s^Y_m is a section of the canonical projection $W_X(Y)_m \Omega^p_X(\log(D+Y))_x \to \mathrm{Gr}^{W_X(Y)}_m \Omega^p_X(\log(D+Y))_x$. Therefore, we obtain a direct sum decomposition

$$\Omega^p_X(\log(D+Y))_x \simeq \bigoplus_{m=0}^{p} \mathrm{Gr}^{W_X(Y)}_m \Omega^p_X(\log(D+Y))_x \tag{3.6.1}$$

and then we have

$$\Omega^p_X(\log(D+Y))_x \simeq \bigoplus_{m=0}^{p} ((a_m)_* \Omega^{p-m}_{Y^m}(\log D \cap Y^m))_x, \tag{3.6.2}$$

by using the residue isomorphism. Via the identification above we have

$$W_X(Y)_l \Omega^p_X(\log(D+Y))_x \simeq \bigoplus_{m=0}^{l} ((a_m)_* \Omega^{p-m}_{Y^m}(\log D \cap Y^m))_x \tag{3.6.3}$$

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and

$$W_X(D)_l\Omega_X^p(\log(D+Y))_x \simeq \bigoplus_{m=0}^{p}((a_m)_* W_{Y^m}(D\cap Y^m)_l\Omega_{Y^m}^{p-m}(\log D\cap Y^m))_x. \tag{3.6.4}$$

In the case that D is equal to a sum of two simple normal crossing divisors D_1 and D_2, that is $D = D_1 + D_2$, then the second isomorphism holds for D_i instead of D for $i = 1, 2$.

COROLLARY (3.7) *Let X be a complex manifold, Y and $D_1, \ldots, D_k$ reduced simple normal crossing divisors on X such that $D_1 + \cdots + D_k + Y$ is a reduced simple normal crossing divisor too. We put $D = D_1 + \cdots + D_k$ for simplicity. Then we have an equality*

$$\left(\sum_{i=1}^{k} W_X(D_i)_{q_i}\right) \cap W_X(Y)_m = \sum_{i=1}^{k}(W_X(D_i)_{q_i} \cap W_X(Y)_m)$$

on $\Omega_X^p(\log(D+Y))$ for integers p, m, and q_i $(i = 1, \ldots, k)$.

Proof. It is sufficient that we prove the equality above for every stalk. On the stalk we have the direct sum decomposition (3.6.1), which implies the result. □

COROLLARY (3.8) *In the situation above, we have*

$$\left(\sum_{i=1}^{k} W_X(D_i)_{q_i} + W_X(D)_q\right) \cap W_X(Y)_m = \sum_{i=1}^{k}(W_X(D_i)_{q_i} \cap W_X(Y)_m) + W_X(D)_q \cap W_X(Y)_m$$

on $\Omega_X^p(\log(D+Y))$ for integers p, q, m and q_i $(i = 1, \ldots, k)$.

Proof. As above. □

COROLLARY (3.9) *Under the assumption in Corollary* (3.7) *we have an equality*

$$\left(\sum_{i=1}^{k} W_X(D_i)_{q_i} + W_X(Y)_q\right) \cap W_X(Y)_m = \sum_{i=1}^{k}(W_X(D_i)_{q_i} \cap W_X(Y)_m) + W_X(Y)_q \cap W_X(Y)_m$$

on $\Omega_X^p(\log(D+Y))$ for integers p, q, m and q_i $(i = 1, \ldots, k)$.

Proof. Easy from Corollary (3.7). □

COROLLARY (3.10) *Under the assumption in Corollary* (3.7), *we have*

$$\left(\sum_{i=1}^{k} W_X(D_i)_{q_i} + W_X(Y)_q + W_X(D)_l\right) \cap W_X(Y)_m$$

$$= \sum_{i=1}^{k}(W_X(D_i)_{q_i} \cap W_X(Y)_m)$$

$$+W_X(Y)_q \cap W_X(Y)_m + W_X(D)_l \cap W_X(Y)_m$$

on $\Omega_X^p(\log(D+Y))$ *for integers* p, q, m, l *and* q_i $(i = 1, \ldots, k)$.

Proof. Easy from Corollary (3.8). □

(3.11) Let X be a complex manifold and $D_1, \ldots, D_k$ reduced simple normal crossing divisors on X such that $D = \sum_{i=1}^{k} D_i$ is a reduced simple normal crossing divisor on X too. We define an $\mathcal{O}_X$-module $B_X(D_1, \ldots, D_k)^{p,q}$ by

$$B_X(D_1, \ldots, D_k)^{p,q} = \Omega_X^{p+|q|+k}(\log D) \Big/ \left(\sum_{i=1}^{k} W_X(D_i)_{q_i}\right)$$

for $p \in \mathbf{Z}_{\geqslant 0}$ and $q = (q_1, \ldots, q_k) \in (\mathbf{Z}_{\geqslant 0})^k$, and by

$$B_X(D_1, \ldots, D_k)^{p,q} = 0$$

if $p < 0$ or $q \in \mathbf{Z}^k \backslash (\mathbf{Z}_{\geqslant 0})^k$. We denote the induced filtrations on $B_X(D_1, \ldots, D_k)^{p,q}$ from the filtrations $W_X(D)$ and $W_X(D_i)$ $(i = 1, \ldots, k)$ on $\Omega_X^{p+|q|+k}(\log(D+Y))$ by the same symbols $W_X(D)$ and $W_X(D_i)$. Then we have

$$W_X(D_k)_m B_X(D_1, \ldots, D_k)^{p,q}$$

$$= \begin{cases} 0 & \text{if } \ m \leqslant q_k \\ B_X(D_1, \ldots, D_k)^{p,q} & \text{if } \ m \geqslant p + q_k + 1, \end{cases} \tag{3.11.1}$$

for every p and q. The first case is trivial and the second can be seen by counting the number of poles.

(3.12) In addition to the assumption in (3.11), we assume that we are given defining functions t_i of D_i over X for all i. Then we define morphisms

$$\frac{dt_i}{t_i}\wedge\colon \Omega_X^p(\log D) \to \Omega_X^{p+1}(\log D)$$

as in (3.4) for every p. By the conditions on the weight filtrations $W_X(D_i)$'s in (3.4) these morphisms induce morphisms

$$d_i = \frac{dt_i}{t_i}\wedge : B_X(D_1, \ldots, D_k)^{p,q} \to B_X(D_1, \ldots, D_k)^{p,q+e_i}$$

for $i = 1, \ldots, k$ and for every $p \in \mathbf{Z}$ and $q \in \mathbf{Z}^k$. On the other hand the differential

$$d: \Omega_X^p(\log D) \to \Omega_X^{p+1}(\log D)$$

induces a morphism

$$d_0: B_X(D_1, \ldots, D_k)^{p,q} \to B_X(D_1, \ldots, D_k)^{p+1,q}$$

for every p and q. Then we can easily see that

$$(B_X(D_1, \ldots, D_k)^{p,q}; d_0, d_1, \ldots, d_k)_{p\in\mathbf{Z}, q\in\mathbf{Z}^k}$$

forms a $(k+1)$-ple complex of $\mathcal{O}_X$-modules. We denote the single complex associated to the $(k+1)$-ple complex above by $sB_X(D_1, \ldots, D_k)$.

(3.13) We define a decreasing filtration F_X and increasing filtrations $L_X(D)$ and $L_X(D_i)$ $(i = 1, \ldots, k)$ on $sB_X(D_1, \ldots, D_k)$ by

$$F_X^p sB_X(D_1, \ldots, D_k)^n = \bigoplus_{\substack{p'+|q|=n \\ p' \geqslant p}} B_X(D_1, \ldots, D_k)^{p',q}$$

$$L_X(D)_m sB_X(D_1, \ldots, D_k)^n = \bigoplus_{p+|q|=n} W_X(D)_{m+2|q|+k} B_X(D_1, \ldots, D_k)^{p,q}$$

$$L_X(D_i)_m sB_X(D_1, \ldots, D_k)^n = \bigoplus_{p+|q|=n} W_X(D_i)_{m+2q_i+1} B_X(D_1, \ldots, D_k)^{p,q}.$$

We can easily see that the definition above actually defines filtrations by complexes on $sB_X(D_1, \ldots, D_k)$ by easy computation.

