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SOLUTIONS OF THE DIOPHANTINE EQUATION $x^y + y^z + z^x = n!$

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Abstract. We prove that the only solutions in coprime positive integers to the equation

$$x^y + y^z + z^x = n!$$

are $(x, y, z) = (n! - 2, 1, 1, n)$, $n \geq 3$.

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0. Introduction. We consider the Diophantine equation

$$x^y + y^z + z^x = n! \tag{1}$$

with positive integers x, y, z, and $\gcd(x, y, z) = 1$. Clearly, this implies that x, y and z are pairwise coprime.

REMARK 1. The left hand side of equation (1) is ≥ 3, therefore $n \geq 3$. Since $n! < n^n$, it follows that at most one of the unknowns x, y and z is greater than or equal to n.

We notice that exactly one of the numbers, say x, must be even. We put $t = \mathrm{v}_2(x)$ and write

$$\boxed{x = 2^t x_1, \text{ with } t > 0 \text{ and } x_1 \text{ odd.}}$$

Here and in what follows, for integers $a > 1$ and m we put $\mathrm{v}_a(m)$ for the largest positive integer k such that $a^k | m$. In particular, we put $\mathrm{v}_a(0) = \infty$.

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Reducing equation (1) modulo 8, we see at once that

$$\boxed{y \equiv 7 \pmod 8.}$$

After a brute force verification which took 2 minutes on a computer, we get

$$\boxed{\max\{x, y, z\} > 500.}$$

Thus $n! > 2^{500}$, which implies

$$\boxed{n \geq 97.}$$

We prove successively the following results.

PROPOSITION 1. *The equation*

$$x^y + y^z + z^x = n!$$

has the infinite family of solutions $(x, y, z, n) = (n! - 2, 1, 1, n)$*,* $n \geq 3$*.*
The above solutions will be called *trivial*.

PROPOSITION 2. *There is no solution with precisely one of the x, y, z equal to 1.*

We next state an intermediate result to show the process of the final proof.

PROPOSITION 3. *If positive integers x even, y and* $z \geq 2$ *with* $\gcd(x, y, z) = 1$ *satisfy the equation*

$$x^y + y^z + z^x = n!$$

for some positive integer n, then:
(1) $\log n < 25.3$*,* $n < yz \log x / \log z$*;*
(2) $x = 2p$ *where p is a prime number greater than n;*
(3) $y \equiv -1 \pmod 8$*,* $y \geq 6{,}000$*;*
(4) $z > 4{,}000$ *is prime.*

Finally, we arrive at the main result of this paper.

THEOREM. *The only positive integer solutions to equation* (1) *with* $\gcd(x, y, z) = 1$ *are the trivial ones.*

Proposition 1 is obvious and we stated it just for the sake of completeness. The proof of Propositon 2 is rather short and it is given in the next section. Several ideas used in its proof will be useful in the rest of the paper which is devoted to a proof of the main result.

"Chinese stairs" is the paradigm of the proof. Using some machinery, one finds bounds for the various components of a solution (x, y, z, n). Feeding these bounds into (the same or a better) machinery, one obtains even tighter margins. One then iterates the procedure as long as significant improvements appear.

Technically speaking, the scheme of our proof is the following.

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• By an argument using the law of reciprocity of Jacobi symbols, we prove that either

(i) $$x > 2n, \quad y < n \quad \text{and} \quad z < \sqrt{n/e},$$

or

(ii) $$x < n/e, \quad y > n \quad \text{and} \quad z < n.$$

• Using a 2–adic linear form in two logarithms, we prove that

$$y \ll \log^4 n$$

with a small constant implied by the above Vinogradov symbol in case (i), whereas $n < 70{,}000$ in case (ii).

• In case (i), using an elementary argument we get

$$n \le yz\frac{\log x}{\log z},$$

from which it is easy to deduce that n is bounded by an absolute constant. Hence, x, y and z are also bounded by absolute constants.

• We then consider essentially the variable z. We verify that there is no solution for $z < 600$. This gives a little improvement on our bound on n; hence, also on the other unknowns. A second verification leads to $z > 4{,}000$ and we prove that this implies that z is prime, which saves a lot of the remaining computation.

• Then the end of the game is relatively easy. We verify that there is no solution.

