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G-GRAPHS AND SPECIAL REPRESENTATIONS FOR BINARY DIHEDRAL GROUPS IN GL(2, $\mathbb{C}$)

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Abstract. Given a finite subgroup $G \subset \mathrm{GL}(2, \mathbb{C})$, it is known that the minimal resolution of singularity $\mathbb{C}^2/G$ is the moduli space $Y = G\text{-Hilb}(\mathbb{C}^2)$ of G-clusters $\mathcal{Z} \subset \mathbb{C}^2$. The explicit description of Y can be obtained by calculating every possible distinguished basis for $\mathcal{O}_{\mathcal{Z}}$ as vector spaces. These basis are the so-called G-graphs. In this paper we classify G-graphs for any small binary dihedral subgroup G in $\mathrm{GL}(2, \mathbb{C})$, and in the context of the special McKay correspondence we use this classification to give a combinatorial description of special representations of G appearing in Y in terms of its maximal normal cyclic subgroup $H \trianglelefteq G$.

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1. Introduction. Motivated by the McKay correspondence [**10**] (read [**15**] for a survey), G-graphs were introduced by Nakamura [**11**] to construct a crepant resolution of quotient $\mathbb{C}^3/G$ for abelian subgroups $G \subset \mathrm{SL}(3, \mathbb{C})$. This resolution is the moduli space G-Hilb of G-clusters introduced by Ito and Nakamura [**6**]. Among other properties, it has been proved to be the minimal resolution of $\mathbb{C}^2/G$ for small finite subgroups $G \subset \mathrm{GL}(2, \mathbb{C})$ in [**4**], and a crepant resolution for subgroups $G \subset \mathrm{SL}(3, \mathbb{C})$ in [**1**].

Since then, G-graphs have been a useful tool in studying G-Hilb when G is abelian (for instance [**5**, **8**] for $G \subset \mathrm{GL}(2, \mathbb{C})$, and [**3**] for $G \subset \mathrm{GL}(3, \mathbb{C})$). The first attempt to extend the notion of G-graph to calculate G-Hilb for non-abelian subgroups was made by Leng in 2002 [**9**] for binary dihedral subgroups $G \subset \mathrm{SL}(2, \mathbb{C})$ and some binary trihedral subgroups $G \subset \mathrm{SL}(3, \mathbb{C})$ (see also [**18**]). In this paper we calculate every possible G-graph for small binary dihedral subgroups $G \subset \mathrm{GL}(2, \mathbb{C})$.

The key idea in the construction of these G-graphs is to consider the action of a dihedral group G on $\mathbb{C}^2$ as the cyclic action by its maximal normal abelian subgroup H of index two, followed by a dihedral involution. This interpretation allows us to construct G-graphs from the union of two H-graphs identified by the action of G/H that we call *qG-graphs*. From a qG-graph, we use the representation theory of G to construct a G-graph in a unique way. In Propositions 5.9, 5.13 and 5.14 we classify the G-graphs arising from qG-graphs into types A and B. In Propositions 5.16 and 5.17 we classify the G-graphs that do not come from qG-graphs into types C and D.

Given any G-graph Γ, there exists a corresponding affine open set U_Γ in $G\text{-Hilb}(\mathbb{C}^2)$, which consists of all G-clusters $\mathcal{Z}$ for which Γ is a basis of the vector space $\mathcal{O}_{\mathcal{Z}}$. In Theorem 6.2 and the following Corollary 6.3 we prove that the number of distinct G-graphs gives us a covering of G-Hilb($\mathbb{C}^2$) with the minimum number of open sets. The explicit equations of these open sets are given in **[13]** using the moduli space of representations of the McKay quiver.

In the last part of the paper we apply the classification of G-graphs to list special representations of any small binary dihedral group $G \subset \mathrm{GL}(2, \mathbb{C})$. We start by using the explicit description of the ideals corresponding to the G-graphs that we give in Section 5 to produce a 1-parameter family of ideals connecting any G-cluster at the exceptional divisor E in G-Hilb($\mathbb{C}^2$). Thanks to the *special* McKay correspondence **[17]**, the exceptional curves appearing in the minimal resolution correspond one-to-one to the special irreducible representations, which now can be expressed in terms of G-graphs. This minimal resolution is G-Hilb($\mathbb{C}^2$) by Ishii's theorem (see **[4]** and Theorem 7.1), which lead us to give in Theorem 7.3 a combinatorial description of the special representations of a small binary dihedral group $G = \mathrm{BD}_{2n}(a)$ in terms of the continued fraction $\frac{2n}{a}$. We note that the classification of the special representations was discovered independently by Iyama and Wemyss in **[7]** using the Cohen–Macaulay modules.

The paper is distributed as follows. In Section 2 we recall the background material on cyclic quotient singularities that is needed. In Section 3 we describe the binary dihedral groups in $\mathrm{GL}(2, \mathbb{C})$ and the resolution of singularities of $\mathbb{C}^2/G$. In Section 4 we define G-graphs and recall the explicit construction of H-Hilb($\mathbb{C}^2$) in terms of H-graphs when H is an abelian subgroup in $\mathrm{GL}(2, \mathbb{C})$. Section 5 is dedicated to the calculation of the G-graphs for the dihedral groups and their classification into types A, B, C and D. In Section 6 we prove that any G-cluster $\mathcal{Z}$ admits as basis of $\mathcal{O}_{\mathcal{Z}}$ a G-graph of type A, B, C or D, thus obtaining an open cover for G-Hilb($\mathbb{C}^2$). Finally, Section 7 is devoted to the special representations and their description in terms of the continued fraction $\frac{2n}{a}$.

I would like to thank M. Reid for introducing me to this topic and for his support during my PhD.

2. Background on cyclic quotient singularities. In this section we introduce some notations needed for the rest of the paper about cyclic quotient singularities. The material is taken from **[14]**.

Let $H = \langle \frac{1}{k}(1, a) \rangle := \langle \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{smallmatrix}\right) | \varepsilon^k = 1 \text{ primitive} \rangle$ be a cyclic group in $\mathrm{GL}(2, \mathbb{C})$ with $(k, a) = 1$. The quotient singularity $X := \mathbb{C}^2/H = \operatorname{Spec} \mathbb{C}[x, y]^H$ and the minimal resolution $Y \to X$ are toric and are completely determined by the continued fractions $\frac{k}{k-a}$ and $\frac{k}{a}$ as follows. Let a_i for $i = 1, \ldots, l$ be the entries of the Hirzebruch–Jung continued fraction

$$\frac{k}{k-a} = a_1 - \frac{1}{a_2 - \ldots \frac{1}{a_l}} = [a_1, \ldots, a_l].$$

The ring $\mathbb{C}[x, y]^H$ of invariants is generated by the monomials u_i for $i = 0, \ldots, l$ that satisfy

$$u_{i-1}u_{i+1} = u_i^{a_i} \text{ for } i = 1, \ldots, l, \tag{2.0.1}$$

where $u_0 = x^k$ and $u_1 = x^{k-a}y$. Then $X \subset \mathbb{C}^{l+2}$ is determined set-theoretically by the relation (2.0.1).

Now consider the lattice $L := \mathbb{Z}^2 + \frac{1}{k}(1, a) \cdot \mathbb{Z} \subset \mathbb{R}^2$ and its dual lattice of invariant monomials M. Define the Newton polygon of L as the convex hull in $\mathbb{R}^2$ of all non-zero lattice points in the positive quadrant. Then the resolution Y is determined explicitly by the continued fraction

$$\frac{k}{a} = [b_1, \dots, b_m]$$

in the following way: Let $e_0 = \frac{1}{k}(0, k) = (0, 1)$, $e_1 = \frac{1}{k}(1, a)$ and $e_{i+1} + e_{i-1} = b_i e_i$ for $i = 1, \dots, m$, be the non-zero lattice points on the boundary of the Newton polygon of L. Then the exceptional divisor $E \subset Y$ consists of m exceptional curves $E_1, \dots, E_m$ where each $E_i \cong \mathbb{P}^1$ with self-intersections $-b_1, \dots, -b_m$ respectively. These rational curves intersect according to the following dual graph of type A:

$$\overset{E_1}{\bullet} \text{—} \overset{E_2}{\bullet} \text{—} \cdots \text{—} \overset{E_m}{\bullet}$$

Furthermore, Y is covered by $m + 1$ affine open sets $Y = Y_0 \cup \ldots \cup Y_m$, where each $Y_i \cong \mathbb{C}^2$ has coordinates λ_i, μ_i defined to be the dual basis in M of the consecutive lattice points e_i, e_{i+1} of L.