(3.14) For the simplicity of the subscript we shift the index k to $k+1$ and denote D_{k+1} by Y for a while. Therefore the index q of $B_X(D_1, \ldots, D_k, Y)^{p,q}$ runs through the set $\mathbf{Z}^{k+1}$ and then $B_X(D_1, \ldots, D_k, Y)^{p,q}$ becomes a $(k+2)$-ple complex. In this situation we also consider the filtration $W_X(D)$ on $B(D_1, \ldots, D_k, Y)^{p,q}$ induced by the filtration $W_X(D)$ on $\Omega_X^{p+|q|+k+1}(\log(D+Y))$. Moreover we also consider the filtration $L_X(D)$ on $sB_X(D_1, \ldots, D_k, Y)$ defined by

$$L_X(D)_m sB_X(D_1, \ldots, D_k, Y)^n$$
$$= \bigoplus_{p+|q|=n} W_X(D)_{m+2|\hat{q}|+k} B_X(D_1, \ldots, D_k, Y)^{p,q},$$

where $\hat{q}$ is an element $(q_1, \ldots, q_k)$ of $\mathbf{Z}^k$ for an element $q = (q_1, \ldots, q_k, q_{k+1})$ of $\mathbf{Z}^{k+1}$.

PROPOSITION (3.15) *Under the assumption above, we have the canonical isomorphism*

$$\begin{aligned}&\mathrm{Gr}_m^{W_X(Y)} B_X(D_1, \ldots, D_k, Y)^{p,q}\\&\simeq \begin{cases}(a_m)_* B_{Y^m}(D_1 \cap Y^m, \ldots, D_k \cap Y^m)^{p+q_{k+1}+1-m, \hat{q}} & \text{if } m > q_{k+1}\\ 0 \quad \text{if } m \leqslant q_{k+1}\end{cases}\end{aligned}$$

for every $p \in \mathbf{Z}_{\geqslant 0}$ and $q = (q_1, \ldots, q_{k+1}) \in (\mathbf{Z}_{\geqslant 0})^{k+1}$. Moreover, the filtration $W_X(D)_l$ on the left-hand side is identified with the filtration $W_{Y^m}(D \cap Y^m)_l$ on the right-hand side via the identification above.

Proof. It is sufficient to prove the case $q_{k+1} < m \leqslant p + q_{k+1} + 1$ by $(3.11.1)$. In this case we have

$$\begin{aligned}&\mathrm{Gr}_m^{W_X(Y)} B_X(D_1, \ldots, D_k, Y)^{p,q}\\&\simeq \mathrm{Gr}_m^{W_X(Y)} \Omega_X^{p+|q|+k+1}(\log(D+Y)) \Big/ \left(\sum_{i=1}^{k} W_X(D_i)_{q_i} + W_X(Y)_{q_{k+1}}\right)\end{aligned}$$

by using Lemma (1.34) and Corollary (3.9), where $W_X(D_i)$ and $W_X(Y)$ above denote the induced filtrations on $\mathrm{Gr}_m^{W_X(Y)} \Omega_X^{p+|q|+k+1}(\log(D+Y))$ from the filtrations $W_X(D_i)$ and $W_X(Y)$ on $\Omega_X^{p+|q|+k+1}(\log(D+Y))$. Then we can easily obtain the first part by the residue isomorphism. Lemma (1.33) and Corollary (3.10) imply the second part. □

PROPOSITION (3.16) *In the situation above, we have*

$$W_X(D+Y)_m = \sum_{\alpha+\beta=m} W_X(D)_\alpha \cap W_X(Y)_\beta$$

on $B_X(D_1, \ldots, D_k, Y)^{p,q}$ for every p, q and m.

Proof. It is easy to see that the equality

$$\begin{aligned}&W_X(D+Y)_m \Omega_X^p(\log(D+Y))\\&= \sum_{\alpha+\beta=m} W_X(D)_\alpha \Omega_X^p(\log(D+Y)) \cap W_X(Y)_\beta \Omega_X^p(\log(D+Y))\end{aligned} \tag{3.16.1}$$

holds. Therefore we have

$$W_X(D+Y)_m \subset \sum_{\alpha+\beta=m} W_X(D)_\alpha \cap W_X(Y)_\beta,$$

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on $B_X(D_1, \dots, D_k, Y)^{p,q}$ for every p, q and m. In order to prove

$$W_X(D+Y)_m \supset W_X(D)_\alpha \cap W_X(Y)_\beta,$$

on $B_X(D_1, \dots, D_k, Y)^{p,q}$ for every p, q, m and α, β with $\alpha+\beta = m$, it is sufficient to prove

$$\begin{aligned}&\left(W_X(D)_\alpha + \sum_{i=1}^{k} W_X(D_i)_{q_i} + W_X(Y)_{q_{k+1}}\right) \cap W_X(Y)_\beta \\ &\quad \subset W_X(D)_\alpha \cap W_X(Y)_\beta + \sum_{i=1}^{k} W_X(D_i)_{q_i} + W_X(Y)_{q_{k+1}},\end{aligned}$$

on $\Omega_X^p(\log(D+Y))$ for every p, q, α and β by the definition of $B_X(D_1, \dots, D_k, Y)^{p,q}$ and by (3.16.1). This is trivial from Corollary (3.10). Thus, we complete the proof. □

(3.17) From now on, we denote $B_X(D_1, \dots, D_k, Y)^{p,q}$ and $B_{Y^m}(D_1 \cap Y^m, \dots, D_k \cap Y^m)^{p,q}$ by $B_X^{p,q}$ and $B_{Y^m}^{p,q}$ respectively for every integer m for simplicity. We use the similar abbreviation for $sB_X(D_1, \dots, D_k, Y)$ and $sB_{Y^m}(D_1 \cap Y^m, \dots, D_k \cap Y^m)$.

PROPOSITION (3.18) *In the situation above, we have*

$$L_X(D+Y)_m sB_X = \sum_{\alpha+\beta=m} L_X(D)_\alpha sB_X \cap L_X(Y)_\beta sB_X.$$

Therefore, we have the canonical isomorphism

$$\bigoplus_{\alpha+\beta=m} \mathrm{Gr}_\alpha^{L_X(D)} \mathrm{Gr}_\beta^{L_X(Y)} sB_X \simeq \mathrm{Gr}_m^{L_X(D+Y)} sB_X$$

Proof. The first equality follows from Proposition (3.16) easily. Then we obtain the second one as Corollary (1.29). □

LEMMA (3.19) *We have an isomorphism*

$$\mathrm{Gr}_\beta^{L_X(Y)} sB_X \simeq \bigoplus_{l \geqslant \max(0,-\beta)} a_* sB_{Y^{\beta+2l+1}}[-\beta - 2l],$$

under which the filtration $L_X(D)$ induced on the left-hand side is identified with the filtration

$$\bigoplus_{l \geqslant \max(0,-\beta)} a_* L_{Y^{\beta+2l+1}}(D \cap Y^{\beta+2l+1}),$$

on the right-hand side. Moreover the filtration F_X on the left is identified with the filtration

$$\bigoplus_{l \geqslant \max(0,-\beta)} a_* F_{Y^{\beta+2l+1}}[-\beta - l],$$

on the right.

Proof. By Proposition (3.15), we have

$$\begin{aligned} \mathrm{Gr}_\beta^{L_X(Y)} sB_X^n &= \bigoplus_{p+|q|=n} \mathrm{Gr}_{\beta+2q_{k+1}+1}^{W_X(Y)} B_X^{p,q} \\ &\simeq \bigoplus_{\substack{p+|\hat{q}|+l=n \\ l \geqslant \max(0,-\beta)}} a_* B_{Y^{\beta+2l+1}}^{p-\beta-l,\hat{q}} \end{aligned}$$

under which the filtration $L_X(D)$ induced on the left is identified with the filtration

$$\bigoplus_{l \geqslant \max(0,-\beta)} a_* L_{Y^{\beta+2l+1}}(D \cap Y^{\beta+2l+1})$$

on the right. Then we obtain the result easily. □

THEOREM (3.20) *We have an isomorphism*

$$\mathrm{Gr}_m^{L_X(D+Y)} sB_X \simeq \bigoplus_{\substack{\alpha+\beta=m \\ l \geqslant \max(0,-\beta)}} a_*(\mathrm{Gr}_\alpha^{L_{Y^{\beta+2l+1}}(D \cap Y^{\beta+2l+1})} sB_{Y^{\beta+2l+1}}[-\beta - 2l]).$$

Moreover the filtration F_X on the left-hand side is identified with the filtration

$$\bigoplus_{\substack{\alpha+\beta=m \\ l \geqslant \max(0,-\beta)}} a_* F_{Y^{\beta+2l+1}}[-\beta - l]$$

on the right-hand side.

Proof. We computed the left-hand side of the identification in Proposition (3.18). Then the result is easily obtained by the lemma above. □

4. Construction of a cohomological mixed Hodge complex

(4.1) In this section we work in the following situation: Let S be the k-dimensional polydisc with the coordinate functions $t_1, \ldots, t_k$. The divisor defined by the function t_i is denoted by T_i for every $i = 1, \ldots, k$ and the divisor $\sum_{i=1}^k T_i$ by T, that is, $T = \sum_{i=1}^k T_i$. Let X be a connected complex manifold of dimension d. We consider a surjective morphism $f: X \to S$ satisfying the following conditions

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(4.1.1) f is smooth over $S \backslash T$

(4.1.2) the divisor $D_i = f^*T_i$ is a reduced simple normal crossing divisor on X for every $i = 1, \ldots, k$

(4.1.3) the divisor $D = \sum_{i=1}^{k} D_i = f^*T$ is a reduced simple normal crossing divisor too.