We conjecture that equation (1) has only finitely many nontrivial solutions integer solutions (x, y, z, n) even without the coprimality condition $\gcd(x, y, z) = 1$. Unfortunately, we did not succeed in finding a finiteness argument for the number of such solutions of equation (1). We leave this problem as a challenge to the reader.

1. Proof of Proposition 2. If $y = 1$, then the equation becomes $x + 1 + z^x = n!$, which shows that $x \equiv 6 \pmod 8$. Thus, x admits a prime divisor $p \equiv 3 \pmod 4$. Note that $p > n$, otherwise p divides both x and $n!$, and reducing the equation modulo p we get $\left(\frac{-1}{p}\right) = 1$, which is a contradiction. Hence, $x \ge 2n + 2$ giving $z < \sqrt{n/e}$ (for the proof of this simple inequality use Lemma 2.1 below). Then, by computing the exponent of z in both sides of the above equation and using Lemma 2.2(b) below, we see that

$$\begin{aligned} v_z(x+1) &\ge \min\{x, v_z(n!)\} \ge \min\left\{x, \left\lfloor \frac{n}{z} \right\rfloor + \left\lfloor \frac{n}{z^2} \right\rfloor\right\} \ge \min\left\{x, \left\lfloor \frac{n}{z} \right\rfloor + 2\right\} \\ &\ge \min\left\{x, \frac{n}{z} + 1\right\} = \frac{n}{z} + 1, \end{aligned}$$

where we used the fact that $n/z^2 > e > 2$ and $x \ge 2n + 3$. Thus,

$$n \log n > \log n! > \log(z^x) = x \log z > x > z^{n/z+1} - 1 \ge 5z^{n/z} - 1 > z^{n/z} \ge 5^{n/5}$$

for $z \ge 5$. This last inequality implies that $n \le 9$. For $z = 3$, one gets $n \log n > 3^{n/3}$, whence $n \le 7$, and now one checks easily that there are no solutions.

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If $z = 1$, then $x^y + y + 1 = n!$, which shows that

$$v_2(y+1) \geq \min\{y, v_2(n!)\} \geq n - \frac{\log(n+1)}{\log 2}$$

(the minimum cannot be y because $v_2(y+1) < y$). Using Lemma 2.2(a) below, we get

$$v_2(y+1) \geq v_2(n!) \geq n - \frac{\log(n+1)}{\log 2}.$$

This implies that

$$y + 1 \geq 2^{v_2(n!)} = 2^{n - \log(n+1)/\log 2} = \frac{2^n}{n+1},$$

and

$$2^n < (n+1)\,\frac{\log n!}{\log 2} < n(n+1)\,\frac{\log n}{\log 2},$$

yielding $n \leq 6$. Since $y \equiv 7 \pmod 8$, we have $y = 7$. Thus, $x = 2$ and we see that this does not lead to a solution for the equation (1). □

2. Preparations.

2.1. Some inequalities.

LEMMA 2.1. *For any integer $n \geq 97$ one has* $\log(n!) < (n + 0.9)\log(n/e)$.

Proof. This is an easy consequence of Stirling's formula $n! < \sqrt{2\pi n}\, e^{\frac{1}{6n}} (n/e)^n$.

LEMMA 2.2. *Let $n \geq 2$ be an integer and $p \leq n$ be a prime number.*
(a) *One has*

$$\frac{n}{p-1} - \frac{\log(n+1)}{\log p} \leq v_p(n!) < \frac{n}{p-1}.$$

(b) *If additionally $n \geq 2p^2$, then*

$$v_p(n!) > \frac{n}{p} + 1.$$

Proof. Part (a) is Lemme 1 of **[2]** and for Part (b) note that

$$v_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots \geq \left\lfloor \frac{n}{p} \right\rfloor + 2 > \frac{n}{p} + 1.$$

2.2. Properties of solutions. We now consider the equation (1) modulo 2, and get easily the following result.

LEMMA 2.3. *The integer y satisfies $y \equiv -1 \pmod{2^{t+2}}$. Furthermore, if $2^w \mid (z^2 - 1)$ then $2^w \mid y + 1$.*