The relation between the entries of the continued fractions $\frac{k}{k-a}$ and $\frac{k}{a}$ is given by an algorithm because of Riemenschneider (see **[16]**): Given the entries $a_1, \dots, a_l$ of the continued fraction $\frac{k}{k-a}$, form l rows with $a_i - 1$ points in each row as follows

$$\begin{array}{l}
\underbrace{\times \times \ldots \times \times}_{a_1-1} \\
\qquad\qquad \underbrace{\times \times \ldots \times \times}_{a_2-1} \\
\qquad\qquad\qquad\qquad \underbrace{\times \times \ldots \times \times}_{a_3-1} \quad \cdots
\end{array}$$

where for $i = 1, \dots, l$ the first point in row i is placed in the same column as the last point in row $i - 1$. Then for $j = 1, \dots, m$, the number of points in column j is $b_j - 1$. Vice versa, given $b_1, \dots, b_m$ we can recover $a_1, \dots, a_l$ in the same way.

EXAMPLE 2.1. Let $k = 46$ $a = 17$ with $\frac{46}{17} = [3, 4, 2, 3]$. Then we write $\begin{array}{ccccc} \times & \times & & & \\ & \times & \times & \times & \\ & & & \times & \\ & & & \times & \times \end{array}$ so that $\frac{46}{29} = [2, 3, 2, 4, 2]$.

3. Dihedral groups in $\mathrm{GL}(2, \mathbb{C})$**.** We consider the following representation of binary dihedral subgroups in $\mathrm{GL}(2, \mathbb{C})$ in terms of their action on the complex plane $\mathbb{C}^2_{x,y}$:

$$\mathrm{BD}_{2n}(a) = \left\langle \alpha = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix}, \ \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \ \varepsilon^{2n} = 1 \text{ primitive}, \right.$$
$$\left. a^2 \equiv 1 \pmod{2n} \right\rangle.$$

In other words, $\mathrm{BD}_{2n}(a)$ is a group of order $4n$ generated by the cyclic group $H := \langle \alpha \rangle = \left\langle \frac{1}{2n}(1, a) \right\rangle \trianglelefteq G$ and the dihedral symmetry β, which sends the coordinates (x, y) to $(-y, x)$. The subgroup H is a choice of maximal cyclic index 2 subgroup of G (note

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that $\beta^2 \in H$). The condition $a^2 \equiv 1 \pmod{2n}$ is equivalent to the classical dihedral condition of $\alpha\beta = \beta\alpha^a$.

We start by giving the definitions of the integers q and k, which appear frequently throughout the paper.

DEFINITION 3.1. Let $q := \frac{2n}{(a-1,2n)}$, and k such that $n = kq$.

An element $g \in G$ is a *quasireflection* if it fixes a hyperplane, and a group G is called *small* if it does not contain any quasireflection. Chevalley et al.'s theorem **[19]** states that if $H \subset \mathrm{GL}(n, \mathbb{C})$ is generated by reflections, then $\mathbb{C}^n/H \cong \mathbb{C}^n$, which traditionally reduces the study of these quotients to small groups.

PROPOSITION 3.2. *The group* $G = \mathrm{BD}_{2n}(a)$ *is small* $\iff \gcd(a+1, 2n) \nmid n$.

Proof. The elements of G are of the form α^i and $\alpha^i\beta$ for $i = 0, \dots, 2n-1$. Since $(2n, a) = 1$, the subgroup H is small, so quasireflections can only occur among the elements of the form $\alpha^i\beta$ (with $i \neq n$). Then

$$\alpha^i\beta \text{ is a quasireflection } \iff \det(\alpha^i\beta - I) = 0 \iff 1 + \varepsilon^{(a+1)i} = 0 \iff (a+1)\, i \equiv n \pmod{2n}.$$

Therefore, G has no quasireflections if and only if there do not exist any solutions to the equation $(a+1)x \equiv n \pmod{2n}$. As a linear congruence, it only has solutions if the $\gcd(a+1, 2n)$ divides n. □

REMARK 3.3. (Brieskorn classification). Small binary dihedral groups in $\mathrm{GL}(2, \mathbb{C})$ were originally classified by Brieskorn in **[2]** as follows:

$$D_{N,q} := \begin{cases} \langle \psi_{2q}, \tau, \phi_{2k} \rangle, & \text{if } k := N - q \equiv 1 \pmod 2 \\ \langle \psi_{2q}, \tau \circ \phi_{4k} \rangle, & \text{if } k \equiv 0 \pmod 2 \end{cases}$$

$$\text{with } \psi_r = \begin{pmatrix} \varepsilon_r & 0 \\ 0 & \varepsilon_r^{-1} \end{pmatrix}, \tau = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \phi_r = \begin{pmatrix} \varepsilon_r & 0 \\ 0 & \varepsilon_r \end{pmatrix}, \varepsilon_r = \exp\frac{2\pi i}{r} \text{ and } |D_{N,q}| = 4kq,$$

where q and k are as in 3.1. The groups $\mathrm{BD}_{2n}(a)$, which are small, correspond to the case k odd. The case k even is obtained by taking $\beta = \left(\begin{smallmatrix} 0 & 1 \\ \varepsilon^q & 0 \end{smallmatrix}\right)$ (see **[12]**, Section 3 for more details). For simplicity, in this paper we only treat $\mathrm{BD}_{2n}(a)$ groups, but emphasize that the methods used here apply to the groups with k even as well.

The group $\mathrm{BD}_{2n}(a)$ has irreducible 1-dimensional representations ρ_j^+ and ρ_j^- of the form

$$\rho_j^{\pm}(\alpha) = \varepsilon^j, \quad \rho_j^{\pm}(\beta) = \begin{cases} \pm i & \text{if } n, j \text{ odd} \\ \pm 1 & \text{otherwise} \end{cases},$$

where ε is a $2n$-th primitive root of unity and j is such that $j \equiv aj \pmod{2n}$. The values r for which $r \not\equiv ar \pmod{2n}$ form in pairs the irreducible 2-dimensional representations V_r of the form

$$V_r(\alpha) = \begin{pmatrix} \varepsilon^r & 0 \\ 0 & \varepsilon^{ar} \end{pmatrix}, \quad V_r(\beta) = \begin{pmatrix} 0 & 1 \\ (-1)^r & 0 \end{pmatrix}.$$

By definition, the natural representation is V_1.

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Noe that the number of 1-dimensional representations coincides with twice the number of scalar diagonal elements in $\frac{1}{2n}(1,a)$. Since $n=kq$, the number of 1-dimensional representations is $4k$. If we call d the number of 2-dimensional irreducible representations, using the formula $|G|=\sum_{\rho\in\mathrm{Irr}\,G}(\dim(\rho))^2$, we have $4n=4k+4d$, which gives $d=n-k$.

Remark 3.4. (Irr G **from** Irr H**).** Irreducible representations of H give rise to irreducible representations of G in the following way. Let $\mathrm{Irr}\,H=\{\rho_0,\dots,\rho_{2n-1}\}$. The group G acts on H by conjugation $g\cdot h:=ghg^{-1}$, for $g\in G, h\in H$. The induced action of G on the characters is given by $g\cdot\chi_{\rho_j}(h):=\chi_{\rho_j}(g^{-1}hg)$. Since the character is a function on conjugacy classes in H, this action is constant in co-sets gH. Therefore, $G/H=\langle\beta\rangle\cong\mathbb{Z}/2\mathbb{Z}$ acts on the characters χ_{ρ_j} of H, which induces an action on Irr H by

$$\beta\cdot\rho_j:=\rho_{aj}.$$

The free orbits under the action of G/H are $\{\rho_j,\rho_{aj}\}$ with $aj\not\equiv j \mod 2n$, and these combine to produce the 2-dimensional representation $V_j\in\mathrm{Irr}\,G$. Every fixed representation ρ_j with $aj\equiv j\pmod{2n}$ splits into two 1-dimensional representations ρ_j^+ and ρ_j^- in Irr G, corresponding to the two characters of $G/H\cong\mathbb{Z}/2\mathbb{Z}$.