The morphism $f \colon X \to S$ above is written, locally on X, in the form

$$f^*t_i = x_{r_{i-1}+1} \cdots x_{r_i},$$

for every $i = 1, \ldots, k$ by an appropriate local coordinate $(x_1, \ldots, x_d)$ on X where $r_0, r_1, \ldots, r_k$ are integers with $0 = r_0 < r_1 < \cdots < r_k \leqslant d$. Therefore the morphism f is flat because every fiber is of the same dimension. In this case we denote the functions f^*t_i simply by t_i if there is no danger of confusion.

(4.2) For the morphism $f \colon X \to S$ given above, the data $X, D_1, \ldots, D_k$ and the functions $t_1 \ldots, t_k$ on X satisfies the assumptions in Sections 2 and 3. Therefore we have a complex of sheaves of **Q**-vector spaces

$$sA_X(D_1, \ldots, D_k; n),$$

for a fixed non-negative integer n and a complex of sheaves of **C**-vector spaces

$$sB_X(D_1, \ldots, D_k),$$

on X.

(4.3) In the situation above, we have a morphism dlog of monoid sheaves

$$\mathrm{dlog}\colon M_X(D) \to \Omega^1_X(\log D)$$

sending a local section a of $M_X(D) \subset \mathcal{O}_X$ to a meromorphic 1-form da/a which is easily seen to be a local section of $\Omega^1_X(\log D)$. The morphism of abelian sheaves

$$M_X(D)^{gp} \to \Omega^1_X(\log D)$$

associated to the morphism above is denoted by the same symbol dlog. Notice that we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "e"] \arrow[d, equal] & M_X(D)^{gp} \arrow[d, "\mathrm{dlog}"] \\ \mathcal{O}_X \arrow[r, "d"] & \Omega^1_X(\log D) \end{tikzcd}$$

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where the morphism e is the composite of the exponential map $\mathcal{O}_X \to \mathcal{O}_X^*$ and the inclusion $\mathcal{O}_X^* \mapsto M_X(D)^{gp}$ as defined in (2.7). From the diagram above, we obtain a commutative diagram

$$\begin{array}{ccc} E_X & \xrightarrow{\varphi_X(D)} & F_X(D) \\ \downarrow & & \downarrow \\ \mathcal{O}_X & \xrightarrow{d} & \Omega_X^1(\log D) \end{array}$$

where $E_X = \mathcal{O}_X \otimes_{\mathbf{Z}} \mathbf{Q}(\simeq \mathcal{O}_X)$ and $F_X(D) = M_X(D)^{gp} \otimes_{\mathbf{Z}} \mathbf{Q}$ as in (2.7). We denote the vertical morphism on the left by $\mu\colon E_X \to \mathcal{O}_X$ and the one on the right by $\nu\colon F_X(D) \to \Omega_X^1(\log D)$.

(4.4) We define a morphism of **Q**-sheaves

$$\phi_X(D)\colon \mathrm{Kos}_X^n(D)^p \to \Omega_X^p(\log D),$$

for every p by

$$\begin{aligned} &\phi_X(D)(a_1^{[l_1]} \cdots a_k^{[l_k]} \otimes b_1 \wedge \cdots \wedge b_p) \\ &\quad = \frac{1}{l_1! \cdots l_k!} \mu(a_1)^{l_1} \cdots \mu(a_k)^{l_k} \nu(b_1) \wedge \cdots \wedge \nu(b_p) \end{aligned}$$

where $l_1, \ldots, l_k$ are integers with $\sum_{i=1}^k l_i = n - p$, $a_1, \ldots, a_k$ local sections of E_X and $b_1, \ldots, b_p$ local sections of $F_X(D)$. Then we can easily see the equality

$$d \circ \phi_X(D) = \phi_X(D) \circ d$$

at every p, which means that we have a morphism $\mathrm{Kos}_X^n(D) \to \Omega_X^{\cdot}(\log D)$ of complexes of **Q**-sheaves on X, which is denoted by $\phi_X(D)$ again. By the construction above, it is easy to see that the morphism $\phi_X(D)$ preserves the filtrations $W_X(D)$ and $W_X(D_i)$ on both sides defined in Sections 2 and 3. Then the morphism $\phi_X(D)$ induces a morphism

$$\psi_X(D_1, \ldots, D_k)\colon A_X(D_1, \ldots, D_k; n)^{p,q} \to B_X(D_1, \ldots, D_k)^{p,q},$$

for a fixed non-negative integer n. Now it is easy to see the equality

$$d_i \circ \psi_X(D_1, \ldots, D_k) = \psi_X(D_1, \ldots, D_k) \circ \mathrm{d}_i,$$

for every i. Therefore we obtain the morphism of $(k+1)$-ple complexes

$$\psi_X(D_1, \ldots, D_k)\colon A_X(D_1, \ldots, D_k; n)^{p,q} \to B_X(D_1, \ldots, D_k)^{p,q}$$

and then the morphism of complexes

$$s\psi_X(D_1,\dots,D_k)\colon sA_X(D_1,\dots,D_k;n) \to sB_X(D_1,\dots,D_k)$$

which preserves the filtrations $L_X(D)$ and $L_X(D_i)$'s for all i on both sides.

(4.5) Now we shift the index k to $k+1$ and denote D_{k+1} by Y as before. We fix a non-negative integer m and denote the canonical morphism

$$a_m\colon Y^m \to X$$

simply by a. Because we have an isomorphism

$$\mathrm{Kos}_X^{n-m}(D)^{p-m} \otimes \bigwedge^m C_X(Y) \to \mathrm{Gr}_m^{W_X(Y)}\mathrm{Kos}_X^n(D+Y)^p$$

by Lemma (1.17) and an isomorphism

$$\mathrm{Kos}_X^{n-m}(D)^{p-m} \otimes \bigwedge^m C_X(Y) \to a_*(a^{-1}\,\mathrm{Kos}_X^{n-m}(D)^{p-m})$$

by the projection formula, we obtain an identification

$$\mathrm{Gr}_m^{W_X(Y)}\mathrm{Kos}_X^n(D+Y)^p \simeq a_*(a^{-1}\mathrm{Kos}_X^{n-m}(D)^{p-m})$$

via these isomorphisms. Moreover we have a morphism

$$a_*(a^{-1}\,\mathrm{Kos}_X^{n-m}(D)^{p-m}) \to a_*(\mathrm{Kos}_{Y^m}^{n-m}(D\cap Y^m)^{p-m})$$

as in (2.11). So we have a morphism

$$\mathrm{Gr}_m^{W_X(Y)}\mathrm{Kos}_X^n(D+Y)^p \to a_*(\mathrm{Kos}_{Y^m}^{n-m}(D\cap Y^m)^{p-m})$$

by the identification above.

LEMMA (4.6) *We have the following commutative diagram*

$$\begin{tikzcd}
\mathrm{Gr}_m^{W_X(Y)}\mathrm{Kos}_X^n(D+Y)^p \arrow[r, "\mathrm{Gr}_m(\phi_X(D+Y))"] \arrow[d] & \mathrm{Gr}_m^{W_X(Y)}\Omega_X^p(\log(D+Y)) \arrow[d] \\
a_*(\mathrm{Kos}_{Y^m}^{n-m}(D\cap Y^m)^{p-m}) \arrow[r, "a_*(\phi_{Y^m}(D\cap Y^m))"] & a_*\Omega_{Y^m}^{p-m}(\log D\cap Y^m),
\end{tikzcd}$$

where the vertical arrow on the left is the morphism above and the one on the right is the residue isomorphism.

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Proof. Easy by direct computation. □

(4.7) Now we come to one of the main points of this article. In the situation in (4.1) we obtained the following data:

(4.7.1) a complex of sheaves of **Q**-vector spaces

$$sA_X(D_1, \ldots, D_k; n)$$

with an increasing filtration $L_X(D)$

(4.7.2) a complex of sheaves of **C**-vector spaces

$$sB_X(D_1, \ldots, D_k)$$

with an increasing filtration $L_X(D)$ and with a decreasing filtration F_X

(4.7.3) a morphism of complexes

$$s\psi_X(D_1, \ldots, D_k): sA_X(D_1, \ldots, D_k; n) \to sB_X(D_1, \ldots, D_k)$$

for a non-negative integer n.