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Proof. Since z is odd, it is clear that the congruence $y^z \equiv -1 \pmod{2^{t+2}}$ holds if $t \leq v_2(n!) - 2$. Since the order of the multiplicative group of odd numbers modulo 2^{t+2} is a power of 2, this congruence implies the statement of the lemma. Hence, it is enough to prove the above inequality. Suppose on the contrary that $t \geq v_2(n!) - 1$. Then, using Lemma 2.2 (a),

$$z^x \geq 3^{2^t} \geq 3^{2^{v_2(n!)-1}} \geq 3^{2^{n-1-\log(n+1)/\log 2}} \geq 3^{2^{n-1}/(n+1)},$$

which implies

$$\frac{2^{n-1}}{n+1} \leq \frac{\log(z^x)}{\log 3} < \frac{\log(n!)}{\log 3},$$

leading to $n \leq 6$, which is a contradiction and finishes the proof of the lemma. □

We now consider equation (1) modulo y. Suppose first that

(i) $$y \mid n!.$$

Reducing equation (1) modulo y, we obtain

$$(-x)^y \equiv z^x \pmod{y}.$$

Since x is even and y is odd, we get

$$\left(\frac{-x}{y}\right) = 1.$$

Since $y \equiv 7 \pmod 8$, it follows that $\left(\frac{-1}{y}\right) = -1$ and $\left(\frac{2}{y}\right) = 1$. Hence, $\left(\frac{x_1}{y}\right) = -1$. It follows that there exists a prime divisor p of x such that

$$\left(\frac{p}{y}\right) = -1. \tag{2}$$

Notice that this formula implies that $x_1 \neq 1$. In particular, x is not a power of 2.

We next prove a lower bound for p, still assuming that $y \mid n!$.

Lemma 2.4. *If $y \mid n!$, then the integer x admits a prime divisor $p > n$. In particular, $x \geq 2n + 2$.*

Proof. Assuming that this is not so, we get that p divides $n!$. We reduce equation (1) modulo p and get

$$y^z + z^x \equiv 0 \pmod{p}.$$

Since x is even and z is odd, this implies

$$1 = \left(\frac{-y}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{y}{p}\right). \tag{3}$$

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Multiplying relations (2) and (3) and using the Quadratic Reciprocity Law for the Jacobi symbol, we get

$$-1 = \left(\frac{-1}{p}\right)\left(\frac{p}{y}\right)\left(\frac{y}{p}\right) = (-1)^{(p-1)/2}(-1)^{\frac{p-1}{2}\frac{y-1}{2}} = 1,$$

where we used the fact that $y \equiv 3 \pmod 4$. This is a contradiction. □

We next prove some elementary upper bounds for the unknowns x, y and z. Notice first that

$$z^x < n! < (n/e)^{n+1},$$

by Lemma 2.1. Thus,

$$z < (n/e)^{\frac{n+1}{x}}.$$

Using the inequalities $z^x < n!$ and $x^y < n!$, respectively, we get

$$x < \frac{n+1}{\log z}\log(n/e) \quad \text{and} \quad y < \frac{n+1}{\log x}\log(n/e).$$

In case (i), using the above lower bounds and Remark 1, we get the upper bounds

$$z < (n/e)^{\frac{n+1}{x}} \leq \sqrt{n/e}, \quad x < \frac{n+1}{\log z}\log(n/e), \quad y < \frac{n+1}{\log x}\log(n/e) < n.$$

Next, we prove an important inequality for n.

Lemma 2.5. *If $y \mid n!$, then*

$$n < yz\frac{\log x}{\log z}.$$

Proof. Let us look again at the equation (1). Since $x > 2n$ and $n > ez^2$, it easily follows that

$$\mathrm{v}_z(x^y + y^z) \geq \min\{\mathrm{v}_z(z^x), \mathrm{v}_z(n!)\} = \mathrm{v}_z(n!) \geq \left\lfloor \frac{n}{z} \right\rfloor + \left\lfloor \frac{n}{z^2} \right\rfloor \geq \left\lfloor \frac{n}{z} \right\rfloor + 2 > \frac{n}{z} + 1 > ez + 1.$$

Hence,

$$x^y + y^z > z^{n/z+1} \geq z^{ez+1}.$$

If $x^y < y^z$, then

$$z\log y > \frac{n}{z}\log z,$$

therefore

$$n < z^2\frac{\log y}{\log z} \leq \frac{n/e}{\log\sqrt{n/e}}\log n,$$

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from which we get that $n < 45$, which is a contradiction. Thus, we must have $x^y > z^{n/z}$, and

$$y \log x > \frac{n}{z} \log z. \qquad \square$$

The inequality given in the previous Lemma 2.5 yields a sharp lower bound for y.