In what follows, we take the notation as in [**21**] (Section 10). Let $V(=V_1)$ be a vector space with basis $\{x,y\}$ where G acts naturally. Define $S=\mathrm{Sym}\,V:=\mathbb{C}[V^*]$ the polynomial ring in variables x and y. Then the action of G extends to S by $g\cdot f(x,y):=f(g(x),g(y))$ for $f\in S, g\in G$.

Definition 3.5. Let $G=\mathrm{BD}_{2n}(a)$ and $f\in S$. A polynomial f is said to belong to $\rho_j^{\pm}$ if

$$\alpha(f)=\varepsilon^j f \text{ and } \beta(f)=\begin{cases}\pm if & \text{if } n,j \text{ odd}\\ \pm f & \text{otherwise}\end{cases}.$$

A pair of polynomials (f,g) is said to belong to V_k if $g=\beta(f)$ and $\alpha(f,\beta(f))=(\varepsilon^k f,\varepsilon^{ak}\beta(f))$.

Let $S_\rho:=\{f\in\mathbb{C}[x,y]: f\in\rho\}$ the S^G-module of ρ-invariants. We say that a polynomial f belongs to ρ, or simply $f\in\rho$ if it belongs to the corresponding module S_ρ. Note that these are precisely the Cohen–Macaulay S^G-modules $S_\rho=(S\otimes\rho^*)^G$ where G acts on S as above, and G acts on a representation ρ by inverse transpose.

Example 3.6. In Table 1 we present the irreducible representations for group $\mathrm{BD}_{12}(7)$ together with some elements belonging to the corresponding modules S_ρ.

3.1. Resolution of dihedral singularities. Let $G=\mathrm{BD}_{2n}(a)$ be a small binary dihedral group with cyclic maximal subgroup $H=\langle\alpha\rangle\trianglelefteq G$. Let us now look at the geometric construction of the resolution of a dihedral singularity $\mathbb{C}^2/G$.

Consider first the action of H on $\mathbb{C}^2$. The quotient affine variety $X=\mathbb{C}^2/H$ is the toric quotient singularity of type $\frac{1}{2n}(1,a)$, with an isolated singular point at the origin. Recall from Section 2 that the resolution of singularities $Y=H\text{-Hilb}(\mathbb{C}^2)\to X$ is determined by the continued fraction $\frac{2n}{a}=[b_1,\dots,b_m]$.

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Table 1. Some semi-invariant elements in each S_ρ for $\mathrm{BD}_{12}(7)$

	α	β	S_ρ
ρ_0^+	1	1	$1, x^{12}+y^{12}, x^5y-xy^5, x^6y^6$
ρ_0^-	1	-1	$x^{12}-y^{12}, x^5y+xy^5, x^3y^3$
ρ_1^+	ε^2	1	x^2+y^2, x^7y-xy^7
ρ_1^-	ε^2	-1	x^2-y^2, x^7y+xy^7
ρ_2^+	ε^4	1	$x^4+y^4, x^9y-xy^9, x^2y^2$
ρ_2^-	ε^4	-1	$x^4-y^4, x^9y+xy^9, x^5y^5$
ρ_3^+	-1	1	$x^6+y^6, x^{11}y-xy^{11}$
ρ_3^-	-1	-1	$x^6-y^6, x^{11}y+xy^{11}$
ρ_4^+	ε^8	1	$x^8+y^8, x^6y^2+x^2y^6, x^4y^4$
ρ_4^-	ε^8	-1	$x^8-y^8, x^6y^2-x^2y^6, xy$
ρ_5^+	ε^{10}	1	$x^{10}+y^{10}, x^3y-xy^3$
ρ_5^-	ε^{10}	-1	$x^{10}-y^{10}, x^3y+xy^3$
V_1	$\left(\begin{smallmatrix}\varepsilon & 0\\ 0 & \varepsilon^7\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$	$(x,y), (y^7,-x^7),$ $(x^6y,-xy^6), (x^2y^5,-x^5y^2)$
V_2	$\left(\begin{smallmatrix}\varepsilon^3 & 0\\ 0 & \varepsilon^9\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$	$(x^3,y^3), (y^9,-x^9),$ $(xy^2,x^2y), (x^8y,-xy^8)$
V_3	$\left(\begin{smallmatrix}\varepsilon^5 & 0\\ 0 & \varepsilon^{11}\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$	$(x^5,y^5), (y^{11},-x^{11}),$ $(xy^4,x^4y), (x^{10}y,-xy^{10})$

LEMMA 3.7. *If $a^2 \equiv 1 \pmod{2n}$ then the entries of the continued fraction $\frac{2n}{a}$ are symmetric with respect to the middle term, that is $b_i = b_{m+1-i}$ for $i = 1, \dots, m$.*

Proof. Let L be the lattice of weights and M the dual lattice of monomials, and consider the continued fractions $\frac{2n}{2n-a} = [a_1, \dots, a_l]$ and $\frac{2n}{a} = [b_1, \dots, b_m]$. If a monomial x^iy^j is H-invariant, then $i + aj \equiv 0 \pmod{2n}$, and by the assumption $a^2 \equiv 1 \pmod{2n}$, x^jy^i is also invariant. Therefore, the continued fraction $\frac{2n}{2n-a}$ is symmetric, i.e. $a_i = a_{l+1-i}$. Indeed, let $u_{i-1} = x^{t'}y^{w'}$, $u_i = x^ty^w$, $u_{i+1} = x^{t''}y^{w''}$ be three consecutive invariant monomials for some integers t and w. Let also $u_{l+1-(i+1)} = x^{w''}y^{t''}$, $u_{l+1-i} = x^wy^t$, $u_{l+1-(i-1)} = x^{w'}y^{t'}$ be their symmetric partners. Since $u_{i-1}u_{i+1} = u_i^{a_i}$, we have $t' + t'' = a_it = a_{l+1-i}t$ and $w' + w'' = a_iw = a_{l+1-i}w$, thus $a_i = a_{l+1-i}$ for all i.

The symmetry in the entries of $\frac{2n}{2n-a}$ implies the symmetry of the entries of $\frac{2n}{a}$ by the algorithm explained in Section 2, thus $b_i = b_{m+1-i}$. □

To complete the action of G on $\mathbb{C}^2$, we act on $Y = H\text{-Hilb}(\mathbb{C}^2)$ with $G/H = \langle\bar{\beta}\rangle \cong \mathbb{Z}/2\mathbb{Z}$. Noe that the symmetry in the continued fraction $\frac{2n}{a}$ given in Lemma 3.7 implies that the coordinates along the exceptional curves E_i in the resolution $Y \to \mathbb{C}^2/H$ are also symmetric, i.e. β identifies the affine subsets $Y_i \cong \mathbb{C}^2_{(x^r/y^s,\, y^u/x^v)}$ with $Y_{m-i} \cong \mathbb{C}^2_{(x^u/y^v,\, y^r/x^s)}$ as well as the rational exceptional curves in E covered by these affine patches.

LEMMA 3.8. *If G is small then the continued fraction expansion of $\frac{2n}{a}$ has an odd number of entries.*

Proof. Suppose that the expansion of $\frac{2n}{a}$ has an even number of elements, and let $\{Y_i\}_{i=1,\dots,2h-1}$ be the open affine covering of H-Hilb$(\mathbb{C}^2)$. Since $(2n, a) = 1$, the abelian action has no quasireflections, so we need to check only the action of β on H-Hilb$(\mathbb{C}^2)$. This action identifies Y_i with Y_{m-i}, thus the only β-fixed part can occur at the middle open set Y_h, which is sent to himself. This open set is defined by the lattice points

$e_{h-1} = \frac{1}{2n}(u, v)$ and $e_h = \frac{1}{2n}(v, u)$ for some $0 < u < v$, so $Y_h \cong \mathbb{C}^2_{\lambda,\mu}$ where $\lambda = x^u/y^v$ and $\mu = y^u/x^v$. Then the action of β on Y_h is of the form $(\lambda, \mu) \mapsto ((-1)^v\mu, (-1)^u\lambda)$.