THEOREM (4.8) *Let $f: X \to S$ be a morphism satisfying the conditions in* (4.1). *Assume, in addition, that all the irreducible components of the divisors D_i are Kähler and that the morphism f is proper. Then the data* (4.7.1)–(4.7.3) *above form a* **Q**-*cohomological mixed Hodge complex over X if the integer n is greater than or equal to d, the dimension of X.*

Proof. We prove it by induction on k. For the case of $k = 1$, the result is proved by Steenbrink in [12] and [13]. Now we proceed to the induction step. So we shift the index from k to $k+1$ and denote the divisor D_{k+1} by Y as before. We write $sA_X(n)$, sB_X and $s\psi_X$ instead of $sA_X(D_1, \ldots, D_k, Y; n)$, $sB_X(D_1, \ldots, D_k, Y)$ and $s\psi_X(D_1, \ldots, D_k, Y)$, and similarly $sA_{Y^m}(n)$, sB_{Y^m} and $s\psi_{Y^m}$ instead of $sA_{Y^m}(D_1 \cap Y^m, \ldots, D_k \cap Y^m; n)$, $sB_{Y^m}(D_1 \cap Y_m, \ldots, D_k \cap Y^m)$ and $s\psi_{Y^m}(D_1 \cap Y^m, \ldots, D_k \cap Y^m)$ for simplicity. Moreover, the filtration $L_X(D+Y)$ on $sA_X(n)$ or sB_X is denoted by L_X and the filtration $L_{Y^m}(D \cap Y^m)$ on $sA_{Y^m}(n)$ or sB_{Y^m} by L_{Y^m} for simplicity. We have a filtered quasi-isomorphism

$$\mathrm{Gr}_m^{L_X} sA_X(n) \to \bigoplus_{\substack{\alpha+\beta=m \\ l \geqslant \max(0,-\beta)}} a_*(\mathrm{Gr}_\alpha^{L_{Y^{\beta+2l+1}}} sA_{Y^{\beta+2l+1}}(n-\beta-l)[-\beta-2l]),$$

by Proposition (2.14). Notice that the integer $n-\beta-l$ on the right hand side is greater than the integer $d-\beta-2l-1 = \dim Y^{\beta+2l+1}$ because $l \geqslant 0$ and because of the assumption $n \geqslant d$. By Theorem (3.20) we have an isomorphism

$$\mathrm{Gr}_m^{L_X} sB_X \simeq \bigoplus_{\substack{\alpha+\beta=m \\ l \geqslant \max(0,-\beta)}} a_*(\mathrm{Gr}_\alpha^{L_{Y^{\beta+2l+1}}} sB_{Y^{\beta+2l+1}}[-\beta-2l]),$$

under which the filtration F_X on the left is identified with the filtration

$$\bigoplus_{\substack{\alpha+\beta=m \\ l\geqslant \max(0,-\beta)}} a_* F_{Y^{\beta+2l+1}}[-\beta-l],$$

on the right-hand side. Moreover we can easily see that the morphism

$$\mathrm{Gr}_m^{L_X} s\psi_X \colon \mathrm{Gr}_m^{L_X} sA_X(n) \to \mathrm{Gr}_m^{L_X} sB_X,$$

is identified with the direct sum of the morphisms

$$\begin{aligned}
&a_*(\mathrm{Gr}_\alpha^{L_{Y^{\beta+2l+1}}} \psi_{Y^{\beta+2l+1}}[-\beta-2l]) : \\
&\quad a_*(\mathrm{Gr}_\alpha^{L_{Y^{\beta+2l+1}}} sA_{Y^{\beta+2l+1}}(n-\beta-l)[-\beta-2l]) \\
&\qquad \to a_*(\mathrm{Gr}_\alpha^{L_{Y^{\beta+2l+1}}} sB_{Y^{\beta+2l+1}}[-\beta-2l]),
\end{aligned}$$

by using Lemma (4.6). By the induction hypothesis for the morphism from $Y^{\beta+2l+1}$ (strictly speaking every connected component of $Y^{\beta+2l+1}$) to the k-dimensional polydisc with the coordinate functions $t_1,\ldots,t_k$, the data

$$\mathrm{Gr}_\alpha^{L_{Y^{\beta+2l+1}}} sA_{Y^{\beta+2l+1}}(n-\beta-l)$$

$$\mathrm{Gr}_\alpha^{L_{Y^{\beta+2l+1}}} sB_{Y^{\beta+2l+1}}$$

$$\mathrm{Gr}_\alpha^{L_{Y^{\beta+2l+1}}} \psi_{Y^{\beta+2l+1}}$$

form a cohomological Hodge complex of weight α on $Y^{\beta+2l+1}$. So we obtain the conclusion by computing the effect on the weight by shifting $[-\beta-2l]$ on the complexes and by shifting $[-\beta-l]$ on the Hodge filtrations F. $\square$

5. Relation to the relative log De Rham complex

(5.1) Let X be a complex manifold and D and Y reduced simple normal crossing divisors on X such that the sum $D+Y$ is a reduced simple normal crossing divisor too. In this situation Y^m is a complex manifold and $D\cap Y^m$ is a reduced simple divisor on Y^m for every non-negative integer m. Therefore we have $\mathcal{O}_{Y^m}$-modules

$$\Omega^p_{Y^m}(\log D\cap Y^m),$$

for all p. It is trivial that we have

$$(a_m)_*\Omega^p_{Y^m}(\log D\cap Y^m) = \bigoplus_{\sigma\in\mathfrak{S}_m^N} \Omega^p_{Y_\sigma}(\log D\cap Y_\sigma),$$

where a_m denotes the canonical morphism from Y^m to X and N the number of the irreducible components of Y. From now on, we omit the symbol $(a_m)_*$ for simplicity. Now we define a morphism

$$\delta\colon \Omega^p_{Y^m}(\log D \cap Y^m) \to \Omega^p_{Y^{m+1}}(\log D \cap Y^{m+1}),$$

for every non-negative integer m by

$$\delta(\omega)_\sigma = \sum_{i=1}^{m+1} (-1)^{i+1} \omega_{\sigma_i}|_{Y_\sigma},$$

where σ is an element of $\mathfrak{S}^N_{m+1}$ and $\omega = (\omega_\sigma)_{\sigma \in \mathfrak{S}^N_m}$ is a local section of

$$\Omega^p_{Y^m}(\log D \cap Y^m) = \bigoplus_{\sigma \in \mathfrak{S}^N_m} \Omega^p_{Y_\sigma}(\log D \cap Y_\sigma).$$

If we have a global defining function u of Y over X in addition, then a morphism

$$\Omega^p_X(\log(D+Y)) \to \Omega^p_X(\log D)$$

is defined by multiplying the function u to a local section ω of the sheaf $\Omega^p_X(\log(D+Y))$. Then it is easy to see that the sequence of the morphisms

$$\begin{aligned}
&0 \to \Omega^p_X(\log(D+Y)) \to \Omega^p_X(\log D) \\
&\xrightarrow{\delta} \Omega^p_{Y^1}(\log D \cap Y^1) \xrightarrow{\delta} \Omega^p_{Y^2}(\log D \cap Y^2) \\
&\xrightarrow{\delta} \cdots \xrightarrow{\delta} \\
&\xrightarrow{\delta} \Omega^p_{Y^m}(\log D \cap Y^m) \xrightarrow{\delta} \Omega^p_{Y^{m+1}}(\log D \cap Y^{m+1}) \\
&\xrightarrow{\delta} \cdots
\end{aligned} \tag{5.1.1}$$

is a complex of $\mathcal{O}_X$-module sheaves.

LEMMA (5.2) *In the situation above, the morphism*

$$\frac{du}{u} \wedge : \mathrm{Gr}^{W_X(Y)}_m \Omega^p_X(\log(D+Y)) \to \mathrm{Gr}^{W_X(Y)}_{m+1} \Omega^{p+1}_X(\log(D+Y))$$

induced by the morphism

$$\frac{du}{u} \wedge : \Omega^p_X(\log(D+Y)) \to \Omega^{p+1}_X(\log(D+Y))$$

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is identified with the morphism

$$(-1)^{p-m}\delta\colon \Omega^{p-m}_{Y^m}(\log D\cap Y^m) \to \Omega^{p-m}_{Y^{m+1}}(\log D\cap Y^{m+1}),$$

via the residue isomorphism

$$\mathrm{Res}_m\colon \mathrm{Gr}_m^{W_X(Y)}\Omega^p_X(\log(D+Y)) \to \Omega^{p-m}_{Y^m}(\log D\cap Y^m)$$

$$\mathrm{Res}_{m+1}\colon \mathrm{Gr}_{m+1}^{W_X(Y)}\Omega^{p+1}_X(\log(D+Y)) \to \Omega^{p-m}_{Y^{m+1}}(\log D\cap Y^{m+1})$$

for every p and m.