LEMMA 2.6. *If* $y \mid n!$, *then*

$$y > \frac{\log(n/e)}{2\log(2n+2)}\sqrt{en}.$$

In particular, for $n > 40{,}000$, *we have* $0.7\sqrt{n} < y$.

Proof. We know that

$$n < yz\frac{\log x}{\log z} < yz\frac{\log\big((n+1)\log(n/e)/\log z\big)}{\log z}. \tag{4}$$

Let us next see that the right hand side of (4) is an increasing function of z when $z \geq 5$. Put $A = \log((n+1)\log(n/e))$. Then the derivative of the function

$$f(z) = \frac{z(A - \log\log z)}{\log z}$$

is

$$\frac{df}{dz} = \frac{(A - \log\log z)\log(z/e) - 1}{(\log z)^2}.$$

Thus, the function $f(z)$ is increasing provided that $(A - \log\log z)\log(z/e) > 1$, which is equivalent to

$$\log((n+1)\log(n/e)/\log z)\log(z/e) > 1.$$

Since $z \geq 5$, and $z < \sqrt{n/e}$, the right hand side above is $\geq \log(2n+2)\log(5/e) \geq \log(2\cdot 98)\log(5e) > 1$. Hence, $f(z)$ is increasing for $z \geq 5$. Using this and the fact that $z < \sqrt{n/e}$, inequality (4) implies that

$$n < \frac{4y^2}{e}\left(\frac{\log(2n+2)}{\log(n/e)}\right)^2.$$

The first lower bound for y asserted by the lemma follows directly, while the second one follows by noting that the function of n from the right-hand side of the previous inequality is decreasing. It remains to deal with the case when $z = 3$. In this case, inequality (4) implies that

$$n < y\frac{3}{\log 3}\log((n+1)\log(n/e)),$$

therefore

$$y > \frac{(\log 3)}{3} \frac{n}{\log((n+1)\log(n/e))}.$$

One checks that the right hand side of the above inequality exceeds the lower bound on y stated by the lemma for $n \geq 97$. This completes the proof of the lemma. $\square$

Suppose now that (i) does not hold. Then obviously $y > n$. The previous argument leads now to bounds on z and x.

To summarize, the two cases are:

(i) $$\boxed{x \geq 2(n+1), \quad \frac{\log(n/e)}{2\log(2n+2)}\sqrt{en} < y < \frac{n+1}{\log x}\log(n/e), \quad z < (n/e)^{\frac{n+1}{x}} < \sqrt{n/e},}$$

and

(ii) $$\boxed{x < (n/e)^{\frac{n+1}{y}} \leq n/e, \quad y \geq n+1, \quad z < \frac{n+1}{\log y}\log(n/e) < n.}$$

2.3. First lower bounds. We have already seen that $n \geq 97$. Hence, equation (1) implies that

$$x^y + y^z + z^x \equiv 0 \pmod{2^{94}}.$$

We now search for lower bounds on both n and on y. Our programs proceed as follows: for y and z fixed, we use the formula

$$x \equiv z\,\frac{\ell og\,(-y)}{\ell og\,z} \pmod{2^{55}},$$

where ℓog stands for the 2-adic logarithm. After less than one hour of computation, we got

$$\boxed{n > 6{,}000 \quad \text{and} \quad y > 6{,}000.}$$

In case (i), combining the lower bound on y stated above in Lemma 2.6 and the upper bound $z < \sqrt{n/e}$, we get

$$\boxed{y > z,}$$

which is trivially true in case (ii) as well.

2.4. Linear forms in p–adic logarithms. The essential ingredient of the proof is a powerful result due to Bugeaud and Laurent [**2**]. More precisely, we use a recent refinement obtained by Bugeaud [**1**], but we state these results in a very particular case, which is sufficient for us in the present situation.