Now notice that u and v have the same parity. Indeed, if we call $e_2 = \frac{1}{2n}(c, d)$, we have $c = b_1$ and $d = a - 2n$. Thus, since a is odd, we perceive that c and d have the parity as b_1. Using the formula $e_{i+1} = b_ie_i - e_{i-1}$, we can use induction to conclude that the entries of every lattice point e_i for the action $\frac{1}{2n}(1, a)$ with $(2n, a) = 1$ have the same parity. In particular, since u and v have the same parity β fixes the line $\lambda = (-1)^u\mu$, i.e. β is a quasireflection, so the group is not small. □

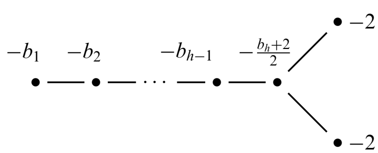
Since we are only interested in the case when G is small, from now on we suppose that $\frac{2n}{a}$ has an odd number of elements, i.e. $\frac{2n}{a} = [b_1, \dots, b_{h-1}, b_h, b_{h-1}, \dots, b_1]$. Then the exceptional divisor $E \subset Y$ has an odd number of irreducible components E_i, and there exists a middle rational curve $E_h \cong \mathbb{P}^1$. This rational curve has coordinate ratio $(x^q : y^q)$ and it is covered by Y_{h-1} and Y_h. Then β identifies Y_{h-1} with Y_h and it is an involution on E_h with two fixed points since there are no quasireflections. In the quotient $Y/\langle\beta\rangle$, these fixed points become two A_1 singularities, and blowing them up we obtain the Dynkin diagram of type D that we were looking for.

REMARK 3.9. For a small finite subgroup $G \subset \mathrm{GL}(2, \mathbb{C})$, by the result of Ishii [**4**], G-Hilb($\mathbb{C}^2$) is the minimal resolution of $\mathbb{C}^2/G$. By the uniqueness of minimal models in dimension 2, the previous construction can be expressed in the following diagram:

$$\begin{array}{ccccc} & & & & \mathbb{C}^2 \curvearrowleft H \trianglelefteq G \subset \mathrm{GL}(2,\mathbb{C}) \\ & & & & \downarrow \\ & & G/H \curvearrowright H\text{-Hilb}(\mathbb{C}^2) & \longrightarrow & \mathbb{C}^2/H \\ & & \downarrow & & \\ G\text{-Hilb}(\mathbb{C}^2) & \longrightarrow & Y/\langle\beta\rangle & & \end{array}$$

In other words, we have G-Hilb($\mathbb{C}^2$) $\cong G/H$-Hilb(H-Hilb($\mathbb{C}^2$)).

PROPOSITION 3.10. *Let $G = \mathrm{BD}_{2n}(a)$ be a small binary dihedral group and E be the exceptional divisor in G-Hilb($\mathbb{C}^2$), the minimal resolution of $\mathbb{C}^2/G$. Then E has the following Dynkin diagram of type D:*



where $\frac{2n}{a} = [b_1, \dots, b_{h-1}, b_h, b_{h-1}, \dots, b_1]$ and b_h is even.

Proof. Let $Y = H$-Hilb($\mathbb{C}^2$), $\pi : Y \to Y/\langle\beta\rangle$ be the quotient map, and E_i, $i = 1, \dots, 2h - 1$ be the exceptional curves in Y with $E_i^2 = E_{2h-i-1}^2 = -b_i$. Denote by E'_j the exceptional curves in $Y/\langle\beta\rangle$ for $j = 1, \dots, h$, and by $\widetilde{E}_i$ the strict transform of E'_i.

For $i \neq h$ the action of β on E_i has no fix points, we have $\widetilde{E}_i^2 = -b_i$. It remains to check as to what happens on the middle rational curve E_h. The curve E_h is covered by affine charts $Y_h \cong \mathbb{C}^2_{(\lambda,\mu)}$ and $Y_{h+1} \cong \mathbb{C}^2_{(\lambda',\mu')}$, where $\lambda = x^i/y^j$, $\mu = y^q/x^q$, $\lambda = x^q/y^q$ and $\mu = y^i/x^j$. Using the fact that $\frac{1}{2n}(i, j)$ and $\frac{1}{2n}(q, q)$ are consecutive points in the Newton polygon of lattice $L := \mathbb{Z}^2 + \frac{1}{2n}(1, a) \cdot \mathbb{Z}$, we know that $i + j = b_hq$, and the action of β is $(\lambda, \mu) \mapsto ((-1)^j\lambda\mu^{b_h}, (-1)^q1/\mu)$.

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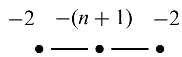
Table 2. Fixed locus in $Y = H\text{-Hilb}(\mathbb{C}^2)$ by the action of β

q	j	b_h	Fixed locus	$\mathrm{BD}_{2n}(a)$
Even	Odd	Even	Points $(0, 1)$ and $(0, -1)$ fixed	Small
		Odd	Point $(0, 1)$ and line $\mu = -1$ fixed	Not small
Odd	Even	$\equiv 0\ (4)$	Lines $\mu = i$ and $\mu = -i$ fixed	Not small
		$\equiv 2\ (4)$	Points $(0, i)$ and $(0, -i)$ fixed	Small
Odd	Odd	$\equiv 0\ (4)$	Points $(0, i)$ and $(0, -i)$ fixed	Small
		$\equiv 2\ (4)$	Lines $\mu = i$ and $\mu = -i$ fixed	Not small

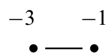
Let us study this action in detail. First note that since i and j have the same parity and $i + j = b_h q$, we therefore have that both b_h and q cannot be odd. In addition, both j and q cannot be even. Indeed, let $(0, 2n), (1, a), \dots, (r, s), (u, v), \dots, (j, i), (q, q), (i, j), \dots, (a, 1), (2n, 0)$ be the sequence of points in the boundary of the Newton polygon (all of them divided by $\frac{1}{2n}$). If q and j (and therefore i) are even, then $u = b_{h-1}j - q$ and $r = b_{m-2}u - j$ are also even. By induction we deduce that 1 is even, which is absurd. Hence, the only possibilities for fixed locus of the action of β on Y are shown in Table 2.

In the case when the group $\mathrm{BD}_{2n}(a)$ is small, we see that b_h is even and $Y/\langle\beta\rangle$ has two singular A_1 points along $E'_h \cong \mathbb{P}^1$ and $E'_h = -b_h/2$. Let $f : \widetilde{Y} \to Y/\langle\beta\rangle$ be the resolution of these two A_1 singularities denoted by C_1, C_2 the corresponding rational curves in $\widetilde{Y}$. Then $-b_h/2 = (\widetilde{E}_h + C_1/2 + C_2/2)^2 = (\widetilde{E}_h)^2 + C_1^2/4 + C_2^2/4 + \widetilde{E}_h C_1 + \widetilde{E}_h C_2 = (\widetilde{E}_h)^2 + 1$, so that $(\widetilde{E}_h)^2 = -(b_h + 2)/2$. □

Remark 3.11. (i) If $a = 1$, the group $\mathrm{BD}_{2n}(1)$ is abelian, and by Proposition 3.2 it is small if and only if n is odd. The continued fraction $\frac{2n}{1} = [2n]$ has only one term $b_h = 2n$ with $-\frac{b_h+2}{2} = -(n + 1)$. Then E has a type A Dynkin diagram of the form



(ii) If we consider the non-small group $\mathrm{BD}_{12}(5)$, the quotient variety $X/\langle\beta\rangle$ is non-singular and the exceptional divisor is of the form



To obtain the minimal resolution, we need to contract the -1-curve, obtaining a single $\mathbb{P}^1$ with self-intersection -2. Explicitly, the subgroup of quasireflections is generated by $H = \langle \alpha\beta, \alpha^3\beta, \alpha^5\beta, \alpha^7\beta, \alpha^9\beta, \alpha^{11}\beta \rangle$, which has order 12. The quotient is $\mathbb{C}^2_{x,y}/H \cong \mathbb{C}^2_{u,v}$ where $u = x^6 - y^6$ and $v = xy$. Now the action of α in new coordinates is $(u, v) \mapsto (\varepsilon^6 u, \varepsilon^6 v)$, i.e. it is of type $\frac{1}{12}(6, 6)$. Since $\mathbb{C}^2_{u,v}/\frac{1}{12}(6, 6) \cong \mathbb{C}^2_{u,v}/\frac{1}{2}(1, 1)$, the exceptional divisor in the minimal resolution of $\mathbb{C}^2/\mathrm{BD}_{12}(5)$ consists of a single $\mathbb{P}^1$ with self-intersection -2.