Proof. Easy computation. □

(5.3) Let X be a complex manifold and $D_1,\dots,D_k$ and Y reduced simple normal crossing divisors on X such that $\sum_{i=1}^k D_i+Y$ is a reduced simple normal crossing divisor on X. We set $D=\sum_{i=1}^k D_i$. Then the divisors D and Y are as in the situation in (5.1). Moreover assume that we are given a global defining function t_i of D_i for every i. We define an $\mathcal{O}_X$-module $\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)(\log Y)$ by

$$\begin{aligned}&\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)(\log Y)\\ &\quad=\Omega^p_X(\log(D+Y))\Big/\left(\sum_{i=1}^k \frac{dt_i}{t_i}\wedge\Omega^{p-1}_X(\log(D+Y))\right),\end{aligned}$$

for every p. (For the case that $Y=0$, we simply use the symbol

$$\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)$$

instead.) Then the differential d on $\Omega^{\cdot}_X(\log(D+Y))$ induces a differential

$$d\colon \Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)(\log Y)\to\Omega^{p+1}_{X/\Sigma_{i=1}^k D_i}(\log D)(\log Y),$$

for every p. Then $(\Omega^{\cdot}_{X/\Sigma_{i=1}^k D_i}(\log D)(\log Y); d)$ forms a complex of sheaves of **C**-vector spaces. If the data D_i and t_i are obtained from the morphism $f\colon X\to S$ in (4.1), then we have

$$\Omega^{\cdot}_{X/S}(\log D)(\log Y)=\Omega^{\cdot}_{X/\Sigma_{i=1}^k D_i}(\log D)(\log Y)$$

as complexes. Since the data $D_1\cap Y^m,\dots,D_k\cap Y^m$ on Y^m for every positive integer m satisfy the conditions as above (with $Y=0$), we have a complex of $\mathcal{O}_{Y^m}$-module sheaves

$$\Omega^{\cdot}_{Y^m/\Sigma_{i=1}^k D_i\cap Y^m}(\log D\cap Y^m),$$

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for every m.

(5.4) Morphisms in the complex (5.1.1) induce morphisms

$$\delta\colon \Omega^p_{Y^m/\Sigma_{i=1}^k D_i\cap Y^m}(\log D\cap Y^m) \to \Omega^p_{Y^{m+1}/\Sigma_{i=1}^k D_i\cap Y^{m+1}}(\log D\cap Y^{m+1})$$

by the trivial reason. If we have a global defining function u of Y, in addition, then we obtain a morphism

$$\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)(\log Y) \to \Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)$$

as before. Then we have a complex of sheaves

$$\begin{aligned}
&0 \to \Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)(\log Y) \to \Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)\\
&\xrightarrow{\delta} \Omega^p_{Y^1/\Sigma_{i=1}^k D_i\cap Y^1}(\log D\cap Y^1) \xrightarrow{\delta} \Omega^p_{Y^2/\Sigma_{i=1}^k D_i\cap Y^2}(\log D\cap Y^2)\\
&\xrightarrow{\delta} \cdots \xrightarrow{\delta}\\
&\xrightarrow{\delta} \Omega^p_{Y^m/\Sigma_{i=1}^k D_i\cap Y^m}(\log D\cap Y^m) \qquad (5.4.1)\\
&\xrightarrow{\delta} \Omega^p_{Y^{m+1}/\Sigma_{i=1}^k D_i\cap Y^{m+1}}(\log D\cap Y^{m+1})\\
&\xrightarrow{\delta}
\end{aligned}$$

as in (5.1). Now a subspace defined by the functions $t_1, \ldots, t_k$ is denoted by $\hat{Z}$, that is, $\mathcal{O}_{\hat{Z}} = \mathcal{O}_X/(t_1, \ldots, t_k)$ and $\hat{Z} = D_1 \cap \cdots \cap D_k$. Tensoring $\mathcal{O}_{\hat{Z}}$ over $\mathcal{O}_X$ to the complex above, we obtain a complex of $\mathcal{O}_{\hat{Z}}$-module sheaves

$$\begin{aligned}
&0 \to \Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)(\log Y)\otimes\mathcal{O}_{\hat{Z}} \to \Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)\otimes\mathcal{O}_{\hat{Z}}\\
&\xrightarrow{\delta} \Omega^p_{Y^1/\Sigma_{i=1}^k D_i\cap Y^1}(\log D\cap Y^1)\otimes\mathcal{O}_{\hat{Z}\cap Y^1} \xrightarrow{\delta} \cdots \xrightarrow{\delta}\\
&\xrightarrow{\delta} \Omega^p_{Y^m/\Sigma_{i=1}^k D_i\cap Y^m}(\log D\cap Y^m)\otimes\mathcal{O}_{\hat{Z}\cap Y^m} \qquad (5.4.2)\\
&\xrightarrow{\delta} \Omega^p_{Y^{m+1}/\Sigma_{i=1}^k D_i\cap Y^{m+1}}(\log D\cap Y^{m+1})\otimes\mathcal{O}_{\hat{Z}\cap Y^{m+1}},\\
&\xrightarrow{\delta} \cdots
\end{aligned}$$

where the symbol $\otimes$ stands for the symbol $\otimes_{\mathcal{O}_X}$, that is, tensor product over $\mathcal{O}_X$.

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LEMMA (5.5) *The complexes* (5.1.1), (5.4.1), *and* (5.4.2) *are exact.*

Proof. Similar to the proof by Friedman [5, Proposition (1.5)]. □

(5.6) Because the sheaf $\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)(\log Y) \otimes \mathcal{O}_{\hat{Z}}$ is a quotient of $\Omega^p_X(\log(D+Y))$, the increasing filtrations W_X, $W_X(D_i)$ $(i = 1, \ldots, k)$ and $W_X(Y)$ on $\Omega^p_X(\log(D+Y))$ induce increasing filtrations on $\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log(D+Y))$ which are denoted by the same letters W_X, $W_X(D_i)$ and $W_X(Y)$ for simplicity. Moreover we denote the stupid filtration on $\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)(\log Y) \otimes \mathcal{O}_{\hat{Z}}$ by F as usual.

(5.7) Let $D_1, \ldots, D_k$ and Y be reduced simple normal crossing divisors on a complex manifold X with global defining functions $t_1, \ldots, t_k$ and u such that $\sum_{i=1}^k D_i + Y$ is a reduced simple normal crossing divisor too. We set $D = D_1 + \cdots + D_k$. We consider the properties of weight filtrations on

$$\Omega^p_{X/\Sigma_{i=1}^k D_i + Y}(\log(D+Y)).$$

(Notice that the sheaf above is different from the sheaf $\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)(\log Y)$!)

LEMMA (5.8) *We have the equality*

$$\begin{aligned}
&\left\{\sum_{i=1}^k \frac{dt_i}{t_i} \wedge \Omega_X^{p-1}(\log(D+Y)) + \frac{du}{u} \wedge \Omega_X^{p-1}(\log(D+Y))\right. \\
&\qquad \left. + \sum_{i=1}^k t_i \Omega_X^p(\log(D+Y))\right\} \cap W_X(Y)_m \\
&\quad = \sum_{i=1}^k \left(\frac{dt_i}{t_i} \wedge \Omega_X^{p-1}(\log(D+Y)) \cap W_X(Y)_m\right) \\
&\qquad\quad + \frac{du}{u} \wedge \Omega_X^{p-1}(\log(D+Y)) \cap W_X(Y)_m \\
&\qquad\quad + \sum_{i=1}^k (t_i \Omega_X^p(\log(D+Y)) \cap W_X(Y)_m)
\end{aligned}$$

on $\Omega^p_X(\log(D+Y))$ *for every* p *and* m.

Proof. Easy from the direct sum decomposition (3.6.1) and Lemma (5.2). □

COROLLARY (5.9) *We also have*

$$\left\{\sum_{i=1}^k \frac{dt_i}{t_i} \wedge \Omega_X^{p-1}(\log(D+Y)) + \frac{du}{u} \wedge \Omega_X^{p-1}(\log(D+Y))\right.$$

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$$+\sum_{i=1}^{k} t_i\Omega_X^p(\log(D+Y)) + u\Omega_X^p(\log(D+Y))\Big\} \cap W_X(Y)_m$$

$$= \sum_{i=1}^{k}\left(\frac{dt_i}{t_i}\wedge\Omega_X^{p-1}(\log(D+Y))\cap W_X(Y)_m\right)$$

$$+\frac{du}{u}\wedge\Omega_X^{p-1}(\log(D+Y))\cap W_X(Y)_m$$

$$+\sum_{i=1}^{k}(t_i\Omega_X^p(\log(D+Y))\cap W_X(Y)_m)$$

$$+u\Omega_X^p(\log(D+Y))\cap W_X(Y)_m$$

for every p *and* m.