Let p be a prime number. Let α_1, α_2 be integers such that $p \nmid \alpha_1\alpha_2$. We look for a lower bound for the exponent of p in

$$\Lambda = \alpha_1^{b_1} - \alpha_2^{b_2},$$

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where b_1 and b_2 are positive integers not both divisible by p. Let g be the smallest positive integer such that

$$\mathrm{v}_p(\alpha_1^g - 1) > 0 \quad \text{and} \quad \mathrm{v}_p(\alpha_2^g - 1) > 0.$$

Assume that there exists a real number E such that

$$\mathrm{v}_p(\alpha_1^g - 1) \geq E > \frac{1}{p-1}.$$

Choose real numbers A_1, A_2 such that

$$\log A_i \geq \max\{\log|\alpha_i|, E\log p\}, \quad i = 1,\ 2.$$

Put

$$b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}.$$

THEOREM 2.1. *Suppose that α_1 and α_2 are multiplicatively independent, and assume that either the prime p is odd or $\mathrm{v}_2(\alpha_2 - 1) \geq 2$. Then we have the upper estimates*

$$\mathrm{v}_p(\Lambda) \leq \frac{36.1g}{E^3(\log p)^4}\max{}^2\left\{\log b' + \log(E\log p) + 0.4, 6E\log p, 5\right\}\log A_1 \log A_2$$

and

$$\mathrm{v}_p(\Lambda) \leq \frac{53.8g}{E^3(\log p)^4}\max{}^2\left\{\log b' + \log(E\log p) + 0.4, 4E\log p, 5\right\}\log A_1 \log A_2.$$

We shall also use the following more precise version of the previous theorem.

THEOREM 2.2. *Let α_1 and α_2, b_1 and b_2 as in Theorem 2.1. Let $K \geq 3$, $L \geq 2$, R_1, R_2, S_1, S_2 be positive integers and set*

$$R = R_1 + R_2 - 1, \qquad S = S_1 + S_2 - 1, \qquad N = KL,$$

$$\gamma_1 = \frac{R+g-1}{2R} - \frac{gN}{6R(S+g-1)}, \quad \gamma_2 = \frac{S+g-1}{2S} - \frac{gN}{6S(R+g-1)}.$$

Denote by p^u the greatest power of p which divides simultaneously b_1 and b_2 and assume that p does not divide b_2/p^u. Put

$$b = \frac{(R-1)b_2 + (S-1)b_1}{2}\left(\prod_{k=1}^{K-1} k!\right)^{-2/(K^2-K)}.$$

Assume that there exist two residue classes c_1 and c_2 modulo g such that:

$$\mathrm{Card}\left\{\alpha_1^{p'r}\alpha_2^{p's};\ 0 \leq r < R_1,\ 0 \leq s < S_1,\ m_1r + m_2s \equiv c_1 \bmod g\right\} \geq L,$$

$$\mathrm{Card}\left\{rb_2 + sb_1;\ 0 \leq r < R_2,\ 0 \leq s < S_2,\ m_1r + m_2s \equiv c_2 \bmod g\right\} > (K-1)L.$$

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Under the condition

$$K(L-1)(E+t)\log p - (1+2e)\log N - (K-1)\log b - \gamma_1 p^t LR\,\mathrm{h}(\alpha_1) - \gamma_2 p^t LS\,\mathrm{h}(\alpha_2) > 0,$$

we have

$$v_p(\Lambda) \le (E+t)(KL-1/2).$$

Further, if p is an odd prime number or if $p = 2$ and $v_2(\alpha_2 - 1) \ge 2$, the condition

$$K(L-1)E\log p - 3\log N - (K-1)\log b - \gamma_1 LR\,\mathrm{h}(\alpha_1) - \gamma_2 LS\,\mathrm{h}(\alpha_2) > 0,$$

implies that

$$v_p(\Lambda) \le E(KL-1/2).$$

3. Upper bounds.

3.1 An upper bound on y. We look at the exponent of 2 in

$$\Lambda_2 = y^z + z^x = z^x - (-y)^z.$$

It satisfies

$$\mathrm{v}_2(\Lambda_2) \ge \min\{\, yt, \mathrm{v}_2(n!)\,\}.$$

To present a detailed example, we shall apply Theorem 2.1 with the choices $p = 2$, $g = 1$, $\alpha_1 = z^2$, $\alpha_2 = -y$. Since $\alpha_1 \equiv \alpha_2 \equiv 1 \pmod 8$, we may take $E = 3$, $b_1 = x/2$, $b_2 = z$, $A_1 = z^2$, $A_2 = y$. Note that α_1 and α_2 are multiplicatively independent since they are coprime and none is ± 1. We assume $n \ge 60{,}000$. Thus, in case (i), we have