4. *G*-Hilb and *G*-graphs.

We start by describing the G-invariant Hilbert scheme G-Hilb, which motivates the definition of G-graph.

Definition 4.1. Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite subgroup. A *G-cluster* is a G-invariant zero-dimensional subscheme $\mathcal{Z} \subset \mathbb{C}^n$ for which $\mathcal{O}_{\mathcal{Z}}$ is isomorphic to the regular representation of G as a $\mathbb{C}G$-module. The *G-Hilbert scheme* $G\text{-Hilb}(\mathbb{C}^n)$ is the moduli space parametrising G-clusters.

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Recall that the regular representation $\mathbb{C}[G] \cong \bigoplus_{\rho \in \operatorname{Irr} G} \rho^{\dim(\rho)}$ and $\mathcal{O}_{\mathcal{Z}} \cong \mathbb{C}[x_1, \ldots, x_n]/I_{\mathcal{Z}}$, where $I_{\mathcal{Z}}$ is the ideal defining $\mathcal{Z}$. Thus, we may pick a vector space basis of $\mathcal{O}_{\mathcal{Z}}$ that contains $\dim(\rho)$ elements in each $\rho \in \operatorname{Irr} G$. To describe a distinguished basis of $\mathcal{O}_{\mathcal{Z}}$ with this property, it is convenient to use the notion of G-graph.

Definition 4.2. Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite subgroup. A *G-graph* is a subset $\Gamma \subset \mathbb{C}[x_1, \ldots, x_n]$ satisfying the following conditions:

(1) It contains $\dim(\rho)$ number of elements in each irreducible representation ρ.
(2) If a monomial $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ is a summand of a polynomial $P \in \Gamma$, then for every $0 \leq \mu_j \leq \lambda_j$ the monomial $x_1^{\mu_1} \cdots x_n^{\mu_n}$ must be a summand of some polynomial $Q_{\mu_1,\ldots,\mu_n} \in \Gamma$.

Note that for any G-cluster $\mathcal{Z}$ we can choose a basis for the vector space $\mathcal{O}_{\mathcal{Z}}$, which is a G-graph. In other words, we can always find a basis of $\mathcal{O}_{\mathcal{Z}}$, which is minimal in the sense of condition 2 in Definition 4.2. Indeed, let $\rho \in \operatorname{Irr} G$ of dimension d and let $g_1, \ldots, g_d \in \rho$ be the basis elements in Γ. Now let $f \in \rho$ and suppose that $f \notin \Gamma$ but $pf \in \Gamma$ for some $p \in \mathbb{C}[x_1, \ldots, x_n]$. Then we have a relation of the form $f \equiv a_1 g_1 + \cdots + a_d g_d$ modulo $I_{\mathcal{Z}}$ for some $a_i \in \mathbb{C}$, which implies that $pf \equiv a_i p g_1 + \cdots + a_d p g_d$ modulo $I_{\mathcal{Z}}$. Since $pf \in \Gamma$, we have $a_j \neq 0$ for at least one j, which allows us to consider the expression $g_j \equiv (1/a_j)f - (a_i/a_j)g_1 - \cdots - (a_d/a_j)g_d$ modulo $I_{\mathcal{Z}}$. Thus, we may choose f to be the basis element in Γ instead of g_j (compared with [**9**], Chapter 2).

For any G-graph Γ there exists a set $U_\Gamma \subset$ G-Hilb($\mathbb{C}^n$) consisting of all G-clusters $\mathcal{Z}$ such that $\mathcal{O}_{\mathcal{Z}}$ admits Γ for basis as a vector space. Since being a basis of an open set is an open condition, the set U_Γ is open. Therefore, given the set of all possible G-graphs Γ, their union covers G-Hilb($\mathbb{C}^n$).

Example 4.3. If we consider the cyclic group $G = \left\langle \frac{1}{5}(1,3) \right\rangle$ then $\Gamma = \{1, x, x^2, y, xy\}$ is a G-graph. For the non-abelian binary dihedral group $D_4 = \left\langle \frac{1}{4}(1,3), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \right\rangle \subset \mathrm{SL}(2,\mathbb{C})$, $\Lambda = \{1, x, y, x^2+y^2, x^2-y^2, y^3, -x^3, x^4-y^4\}$ is a D_4-graph (note that $(x,y), (y^3, -x^3) \in V_1$).

Definition 4.4. Let I_Γ be the ideal generated by the polynomials $f \in \rho$ which are not in Γ, for any $\rho \in \operatorname{Irr} G$. We say that the G-graph Γ is *represented by the ideal* I_Γ. Note that I_Γ determines Γ uniquely. We may also say that a G-cluster in an open set $U_\Gamma \subset G$-Hilb($\mathbb{C}^n$) is represented by the G-graph Γ.

The *representation* of a G-graph Γ is the Young diagram in the lattice M consisting of monomials, which are summands of polynomials in Γ modulo I_Γ. This representation reflects the nature of a G-graph and is useful to describe visually how a G-graph varies through G-Hilb($\mathbb{C}^2$).

Example 4.5. In Example 4.3 the G-graph Γ is represented by the ideal $I_\Gamma = \langle x^3, x^2y, y^2 \rangle$, and Λ is represented by the ideal $I_\Lambda = \langle xy, x^4+y^4 \rangle$. The representations of Γ and Λ are shown in Figure 1.

The relation $x^4 + y^4 \in I_\Lambda$ identifies x^4 and y^4 in $\mathbb{C}[x,y]/I_\Lambda$ and we say that x^4 and y^4 are '*twins*'.

4.1. The cyclic case. In this section we recall the construction of the minimal resolution $Y := H$-Hilb($\mathbb{C}^2$) of $\mathbb{C}^2/H$ for an abelian subgroup $H \subset \mathrm{GL}(2,\mathbb{C})$ using the Hirzebruch–Jung continued fractions and the relation with H-graphs (see [**8**]).

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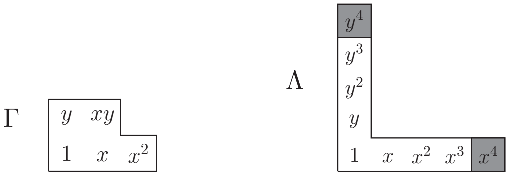


Figure 1. Representation of the G-graphs Γ and Λ.

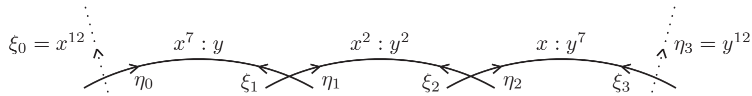
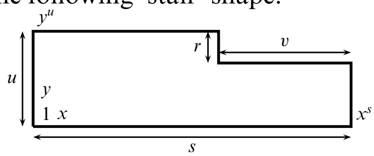


Figure 2. Resolution of singularities Y of the cyclic singularity of type $\frac{1}{12}(1, 7)$.

As it was introduced in Section 2, Y is the union of $m+1$ open sets $Y_i \cong \mathbb{C}^2$ defined by two consecutive boundary lattice points e_i, e_{i+1} of L. If we denote $e_i = \frac{1}{k}(r, s)$ and $e_{i+1} = \frac{1}{k}(u, v)$, then the corresponding open set in Y is of the form $Y_i \cong \mathbb{C}^2_{(\xi_i,\eta_i)}$, where $\xi_i = x^s/y^r$ and $\eta_i = y^u/x^v$. Every point $(\xi_i, \eta_i) \in Y_i$ corresponds to the H-cluster $\mathcal{Z}_{\xi_i,\eta_i}$ defined by the ideal

$$I_{\xi_i,\eta_i} = \langle x^s - \xi_i y^r,\ y^u - \eta_i x^v,\ x^{s-v}y^{u-r} - \xi_i\eta_i\rangle.$$

The H-graph Γ_i corresponding to the open set Y_i is determined by setting $\xi_i = \eta_i = 0$. In other words, Γ is represented by the ideal $I_\Gamma = I_{0,0} = \langle x^s, y^u, x^{s-v}y^{u-r}\rangle$, which pictorially is given by the following 'stair' shape:



For $(\xi_i, \eta_i) \in \mathbb{C}^2$, any monomial in $\mathbb{C}[x, y]$ can be written in terms of elements in Γ modulo the ideal $I_{\mathcal{Z}_{\xi_i,\eta_i}}$. In other words, Γ is a basis for the vector space $\mathbb{C}[x, y]/I_{\mathcal{Z}_{\xi_i,\eta_i}}$. Note also that $k = su - rv$, so the number of elements in Γ agrees with the order of the group. Thus, $\mathbb{C}^2_{\xi_i,\eta_i}$ is an open set in H-Hilb$(\mathbb{C}^2)$.