Proof. Easy from the inclusion $u\Omega_X^p(\log(D+Y)) \subset W_X(Y)_m$ for every $m \geqslant 0$ and from the lemma above on $\Omega_X^p(\log(D+Y))$. □

PROPOSITION (5.10) *In the situation above, we denote the subspace of* X *defined by the functions* $t_1,\ldots,t_k$ *and* u *by* Z*, and the one defined by* $t_1,\ldots,t_k$ *by* $\hat{Z}$*. Then the sheaf*

$$\mathrm{Gr}_m^{W_X(Y)}\Omega^p_{X/\Sigma_{i=1}^k D_i+Y}(\log(D+Y))\otimes_{\mathcal{O}_X}\mathcal{O}_Z$$

is isomorphic to

$$\Omega^{p-m}_{Y^m/\Sigma_{i=1}^k D_i\cap Y^m}(\log D\cap Y^m)\otimes_{\mathcal{O}_{Y^m}}\mathcal{O}_{\hat{Z}\cap Y^m}/\mathrm{Im}\,\delta, \tag{5.10.1}$$

if $1\leqslant m\leqslant p$ *and isomorphic to*

$$\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)\otimes_{\mathcal{O}_X}\mathcal{O}_{\hat{Z}}/u\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)\otimes_{\mathcal{O}_X}\mathcal{O}_{\hat{Z}}, \tag{5.10.2}$$

if $m=0$*, where* $\mathrm{Im}\,\delta$ *denotes the image of the morphism*

$$\delta\colon \Omega^{p-m}_{Y^{m-1}/\Sigma_{i=1}^k D_i\cap Y^{m-1}}(\log D\cap Y^{m-1})\otimes_{\mathcal{O}_{Y^{m-1}}}\mathcal{O}_{\hat{Z}\cap Y^{m-1}}$$

$$\to \Omega^{p-m}_{Y^m/\Sigma_{i=1}^k D_i\cap Y^m}(\log D\cap Y^m)\otimes_{\mathcal{O}_{Y^m}}\mathcal{O}_{\hat{Z}\cap Y^m}.$$

Proof. Easy by the corollary above and Lemma (1.34). □

(5.11) Let $D_1,\ldots,D_k$ be reduced simple normal crossing divisors on a complex manifold X with global defining functions $t_1,\ldots,t_k$ over X, such that the sum

$D = \sum_{i=1}^{k} D_i$ is a reduced simple normal crossing divisor on X too. Then we have the $(k+1)$-ple complex of sheaves

$$(B_X(D_1, \dots, D_k)^{p,q})_{p \in \mathbf{Z}, q \in \mathbf{Z}^k}$$

as in Section 3 and the single complex

$$sB_X(D_1, \dots, D_k)$$

associated to it. We define a morphism

$$\Omega_X^p(\log D) \to B_X(D_1, \dots, D_k)^{p,0} = \Omega_X^{p+k}(\log D) / \sum_{i=1}^{k} W_X(D_i)_0,$$

by sending a local section ω of the left-hand side to the class of the element

$$\omega \wedge \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_k}{t_k},$$

of $\Omega_X^{p+k}(\log D)$ on the right-hand side. Then it is easy to see that this morphism induces a morphism

$$\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D) \otimes \mathcal{O}_Z \to B_X(D_1, \dots, D_k)^{p,0}$$

which is denoted by θ. Then we trivially have

$$\theta(W_X(D_i)_m) \subset W_X(D_i)_{m+1}, \tag{5.11.1}$$

for every m. We see the equalities

$$d_0 \circ \theta = \theta \circ d_0$$

$$d_i \circ \theta = 0,$$

easily. Therefore the morphisms θ for all p form a morphism of complexes

$$\theta \colon \Omega^{\cdot}_{X/\Sigma_{i=1}^k D_i}(\log D) \otimes \mathcal{O}_Z \to sB_X(D_1, \dots, D_k).$$

This morphism preserves the filtration F on both sides.

(5.12) Here we define another increasing filtration M on $sB_X(D_1, \dots, D_k)$ by

$$M_m sB_X(D_1, \dots, D_k)^n = \bigoplus_{p+|q|=n} W_X(D_k)_{m+q_k+1} B_X(D_1, \dots, D_k)^{p,q}$$

for every integer m. It is easy to see that this defines a filtration as a subcomplex. We easily see the equality

$$\theta(W_X(D_k)_m) \subset M_m sB_X(D_1, \dots, D_k)$$

for every integer m by (5.11.1).

THEOREM (5.13) *In the situation* (5.11)*, let Z be the subspace of X defined by the functions $t_1, \dots, t_k$. Then the morphism*

$$\theta_X\colon \Omega^{\cdot}_{X/\Sigma_{i=1}^k D_i}(\log D) \otimes \mathcal{O}_Z \to sB_X(D_1, \dots, D_k),$$

induces a filtered quasi-isomorphism

$$\mathrm{Gr}_F^p(\theta_X)\colon \Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D) \otimes \mathcal{O}_Z[-p] \to \mathrm{Gr}_F^p sB_X(D_1, \dots, D_k),$$

with respect to the filtrations $W_X(D_k)$ on the left and M on the right.

Proof. We prove it by induction on k. For the case of $k = 1$ it is proved in Steenbrink [12] and Zucker [16]. Now we proceed to the induction step. For simplicity on the subscript, we shift the number k to $k+1$, and denote the divisor D_{k+1} by Y and the defining function t_{k+1} by u. Therefore the subspace Z of X is defined by the functions $t_1, \dots, t_k$ and u. Moreover we denote the subspace of X defined by $t_1, \dots, t_k$ by $\hat{Z}$ as in Proposition (5.10). We abbreviate $sB_X(D_1, \dots, D_k, Y)$ and $sB_{Y^m}(D_1 \cap Y^m, \dots, D_k \cap Y^m)$ to sB_X and sB_{Y^m} as before. It is sufficient to prove that $\mathrm{Gr}_F^p(\theta_X)$ induces an isomorphism

$$\mathrm{Gr}_m^{W_X(Y)} \Omega^p_{X/\Sigma_{i=1}^k D_i + Y}(\log(D+Y)) \otimes \mathcal{O}_Z \to \mathrm{Gr}_m^M \mathrm{Gr}_F^p sB_X[p],$$

for every integer m. For the case $m < 0$ is trivial from the definition. For the case of $m > p$, it is clear because of the equality $\mathrm{Gr}_m^M \mathrm{Gr}_F^p sB_X = 0$ in (3.11.1). Therefore we assume the inequality $0 \leqslant m \leqslant p$. Computation as in (3.19) shows that the complex $\mathrm{Gr}_m^M \mathrm{Gr}_F^p sB_X[p]$ is isomorphic to the single complex associated to the double complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Gr}_F^{p-m} sB_{Y^{m+1}}[p-m]^0 \arrow[r] \arrow[d] & \mathrm{Gr}_F^{p-m} sB_{Y^{m+2}}[p-m]^0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathrm{Gr}_F^{p-m} sB_{Y^{m+1}}[p-m]^1 \arrow[r] \arrow[d] & \mathrm{Gr}_F^{p-m} sB_{Y^{m+2}}[p-m]^0 \arrow[r] \arrow[d] & \cdots \\
 & {} & {} &
\end{tikzcd}$$

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by using the residue isomorphism, where the horizontal arrows coincide with the morphism δ up to sign and the vertical arrows are the differential of the complex

$$\mathrm{Gr}_F^{p-m} sB_{Y^{m+l+1}}[p-m],$$

for $l \geqslant 0$. Because of the induction hypothesis the morphism

$$\mathrm{Gr}_F^{p-m}(\theta_{Y^{m+l+1}})\colon \Omega^{p-m}_{Y^{m+l+1}/\Sigma_{i=1}^k D_i \cap Y^{m+l+1}}(\log D \cap Y^{m+l+1}) \otimes \mathcal{O}_{\hat{Z} \cap Y^{m+l+1}}$$
$$\to \mathrm{Gr}_F^{p-m} sB_{Y^{m+l+1}}[p-m]$$

is a quasi-isomorphism, therefore the sequence

$$0 \to \Omega^{p-m}_{Y^{m+l+1}/\Sigma_{i=1}^k D_i \cap Y^{m+l+1}}(\log D \cap Y^{m+l+1}) \otimes \mathcal{O}_{\hat{Z} \cap Y^{m+l+1}}$$
$$\to \mathrm{Gr}_F^{p-m} sB_{Y^{m+l+1}}[p-m]^0 \to \mathrm{Gr}_F^{p-m} sB_{Y^{m+l+1}}[p-m]^1$$
$$\to \cdots$$

is exact. Therefore the complex $\mathrm{Gr}_m^M \mathrm{Gr}_F^p sB_X$ is quasi-isomorphic to the complex