$$b' = \frac{z}{2\log z} + \frac{x}{2\log y} < \frac{\sqrt{n/e}}{\log(n/e)} + \frac{x}{2\log y} < \frac{x + 2\sqrt{n}}{2\log y} < \frac{0.51x}{\log y},$$

while in case (ii), we have

$$b' < \frac{n/e}{2\log n} + \frac{n}{2\log n} < \frac{0.69n}{\log n}\,.$$

If we suppose that

$$yt \ge \mathrm{v}_2(n!)$$

(which certainly holds in case (ii)), then using Lemma 2.2(a) and the second case of Theorem 2.1, we get

$$n - \frac{\log(n+1)}{\log 2} < 17.2643\left(\max\{\log b' + 1.133, 8.318\,\}\right)^2 \log y\, \log n.$$

Here, we have also used the fact that $A_1 < n$. By iteration, we see that this implies $n < 353{,}000$. Thus, for $n \ge 353{,}000$, applying now the first case of Theorem 2.1, we conclude that

$$yt < 11.5844\,(\max\{\log b' + 1.133, 12.477\,\})^2 \log y\, \log z.$$

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This example was just given to show how we use upper bounds for p–adic linear forms in logarithms. Indeed our program uses (several iterations of) the more powerful Theorem 2.2.

Treating separately the cases $yt \le v_2(n!)$ and $yt \ge v_2(n!)$, and noticing that $yt > v_2(n!)$ in case (ii), we arrive at the following conclusion. For the last iteration, in the case $yt > v_2(n!)$, we take $L = 34$, $K = 1142$, $R = 162$ and $S = 134$ in Theorem 2.2.

Lemma 3.1.

(a) *If $y \mid n!$, then either*

$$yt < 11.5844(\max\{\log x - \log\log y + 0.459,\ 12.477\})^2 \log y \log z,$$

or $n < 70{,}000$.

(b) *In case (ii), we have* $n < 70{,}000$ *and* $x < 26{,}000$.

3.2 An upper bound on *n*. At the beginning of this subsection, we assume that we are in case (i). Recall that in Lemma 2.5, we have obtained the important inequality

$$n < yz\frac{\log x}{\log z}.$$

Combining it with Lemma 3.1, we get that n is bounded by an absolute constant. Hence, x, y and z are also bounded by absolute constants. Applying a few iterations of Theorem 2.2 (the last one with the choice $L = 34$, $K = 2242$, $R = 285$ and $S = 279$), we arrive at the following conclusion.

Lemma 3.2. *We have* $\log n < 25.3$ *and either* $x < 4n$ *and* $z < 190{,}000$*, or* $4n \le x$ *and* $z < 450$.

We keep the information

$$\boxed{\log n < 25.3 \text{ and } z < 190{,}000.}$$

Since $z < \sqrt{n/e}$ and $x < (\log n!)/\log 3$, we also have

$$\boxed{\log x < 41 \log 2.}$$

We have seen above that all these bounds hold also in case (ii), so they are always true regardless of which case we are in.

One can improve on these bounds when $16 \mid (y + 1)$. Namely, by using Theorem 2.2 with $p = 2$ and $E = 4$, we get the following result.

Lemma 3.3. *When* $16 \mid (y + 1)$ *we have* $z < 500$.

Combining this result with the second part of Lemma 2.3, we deduce

Corollary 3.4. *When* $z > 500$*, we have* $8 \parallel (z^2 - 1)$*; hence,* $z \equiv \pm 3 \pmod 8$.