Example 4.6. Consider the group $H = \left\langle \frac{1}{12}(1, 7)\right\rangle$. We have $\frac{12}{7}$=[2,4,2] and therefore $Y = H\text{-Hilb}(\mathbb{C}^2)$ is of the form $Y = Y_0 \cup Y_1 \cup Y_2 \cup Y_3$, where $Y_i \cong \mathbb{C}^2_{(\xi_i,\eta_i)}$, $i = 0, \ldots, 3$ (see Figure 2). The corresponding H-clusters for these affine pieces are defined by the following ideals:

$$I_{\xi_0,\eta_0} = \langle x^{12} - \xi_0, y - \eta_0 x^7\rangle \qquad I_{\xi_2,\eta_2} = \langle x^2 - \xi_2 y^2, y^7 - \eta_2 x, xy^5 - \xi_2\eta_2\rangle$$
$$I_{\xi_1,\eta_1} = \langle x^7 - \xi_1 y, y^2 - \eta_1 x^2, x^5 y - \xi_1\eta_1\rangle \quad I_{\xi_3,\eta_3} = \langle x - \xi_3 y^7, y^{12} - \eta_3\rangle$$

and the representation of H-graphs is shown in Figure 3.

5. G-graphs for $\mathrm{BD}_{2n}(a)$ groups. Let $G = \mathrm{BD}_{2n}(a)$ be a small binary dihedral group, and let H be the maximal normal cyclic subgroup of G. As we have seen in Section 3.1, the minimal resolution Y of $\mathbb{C}^2/G$ is obtained by acting with β on

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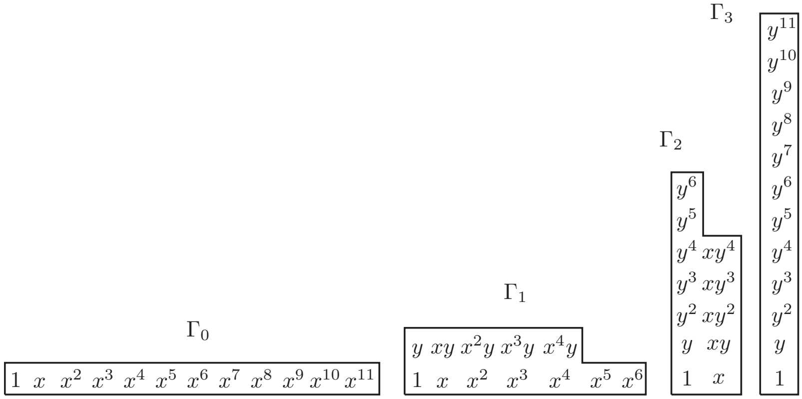

Figure 3. H-graphs for the group $H = \langle \frac{1}{12}(1, 7)\rangle$.

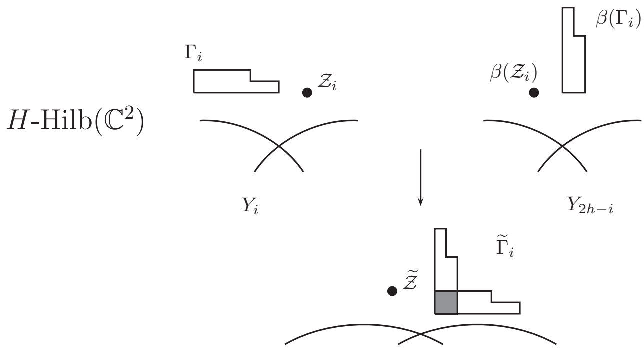

Figure 4. Action of β on H-Hilb$(\mathbb{C}^2)$ in terms of H-graphs.

H-Hilb$(\mathbb{C}^2)$. The G-graphs are constructed in the same way by translating the action of β into the H-graphs.

The symmetry along the coordinates of the exceptional divisor $E \subset H$-Hilb$(\mathbb{C}^2) = \bigcup_{i=0}^{2h} Y_i$ implies that β identifies Y_i with Y_{2h-i}, as well as identifying the corresponding H-graphs Γ_i and Γ_{2h-i}. The union $\Gamma \cup \beta(\Gamma)$ of two H-graphs identified by β is what we call a *qG-graph*.

5.1. ***qG*-graphs.** Let $\mathcal{Z}_i$ be an H-cluster in Y_i and $\beta(\mathcal{Z}_i)$ be its image under β in Y_{2h-i} with ideals $I_{\mathcal{Z}_i}$ and $I_{\beta(\mathcal{Z}_i)}$, respectively. Denote also by $\widetilde{\mathcal{Z}}$ the point in the quotient $\widetilde{X} := H$-Hilb$(\mathbb{C}^2)/\langle\beta\rangle$ corresponding to the orbit $\{\mathcal{Z}_i, \beta(\mathcal{Z}_i)\}$ (see Figure 4). Suppose $\mathcal{Z}_i$ and $\beta(\mathcal{Z}_i)$ are not one of the two points in H-Hilb$(\mathbb{C}^2)$ fixed by β, which means that $\widetilde{\mathcal{Z}}$ is not one of the singular A_1 points $P_1, P_2 \in \widetilde{X}$. Now $\pi : G$-Hilb$(\mathbb{C}^2) \to \widetilde{X}$ is the minimal resolution with exceptional divisor E consisting of two rational curves. Therefore, there is an isomorphism G-Hilb$(\mathbb{C}^2)\backslash E \cong \widetilde{X}\backslash\{P_1, P_2\}$, which means that there exists a unique G-cluster $\mathcal{Z}$ corresponding to $\widetilde{\mathcal{Z}}$.

As clusters in $\mathbb{C}^2$, we have $\mathcal{Z} \supset \mathcal{Z}_i \cup \beta(\mathcal{Z}_i)$, or equivalently $I_{\mathcal{Z}} \subset I_{\mathcal{Z}_i} \cap I_{\beta(\mathcal{Z}_i)}$. In terms of graphs if we denote by Γ, Γ_i and $\beta(\Gamma_i)$ the graphs corresponding to the ideals

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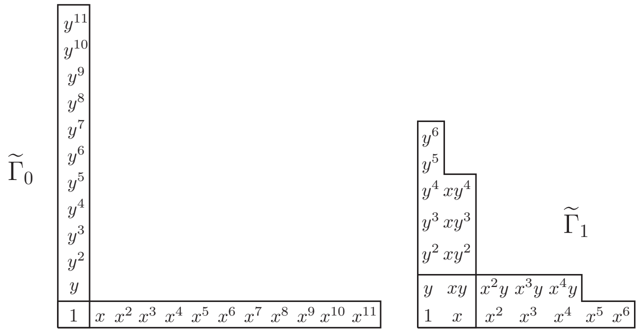

Figure 5. qG-graphs for the group $\mathrm{BD}_{12}(7)$.

$I_{\mathcal{Z}}$, $I_{\mathcal{Z}_i}$ and $I_{\beta(\mathcal{Z}_i)}$, respectively, we have

$$\Gamma \supset \Gamma_i \cup \beta(\Gamma_i) = \widetilde{\Gamma}_i$$

where (abusing the notation) by the inclusion we mean that every monomial in $\widetilde{\Gamma}_i$ is a summand of a polynomial in Γ.

But note that $\widetilde{\Gamma}_i$ is not a G-graph. Indeed, Γ_i and $\beta(\Gamma_i)$ have an overlap (common basis elements shaded in Figure 4), so the number of elements of $\widetilde{\Gamma}_i$ is always smaller than $|G| = 2 \cdot |H|$. Nevertheless, we will see in the next section that there is a unique way of extending $\widetilde{\Gamma}_i$ into a G-graph Γ. We call these new graphs $\widetilde{\Gamma}_i$ '*quasi G-graph*' (*qG-graphs*).