$$0 \to \Omega^{p-m}_{Y^{m+1}/\Sigma_{i=1}^k D_i \cap Y^{m+1}}(\log D \cap Y^{m+1}) \otimes \mathcal{O}_{\hat{Z} \cap Y^{m+1}}$$
$$\to \Omega^{p-m}_{Y^{m+2}/\Sigma_{i=1}^k D_i \cap Y^{m+2}}(\log D \cap Y^{m+2}) \otimes \mathcal{O}_{\hat{Z} \cap Y^{m+2}} \to \cdots,$$

where the morphisms coincide with the morphism δ up to sign. We denote this complex by C here for a while. Then, more precisely, we have a quasi-isomorphism

$$C \to \mathrm{Gr}_m^M \mathrm{Gr}_F^p sB_X[p] \tag{5.13.1}$$

induced from the morphism $\theta_{Y^{m+l+1}}$. For the case of $m \geqslant 1$,

$$\mathrm{Gr}_m^{W_X(Y)} \Omega^p_{X/\Sigma_{i=1}^k D_i + Y}(\log(D+Y)) \otimes \mathcal{O}_Z$$

is isomorphic to

$$\Omega^{p-m}_{Y^m/\Sigma_{i=1}^k D_i \cap Y^m}(\log D \cap Y^m) \otimes_{\mathcal{O}_{Y^m}} \mathcal{O}_{\hat{Z} \cap Y^m}/\mathrm{Im}\,\delta,$$

by using the residue isomorphism of Proposition (5.10). Therefore the morphism

$$\delta\colon \Omega^{p-m}_{Y^m/\Sigma_{i=1}^k D_i \cap Y^m}(\log D \cap Y^m) \otimes_{\mathcal{O}_{Y^m}} \mathcal{O}_{\hat{Z} \cap Y^m}$$
$$\to \Omega^{p-m}_{Y^{m+1}/\Sigma_{i=1}^k D_i \cap Y^{m+1}}(\log D \cap Y^{m+1}) \otimes_{\mathcal{O}_{Y^{m+1}}} \mathcal{O}_{\hat{Z} \cap Y^{m+1}}$$

induces a morphism of complexes

$$\begin{aligned}&\mathrm{Gr}_m^{W_X(Y)}\Omega^p_{X/\Sigma_{i=1}^k D_i+Y}(\log(D+Y))\otimes\mathcal{O}_Z\\ &\quad\simeq\Omega^{p-m}_{Y^m/\Sigma_{i=1}^k D_i\cap Y^m}(\log D\cap Y^m)\otimes_{\mathcal{O}_{Y^m}}\mathcal{O}_{\hat{Z}\cap Y^m}/\mathrm{Im}\,\delta\to C,\end{aligned}$$

which is a quasi-isomorphism by the exactness of the sequence (5.4.1) in Lemma (5.5). For the case of $m=0$,

$$\mathrm{Gr}_m^{W_X(Y)}\Omega^p_{X/\Sigma_{i=1}^k D_i+Y}(\log(D+Y))\otimes\mathcal{O}_Z$$

is isomorphic to the complex

$$\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)\otimes_{\mathcal{O}_X}\mathcal{O}_{\hat{Z}}/u\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)\otimes_{\mathcal{O}_X}\mathcal{O}_{\hat{Z}},$$

by Proposition (5.10) as before. Then we have a quasi-isomorphism

$$\begin{aligned}&\mathrm{Gr}_0^{W_X(Y)}\Omega^p_{X/\Sigma_{i=1}^k D_i+Y}(\log(D+Y))\otimes\mathcal{O}_Z\\ &\quad\simeq\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)\otimes_{\mathcal{O}_X}\mathcal{O}_{\hat{Z}}/u\Omega^p_{X/\Sigma_{i=1}^k D_i}(\log D)\otimes_{\mathcal{O}_X}\mathcal{O}_{\hat{Z}}\to C\end{aligned}$$

as before. Combining these two cases, we have a quasi-isomorphism

$$\mathrm{Gr}_m^{W_X(Y)}\Omega^p_{X/\Sigma_{i=1}^k D_i+Y}(\log(D+Y))\otimes\mathcal{O}_Z\to C \tag{5.13.2}$$

for every m. Now it is easy to check the commutativity of the three morphisms of complexes

$$\mathrm{Gr}_m^{W_X(Y)}\Omega^p_{X/\Sigma_{i=1}^k D_i+Y}(\log(D+Y))\otimes\mathcal{O}_Z\to\mathrm{Gr}_m^M\,\mathrm{Gr}_F^p sB_X[p]$$

$$\mathrm{Gr}_m^{W_X(Y)}\Omega^p_{X/\Sigma_{i=1}^k D_i+Y}(\log(D+Y))\otimes\mathcal{O}_Z\to C$$

$$C\to\mathrm{Gr}_m^M\,\mathrm{Gr}_F^p sB_X[p],$$

where the first morphism is the one induced by θ_X, the second is the one in (5.13.2) with appropriate sign and the third is the one in (5.13.1). Thus we obtain the result because the second and the third are quasi-isomorphisms. □

COROLLARY (5.14) *Let X, S and $f\colon X\to S$ be as in* (4.1), *the data $D_1,\ldots,D_k$ and $t_1,\ldots,t_k$ defined in the same way as in* (4.1). *We denote the fiber $f^{-1}(0)$ by Z. In this situation we have a filtered quasi-isomorphism*

$$\Omega^{\cdot}_{X/S}(\log D)\otimes_{\mathcal{O}_X}\mathcal{O}_Z\to sB_X(D_1,\ldots,D_k),$$

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with respect to the stupid filtration F on the left and the filtration F on the right.

Proof. Easy by the theorem above. □

COROLLARY (5.15) *In addition to the situation in the corollary above, assume that the morphism $f: X \to S$ is proper and that all the irreducible components of the divisors D_i are Kähler. Then the spectral sequence*

$$E_1^{pq} = H^q(Z, \Omega^p_{X/S}(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_Z) \Longrightarrow H^{p+q}(Z, \Omega^{\cdot}_{X/S}(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_Z)$$

obtained from the stupid filtration on the complex

$$\Omega^{\cdot}_{X/S}(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_Z,$$

degenerates at E_1.

Proof. Easy by the corollary above, Theorem (4.8) and Deligne [3, Scholie (8.1.9)]. □

6. Application

(6.1) Let (X, D) and (S, T) be pairs of connected complex manifolds X, S and reduced simple normal crossing divisors D, T on X and S respectively. A morphism $f: (X, D) \to (S, T)$ of pairs is a morphism $f: X \to S$ of complex manifolds satisfying the condition $D = f^*T$. For such a morphism $f: (X, D) \to (S, T)$ we have a morphism of $\mathcal{O}_X$-module sheaves

$$f^*\Omega^1_S(\log T) \to \Omega^1_X(\log D) \tag{6.1.1}$$

as in (0.7).

DEFINITION (6.2) A morphism of pairs $f: (X, D) \to (S, T)$ is said to be of generalized semi-stable type, if the morphism (6.1.1) is injective and if the cokernel of the morphism (6.1.1) is locally free (that is, $f^*\Omega^1_S(\log T)$ is a subbundle of $\Omega^1_X(\log D)$ via the morphism (6.1.1)).

REMARK (6.3) To a given pair (X, D), we can associate a ‘log structure’ on X (see K. Kato [10]). Then a morphism of pairs $f: (X, D) \to (S, T)$ turns out to be a morphism of log complex analytic spaces with respect to the ‘log structure’ above. Such a morphism f is of generalized semi-stable type if and only if f is log smooth (for the definition and the proof, see [10]).

(6.4) Assume that a morphism of pairs $f: (X, D) \to (S, T)$ of generalized semi-stable type is given. Take a point x of X. Then there exists a local coordinate system $(t_1, \ldots, t_k)$ centered at the point $f(x)$ on S such that the divisor T is defined by the function $t_1 \cdots t_l$ for some l with $1 \leqslant l \leqslant k$. The divisor defined by the function t_i is denoted by T_i for every i, and the divisor f^*T_i by D_i. Then we have $T = \sum_{i=1}^{l} T_i$ and $D = \sum_{i=1}^{l} D_i$ locally on S.

LEMMA (6.5) *In the situation above, there exists a local coordinate system $(x_1, \ldots, x_d)$ centered at the point x such that the morphism f is written in the form*

$$
\begin{aligned}
t_1 &= x_1 \cdots x_{r_1} \\
t_2 &= x_{r_1+1} \cdots x_{r_2} \\
&\cdots \\
t_l &= x_{r_{l-1}+1} \cdots x_{r_l} \\
t_{l+1} &= x_{r_l+1} \\
t_{l+2} &= x_{r_l+2} \\
&\cdots \\
t_k &= x_{r_l+k-l},
\end{aligned}
$$

for some integers $r_1, \ldots, r_l$ with $1 \leqslant r_1 < \cdots < r_l \leqslant d$. Therefore a morphism of pairs of generalized semi-stable type is flat over S and smooth over $S \backslash T$.