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3.3. A lower bound on x and consequences. Consider a fixed value of $x \le 26{,}000$. Then, for z fixed coprime to x, with the help of Lemma 3.1 we compute a bound, say B_y, on y, and using the formula

$$y = -exp\big((x/2)\,\ell og\,(z^2)/z\big),$$

where exp is the 2–adic exponential, we get a positive integer y such that

$$y^z + z^x \equiv 0 \pmod{2^{51}}.$$

Of course, much more is actually true since $\min\{n, y\} > 6{,}000$, so the congruence holds modulo $2^{5\,000}$, but our choice of modulus is convenient for computational purposes. Now we test whether y satisfies the three conditions $y < B_y$, $\gcd(y, xz) = 1$, and

$$x^y + y^z + z^x \equiv 0 \pmod{33!}.$$

The time of computation was about one day and the program found no solutions. This shows that case (ii) cannot occur. From now on, we work under hypothesis (i) without further mention but we record our conclusion as follows.

LEMMA 3.5. *There is no solution with $n!$ not divisible by y. Moreover, we always have $x > 26{,}000$ and $n > 4{,}000$.*

3.4. Bounds on z. Let r be the smallest prime divisor of z. Notice first that $r^2 \le z$ when z is composite. We have

$$\mathrm{v}_r(z^x) \ge x.$$

But

$$\mathrm{v}_r(n!) \le \frac{n}{r-1} \le \frac{n}{2} < x.$$

In the above inequality we used the fact that $2n < x$. Comparing the previous estimates, we get

$$\mathrm{v}_r(x^y + y^z) = \mathrm{v}_r(n!) > n/r,$$

using the fact that $n > z^2$.

We put

$$\Lambda_r = x^y + y^z = y^z - (-x)^y = n! - z^x,$$

and using the first case of Theorem 2.1 with $g = r - 1$ and $E = 1$, we get

$$\mathrm{v}_r(x^y + y^z) < \frac{36.1\,r}{(\log r)^4}\max{}^2\{\log b' + \log\log r + 0.4,\ 6\log r\}\ \log x\ \log y$$

(since $\max\{y, r\} = y$ because $r \le z < y$), where

$$b' = \frac{y}{\log y} + \frac{z}{\log x}.$$

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Using the inequalities $z^2 < n/e$, $y < n$ and $x < (\log n!)/\log z$, we find, after several iterations, that

$$z \text{ composite} \implies \log n < 16.84.$$

From $z < \sqrt{n/e}$, we infer that

$$z \text{ composite} \implies z < 3{,}000.$$

Here, for the last iteration of Theorem 2.2, we take $L = 11$, $K = 24{,}700$, $R = 3{,}800$ and $S = 5{,}300$.

LEMMA 3.6. *If z is composite, then* $\log n < 16.84$ *and* $z < 3{,}000$.

We add the following easy result which saves one third of the time of computation.

LEMMA 3.7. *For* $z > 60$*, we have* $y \equiv 7$*, or* $15 \pmod{24}$.

Indeed, if $3 \mid x$, then $x \geq 6n$ and $z < 60$. Besides, it is easy to check that $y \equiv 2 \pmod 3$ implies $3 \mid x$; hence, the result. □

These two remarks enable us to save a lot of computation.

3.5. Remarks on *y*. We already know that $y > 6{,}000$. To get more precise information for y, we apply Theorem 2.2 to the linear form

$$\Lambda_q = x^y + z^x = z^x - (-x)^y = n! - y^z,$$

where q is the smallest prime divisor of y.

A priori, we have to distinguish two cases:

$$z \leq \mathrm{v}_q(n!), \qquad \text{or} \qquad z > \mathrm{v}_q(n!).$$

But we know that $n > ez^2$ thus, since $\mathrm{v}_q(n!) > n/q - 1$, the second case does not hold if

$$q < \frac{n}{1 + \sqrt{n/e}}.$$

For example, this inequality holds for $q \leq 11$ when $n \geq 65$.

Now, we study Λ_q in more detail. We have

$$\Lambda_q = z^x - (-x)^y = n! - y^z \equiv 0 \pmod{q^2}.$$

We have to find a positive integer g such that

$$(-x)^g \equiv z^g \equiv 1 \pmod q.$$

Notice that

$$(-x)^y \equiv z^x \pmod{q^2},$$

where x is even, thus $-x$ is a quadratic residue modulo q. If $q \equiv 3 \pmod 4$, then either z or $-z$ is a quadratic residue mod q. Thus, provided that we put $\alpha_1 = \varepsilon z$ (in the

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notation of Theorem 2.2), where $\varepsilon = \pm 1$ is such that εz is a quadratic residue mod q, we can choose

$$g = \frac{q-1}{2}.$$

In any case, if

$$(-x)^g \equiv z^g \equiv 1 \pmod{q},$$

we have

$$z^{gx} \equiv (-x)^{gy} \equiv 1 \pmod{q^2},$$

since q divides y.