For the two fixed points $\mathcal{Z}_1, \mathcal{Z}_2 \in E_h \subset H\text{-Hilb}(\mathbb{C}^2)$, where $\mathcal{Z}_i = \beta(\mathcal{Z}_i)$, we have $I_{\mathcal{Z}_i} = I_{\beta(\mathcal{Z}_i)}$ and we can choose an H-graph Γ_i for $\mathcal{Z}_i$ such that $\Gamma_i = \beta(\Gamma_i)$ for $i = 1, 2$. This implies that the qG-graph $\Gamma_i \cup \beta(\Gamma_i)$ is all overlap, and the extension to a G-graph in this case is not unique. In fact, for each of the two β-fixed points there is a projective line of G-clusters corresponding to the exceptional curves of the blow-up, which do not come from orbits of H-graphs. In other words, because of the presence of fixed points, to define an open cover of G-Hilb, we need to treat the case where $\widetilde{\Gamma}$ is the union of the two middle H-graphs separately. This lead us to G-graphs of type C and D (see Section 5.5).

REMARK 5.1. Since every H-graph is given by two consecutive points e_i, e_{i+1} in the boundary of the Newton polygon of L, every qG-graph is given by the consecutive pair e_i, e_{i+1} together with its symmetric pair with respect to the diagonal. Therefore, in order to calculate all possible qG-graphs, we just have to look at the qG-graphs coming from consecutive points of the list: $e_0 = \frac{1}{2n}(0, 2n)$, $e_1 = \frac{1}{2n}(1, a)$, …, $e_h = \frac{1}{2n}(q, q)$.

EXAMPLE 5.2. Continuing with Example 4.6, the action of β glues together Γ_0 with Γ_3 and Γ_1 with Γ_2, obtaining the qG-graphs $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$. See Figure 5.

Now we look at how two H-graphs identified by the action of β can merge together into a qG-graph. The following proposition shows that there are only two types of gluing.

PROPOSITION 5.3. *Let $G = \mathrm{BD}_{2n}(a) \subset \mathrm{GL}(2, \mathbb{C})$ be a small binary dihedral group. Let Γ be the H-graph defined by the consecutive Newton polygon points $e_i = \frac{1}{2n}(r, s)$ and $e_{i+1} = \frac{1}{2n}(u, v)$. Then corresponding qG-graph $\widetilde{\Gamma}$ is either of type A if $s - v > u$, or of type B if $s - v = u - r$.*

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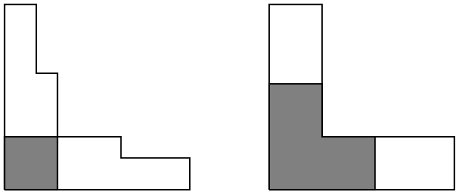
We may also say that the H-graph Γ is of type A or B. Their shapes are shown in the following diagram, where the shaded area represents the overlap between H-graphs:



Type A Type B

Proof. Since any qG-graph $\widetilde{\Gamma}$ is the union of two H-graphs identified by the action of β, the shape of the qG-graph will depend on the relation between $s - v$ and u.

Suppose that $s - v \leq u$. Since $s < v$, this means that $s - v = u - p$ for some $0 \leq p < u$. Let $e_{i+2} = \frac{1}{2n}(t, w)$ be the next point in the boundary of the Newton polygon. Then, since $w = b_{i+1}v - s$, we have $p = u + v - s = u + (1 - b_{i+1})v + w$, which implies that $b_{i+1} = 2$ (otherwise $p < u - 2v + w$, and since $v \geq u$ and $v > w$, then $p < 0$ a contradiction). Thus, $p = u - v + w$, and applying the same argument to w, we obtain $b_{i+1} = b_{i+2} = \ldots = b_h = 2$, that is $\frac{2n}{a} = [b_0, \ldots, b_i, 2, \ldots, 2, b_i, \ldots, b_0]$.

Finally, the chain of 2s in the middle of the continued fraction $\frac{2n}{a}$ gives us the value of p. Indeed, let $e_{h-1} = \frac{1}{2n}(c, d)$, $e_h = \frac{1}{2n}(q, q)$ and $e_{h+1} = \frac{1}{2n}(d, c)$ be the three middle Newton polygon points. Then proceeding with the previous argument we get

$$\begin{aligned} p &= u - d + q \\ &= u - (2q - c) + q = u - q + c \\ &\vdots \\ &= u - t + u = 2u - t \\ &= r \end{aligned}$$

which gives the qG-graph of type B. □

From the previous proof, we can deduce that the distribution of qG-graphs of type A and B in the exceptional locus depend on the number of 2s in the middle of the continued fraction $\frac{2n}{a}$.

Corollary 5.4. *Let $\widetilde{\Gamma}_0, \ldots, \widetilde{\Gamma}_{h-1}$ be the sequence of qG-graphs for a given group $G = \mathrm{BD}_{2n}(a)$, and let $\frac{2n}{a} = [b_0, \ldots, b_h, \ldots, b_0]$.*

(i) *If $b_i = 2$ for $k \leq i \leq h-1$ and $b_{k-1} \neq 2$, we have that $\widetilde{\Gamma}_0, \ldots, \widetilde{\Gamma}_{k-1}$ are of type A, and $\widetilde{\Gamma}_k, \ldots, \widetilde{\Gamma}_{h-1}$ are of type B.*

(ii) *There are no type B qG-graphs if and only if $b_h \neq 2$.*

For example, the continued fraction $[\ldots, b, 2, 2, 2, 2, 2, b, \ldots]$ with $b \neq 2$ gives three qG-graphs of type B. As a consequence we get the following corollary. In the case when $a = 2n - 1$, i.e. $\mathrm{BD}_{2n}(2n-1) \subset \mathrm{SL}(2, \mathbb{C})$, the coefficients of the continued fraction $\frac{2n}{2n-1}$ are all 2, and every qG-graph is of type B.

5.2. From qG-graphs to G-graphs. In this section we construct the G-graph corresponding to a given qG-graph. First notice that every monomial of the qG-graph is in general a summand of a polynomial in the G-graph. More precisely, let

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$\widetilde{\Gamma} = \Gamma \cup \beta(\Gamma)$ be a qG-graph, and suppose that $x^l y^m \in \rho_j$ is an element in Γ for some j. Then by Remark 3.4 we have $\beta(x^l y^m) = (-1)^m x^m y^l \in \rho_{aj}$. If $j \equiv aj \pmod{2n}$, the polynomial that we obtain in the 1-dimensional representation $\rho_j^{\pm}$ is

$$\begin{aligned} &\text{either } x^l y^m \pm i(-1)^m x^m y^l \text{ if } n, j \text{ odd}, \\ &\text{or } x^l y^m \pm (-1)^m x^m y^l \text{ otherwise.} \end{aligned} \tag{5.2.1}$$

If $j \not\equiv aj \pmod{2n}$, we obtain the pair $(x^l y^m, (-1)^m x^m y^l)$ in V_j. Moreover, since Γ is an H-graph, there always exists another monomial $f \in \Gamma$ belonging to ρ_{aj}, which gives a second element in V_j, namely $(f, \beta(f))$. These two elements in V_l are the same if and only if $f = x^l y^m$ is contained in the overlap $O := \Gamma \cap \beta(\Gamma)$.

Therefore, for every representation $\rho_j \in \operatorname{Irr} H$ not contained in the overlap O, we obtain one element in every 1-dimensional representation $\rho_j^{\pm}$ of G, and two elements in every 2-dimensional irreducible representation V_j of G as desired. In O we only get one element either in ρ_j^+ or ρ_j^-, and only one pair in V_r. Thus, to form a G-graph from a qG-graph we need to add new elements in the representations of G arising from the representations of H contained in O.

REMARK 5.5. Let I be an ideal defining a G-cluster, and let Γ be a choice of G-graph represented by I. Suppose for simplicity ρ is a 1-dimensional irreducible representation of G, and let $g \in \rho$ be an element in Γ. Since modulo I, the module S_ρ can be considered as a 1-dimensional vector space with basis g, for any other polynomial $f \in \rho$ not in Γ we have a relation of the form $f \equiv a \cdot g \pmod{I}$ for some $a \in \mathbb{C}$. Doing the same for every irreducible representation of G we obtain the relation of any semi-invariant polynomial as a $\mathbb{C}$-linear combination of elements in $\Gamma \pmod{I}$. We can (and do) choose to take all coefficients equal to zero. In other words, we take the ideal representing Γ, which geometrically corresponds to look at the intersection points of two of the exceptional curves in G-Hilb$(\mathbb{C}^2)$ plus the strict transform of the coordinate axis in $\mathbb{C}^2$.