Proof. We can take a local coordinate system $(x'_1, \ldots, x'_d)$ on X centered at the point x such that we have the equality

$$t_i = x'_{r_{i-1}+1} \cdots x'_{r_i},$$

for every i for some integers $r_1, \ldots, r_l$ with $1 \leqslant r_1 < \cdots < r_l \leqslant d$ because the divisor $D = f^*T$ is a reduced simple divisor on X. Now we may work on these local coordinates on X and S. Then the sheaves $\Omega^1_S(\log T)$ and $\Omega^1_X(\log D)$ are free with basis

$$\frac{dt_1}{t_1}, \ldots, \frac{dt_l}{t_l}, dt_{l+1}, \ldots, dt_k$$

and

$$\frac{dx'_1}{x'_1}, \ldots, \frac{dx'_{r_l}}{x'_{r_l}}, dx'_{r_l+1}, \ldots, dx'_d.$$

Because f is of generalized semi-stable type, the exact sequence

$$0 \to f^*\Omega^1_S(\log T) \to \Omega^1_X(\log D) \to \Omega^1_{X/S}(\log D) \to 0,$$

splits near the point x. Therefore taking tensor product with the residue field $\mathbf{C}(x)$ of the point x, we obtain an exact sequence

$$\begin{aligned} 0 \;\to\; & f^*\Omega^1_S(\log T) \otimes \mathbf{C}(x) \to \Omega^1_X(\log D) \otimes \mathbf{C}(x) \\ & \to \Omega^1_{X/S}(\log D) \otimes \mathbf{C}(x) \to 0 \end{aligned}$$

of $\mathbf{C}$-vector spaces. By using the bases above, we express the first morphism by a matrix and then we can easily see that the matrix

$$\left(\frac{\partial f^* t_{l+j}}{\partial x'_{r_l+i}}(x)\right)_{\substack{1 \leqslant i \leqslant d-r_l \\ 1 \leqslant j \leqslant k-l}}$$

is of rank $k-l$, where

$$\frac{\partial f^* t_{l+j}}{\partial x'_{r_l+i}}(x)$$

denotes the value of the function

$$\frac{\partial f^* t_{l+j}}{\partial x'_{r_l+i}}$$

in the residue field $\mathbf{C}(x) = \mathbf{C}$. Therefore we can choose a local coordinate system $(x_1, \ldots, x_d)$ centered at the point x satisfying

$$x_i = x'_i \quad \text{for } \; i = 1, \ldots, r_l$$

$$x_{r_l+j} = f^* t_{l+j} \quad \text{for } \; j = 1, \ldots, k-l$$

because of the implicit function theorem. $\square$

REMARK (6.6) By the lemma above, a morphism of pairs of generalized semi-stable type is a morphism of quasi-semistable type (without 'horizontal divisors') in the sense of F. Kato [9] with respect to the 'log structures' in Remark (6.3). Such a morphism is treated by Illusie in [8] for the algebraic case.

COROLLARY (6.7) *If a morphism of pairs $f\colon (X, D) \to (S, T)$ of generalized semi-stable type is proper, then any point s of S has a coordinate neighborhood U centered at the point s with the local coordinate functions $t_1, \ldots, t_k$ such that the following conditions hold:*

(6.7.1) *the divisor T is defined by the function $t_1 \cdots t_l$ for some l with $1 \leqslant l \leqslant k$ in U*

(6.7.2) *the divisors* D_i *defined by the functions* f^*t_i *are smooth subvarieties in* $f^{-1}(U)$ *for all* i *with* $l+1 \leqslant i \leqslant k$

(6.7.3) $D + \sum_{i=l+1}^{k} D_i$ *is a reduced simple normal crossing divisor on* $f^{-1}(U)$.

Proof. Easy by Lemma (6.5). □

LEMMA (6.8) *We are given a morphism of pairs* $f\colon (X, D) \to (S, T)$ *which is proper and of generalized semi-stable type. We assume that all the irreducible components of the divisor* D *are Kähler. Then, for any point* s *on* S, *the spectral sequence*

$$E_1^{pq} = H^q(X_s, \Omega^p_{X/S}(\log D) \otimes \mathcal{O}_{X_s}) \Longrightarrow H^{p+q}(X_s, \Omega^{\cdot}_{X/S}(\log D) \otimes \mathcal{O}_{X_s}),$$

obtained from the stupid filtration on $\Omega^{\cdot}_{X/S}(\log D) \otimes \mathcal{O}_{X_s}$, *degenerates at* E_1, *where* X_s *denotes the fiber* $f^{-1}(s)$.

Proof. By the corollary above, we may assume that S is a k-dimensional polydisc with the coordinate functions $t_1, \ldots, t_k$ satisfying the conditions (6.7.1)–(6.7.3) and that the point s is the origin in this polydisc. The divisor defined by f^*t_i is denoted by D_i for every i. Then we have $D = \sum_{i=1}^{l} D_i$. We set $\overline{T} = \sum_{i=l+1}^{k} T_i$ and $\overline{D} = \sum_{i=l+1}^{k} D_i$. Considering $D + \overline{D}$ and $T + \overline{T}$ instead of D and T, the morphism f satisfies the conditions (4.1.1)–(4.1.3). Therefore the spectral sequence

$$\begin{aligned} E_1^{pq} &= H^q(X_s, \Omega^p_{X/S}(\log(D + \overline{D})) \otimes \mathcal{O}_{X_s}) \\ &\Longrightarrow H^{p+q}(X_s, \Omega^{\cdot}_{X/S}(\log(D + \overline{D})) \otimes \mathcal{O}_{X_s}) \end{aligned}$$

degenerates at E_1 by Corollary (5.15). On the other hand, we can easily check that

$$\Omega^{\cdot}_{X/S}(\log(D + \overline{D})) = \Omega^{\cdot}_{X/S}(\log D),$$

because the divisor $D_i = f^*T_i$ is smooth for every $i = l+1, \ldots, k$. Therefore we obtain the result. □

(6.9) Let $f\colon (X, D) \to (S, T)$ be a morphism of pairs which is proper and of generalized semi-stable type. By the result of F. Kato [9, Corollary 4.6] and by the base change theorem, we know that the sheaf

$$R^q f_* \Omega^{\cdot}_{X/S}(\log D)$$

is a locally free $\mathcal{O}_S$-module of finite rank on S for every q and that it commutes with base change, in particular, we have

$$R^q f_* \Omega^{\cdot}_{X/S}(\log D) \otimes \mathbf{C}(s) \simeq H^q(X_s, \Omega^{\cdot}_{X/S}(\log D) \otimes \mathcal{O}_{X_s}),$$

for every point s on S, where $\mathbf{C}(s)$ denotes the residue field of the point s. From this fact, we obtain the following result by Theorem (4.8), Corollary (5.15), the lemma above and by the standard argument.

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THEOREM (6.10) *Assume that we are given a morphism of pairs* $f\colon (X,D) \to (S,T)$ *which is proper and of generalized semi-stable type. Assume, in addition, that all the irreducible components of the divisor* D *are Kähler. Then we have the following:*

(6.10.1) *for any point* s *on* S *and for any integer* q*, there exists a* $\mathbf{Q}$*-mixed Hodge structure* $(H_{\mathbf{Q}}, W, F)$ *such that the* $\mathbf{C}$*-vector space* $H_{\mathbf{C}} = H_{\mathbf{Q}} \otimes \mathbf{C}$ *is isomorphic to the cohomology group*

$$H^q(X_s, \Omega^{\cdot}_{X/S}(\log D) \otimes \mathcal{O}_{X_s}) \simeq R^q f_* \Omega^{\cdot}_{X/S}(\log D) \otimes \mathbf{C}(s)$$

and that the filtration F *on* $H_{\mathbf{C}}$ *coincides with the filtration on this cohomology group obtained from the stupid filtration on* $\Omega^{\cdot}_{X/S}(\log D) \otimes \mathcal{O}_{X_s}$

(6.10.2) $R^q f_* \Omega^p_{X/S}(\log D)$ *is locally free of finite rank for every* p *and* q

(6.10.3) $R^q f_* F^p \Omega^{\cdot}_{X/S}(\log D)$ *is a subbundle of the vector bundle* $R^q f_* \Omega^{\cdot}_{X/S}(\log D)$*, where* F *denotes the stupid filtration on* $\Omega^{\cdot}_{X/S}(\log D)$

(6.10.4) *the spectral sequence*

$$E_1^{pq} = R^q f_* \Omega^p_{X/S}(\log D) \Longrightarrow R^{p+q} f_* \Omega^{\cdot}_{X/S}(\log D)$$

degenerates at E_1.

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