But it is well-known (and easy to prove), that if ω is the order of some integer mod q, then its order mod q^2 is either ω or ωq. This remark proves that we always have

$$(-x)^g \equiv 1 \pmod{q^2}.$$

Hence, we can take $E = 2$ in Theorem 2.1. We then get an upper bound on z. For example, for $q = 3$, we get

$$z < 37{,}000 \quad \text{if} \quad 3 \mid y, \tag{5}$$

by taking $p = 3$, $g = 1$, $E = 2$, $L = 20$, $K = 917$, $R = 212$ and $S = 92$ in Theorem 2.2.

4. Conclusion.

4.1. First lower bounds on *z*. At the beginning of this section, we supposed that $z < 600$. Using the bound $n < (yz \log x)/\log z$, we get $n < e^{20.7}$. Hence, $x < (\log n!)/\log 3 < e^{23.6} < 2^{35}$. We put $B_x = 2^{35}$.

Consider a fixed value of z. We have previously seen that

$$\mathrm{v}_2(y+1) \geq \mathrm{v}_2(z^2-1).$$

Thus, we have

$$y \equiv -1 \pmod{2^w}, \quad \text{where } w = \mathrm{v}_2(z^2-1) \geq 3.$$

Then, for z and y fixed, we obtain x by the congruence

$$x \equiv \frac{z\,\ell og\,(-y)}{\ell og\, z} \pmod{2^{55}}$$

(recall that $x < 2^{35}$). Afterwards, we test whether x satisfies the three conditions $x < B_x$, $\gcd(x, yz) = 1$, and

$$x^y + y^z + z^x \equiv 0 \pmod{33!}.$$

Proceeding this way, we checked in about one hour that for any solution one has

$$z \geq 600.$$

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Combining Lemma 3.4 and Lemma 3.7, we deduce that

$$\boxed{y \equiv 7, \quad \text{or} \quad 39 \pmod{48}.}$$

Using this fact, we treat the range $600 < z < 4{,}000$. After about another hour of computation, we verify that there are no solutions in this interval. Hence, by Lemma 3.6, we know that

$$\boxed{z > 4{,}000 \text{ is prime.}}$$

4.2. Further estimates on z. Recall that we have proved in Lemma 2.5 that $x^y > z^{n/z}$, which implies

$$\frac{y}{z} > \frac{n}{z^2}\frac{\log z}{\log x} > \frac{n}{z^2}\frac{\log z}{\log\big((\log n!)/\log z\big)}.$$

As in the proof of Lemma 2.6, one shows that the rightmost term is a decreasing function of z. So, since $z < \sqrt{n/e}$ and $n! < (n/e)^{n+1}$, we get

$$\frac{y}{z} > \frac{e}{2}\frac{\log(n/e)}{\log(2n+2)}.$$

As $n > ez^2$, we obtain

$$y > 1.233\, z.$$

Thus, we now know that

$$4{,}000 < z < 0.811\, y, \qquad ez^2 < n < x/2, \qquad x < 3\,n,$$

the last inequality being a consequence of $z^x < n!$.

We also have

$$n < yz\,\frac{\log x}{\log z} < 3.43\, yz < 2.79\, y^2.$$

4.3. Final computations. We are now ready to do the final computations.

We return to bounding z from below. We know that $y < 1{,}160{,}000$, and we can consider the procedure explained before for each pair (y, z) with a fixed value for z. We also know that x always satisfies the inequality $x < 3\,n < e^{26.4}$. Repeating a previous computation, but now in the range $4{,}000 < z < 40{,}000$, we found no solutions after a little more than six hours of computation. By property (5), this lower bound implies that 3 does not divide y; hence,

$$\boxed{y \equiv 7 \pmod{48}.}$$

Moreover, we verify that now $y > 1.2585\, z$. Then we just used the computer to fill in the gap. This last verification took about 20 hours.

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