If Γ is a G-graph and G is abelian, we can always choose monomials for elements in Γ. In our case we need to deal with combinations of monomials identified by β, which creates the new phenomenon of *twin elements* in G-graphs (see Example 4.5). For instance, let $f = x^i y^j - x^j y^i$ with $i > j$ be an element in some 1-dimensional representation ρ. If $f \notin \Gamma$ by Remark 5.5, we have $x^i y^j = x^j y^i$ in $\mathbb{C}[x, y]/I$, so both monomials become identified as twins. Moreover, multiplying f by $x^p y^q$ for any positive integers p and q, we have $x^{i+p} y^{j+q} = x^{j+p} y^{i+q}$, and we get a pair of symmetric *twin regions*. We call such an f a *twin relation*. Note that for the purpose of counting basis elements, a pair of twin elements will count as a single basis element.

Let $\widetilde{\Gamma}$ be the qG-graph defined by $e_i = \frac{1}{2n}(r, s)$ and $e_{i+1} = \frac{1}{2n}(u, v)$. We denote by $\Gamma(r, s; u, v)$ or simply Γ the corresponding G-graph and I its defining ideal.

5.3. ***G*-graphs of type *A*.** Suppose that $\widetilde{\Gamma}$ is a qG-graph of type A. Then $\widetilde{\Gamma} = \Gamma_I \cup \Gamma_{\beta(I)}$, where $I = \langle x^s, y^u, x^{s-v} y^{u-r} \rangle$ and $\beta(I) = \langle x^u, y^s, x^{u-r} y^{s-v} \rangle$. In addition, we have the inequalities $r < u \leq v < s$ and $u < s - v$ coming, respectively, from the lattice L and the type A condition in Proposition 5.3. Following is the general shape of qG-graph of type A:

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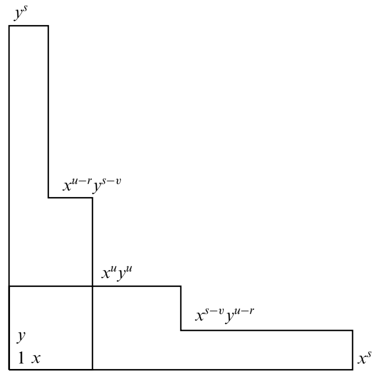

Clearly, $\#\widetilde{\Gamma} = 2 \cdot \#\Gamma_I - \#O < 2n$, with $O = \Gamma_I \cap \Gamma_{\beta(I)}$ the overlap. Then $\widetilde{\Gamma}$ needs to be extended by $\#O = u^2$ elements to become a G-graph, where these extra elements belong to the irreducible representations appearing in O. The following lemmas show that we just need to look at some key representations.

LEMMA 5.6. *The polynomial $x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}$ is G-invariant, i.e. it belongs to I.*

Proof. We need to show that $x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}$ is invariant under the action of α and β. We know that $ar \equiv s$ and $au \equiv v \pmod{2n}$, then $\alpha(x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}) = \varepsilon^{s-v+au-ar}x^{s-v}y^{u-r} + (-1)^{u-r}\varepsilon^{u-r+as-av}x^{u-r}y^{s-v} = x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}$.

Since a is odd, the congruence $ar \equiv s \pmod{2n}$ also implies that r and s have the same parity (similarly for the pair u, v). Therefore, $s - v$ and $u - r$ must have the same parity and then $\beta(x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}) = x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}$ as required. □

LEMMA 5.7. *The monomial $x^u y^u$ is not in the G-graph Γ.*

Proof. Suppose $x^u y^u \in \Gamma$. Note that $x^u y^u, x^{u+v} + (-1)^u y^{u+v} \in \rho_{u+v}^{(-1)^u}$, so $x^{u+v} + (-1)^u y^{u+v} \in I$ and it forms a twin region. Observe also that $(x^u y^{u+1}, (-1)^u x^{u+1} y^u)$ and $(y^{u+v+1}, -x^{u+v+1})$ belong to the same 2-dimensional representation V_l, and the monomials $x^{u+v+1}, y^{u+v+1} \in \widetilde{\Gamma}$ do not lie on the overlap O. Then there must be another monomial pair in the qG-graph $\widetilde{\Gamma}$ outside O that lie in the same representation V_l, so it has to be included in Γ. Therefore, $x^u y^{u+1}, x^{u+1}y^u \in I$, which implies that on diagonal the qG-graph can only be extended by the element $x^u y^u$.

We claim that along the sides $\widetilde{\Gamma}$ cannot be extended enough to obtain a G-graph. Indeed, by the previous lemma we know that $x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v} \in I$, so it forms the pair of twin regions T_1. Also, $x^r(x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}) = x^{s+r-v}y^{u-r} + (-1)^{u-r}x^u y^{s-v} \in I$. Now the type A condition $u < s - v$ implies that $x^u y^{s-v} \in I$, and therefore $x^{s+r-v}y^{u-r} \in I$ (same for $x^{u-r}y^{s+r-v}$). Thus, the twin regions T_1 have each block of r^2 elements. In the same way, since $x^u y^u \in \Gamma$, the polynomial $x^{u+v} + (-1)^u y^{u+v}$ forms the twin region T_2, and because $u + u < s$, their size is at most $(u - r)^2 - (u - r)$. Thus, the qG-graph is extended with at most $r^2 + (u - r)^2 - (u - r) + 1 < u^2 = \#M$ elements, which is a contradiction. □

As a consequence we have that $x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}$ creates the twin region T_1 of size r^2. The following lemma shows that there exists another polynomial creating a second twin region.

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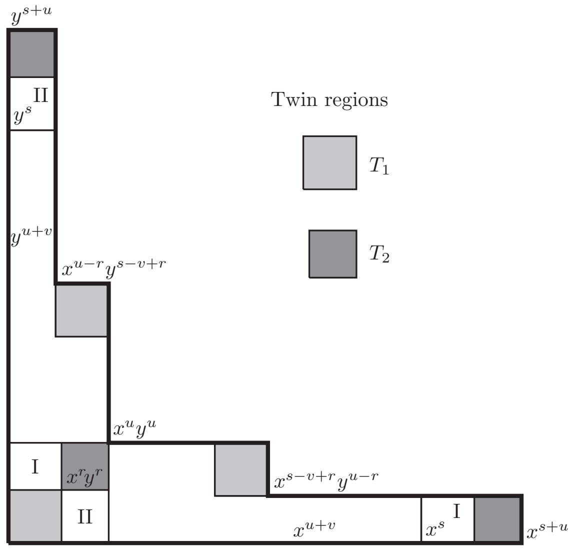

Figure 6. Representation of $\Gamma_A(r, s; u, v)$ from the extension of a qG-graph by the elements in the overlap.

Lemma 5.8. *The polynomial* $x^{r+s} + (-1)^r y^{r+s} \in I$ *and creates the twin region* T_2 *of size* $(u - r)^2$.

Proof. It is easy to see that $x^r y^r, x^{r+s} + (-1)^r y^{r+s} \in \rho_{r+s}^{(-1)^r}$. However, $x^r y^r$ belongs to the qG-graph, so it must be in Γ, and therefore $x^{r+s} + (-1)^r y^{r+s} \in I$. Now combining the previous lemmas we see that $x^{s+u}, y^{s+u} \in I$ and the size of the twin region T_2 is equal to $(u - r)^2$. □

Hence, the extension to a G-graph from any type A qG-graph has two twin regions T_1 and T_2, except for $\Gamma_A(0, 2n; 1, a)$, where only T_2 appears (r is 0 in this case). Figure 6 represents the shape of the G-graph that we were looking for.

Proposition 5.9. *Every free* G/H*-orbit of an* H*-cluster defined by an* H*-graph of type* A *is represented by an ideal of the form*

$$I_A = \langle x^u y^u, x^{s-v} y^{u-r} + (-1)^{u-r} x^{u-r} y^{s-v}, x^{r+s} + (-1)^r y^{r+s} \rangle.$$

G*-graphs represented by ideals of the form* I_A *are said to be of type* A *and are denoted by* $\Gamma_A(r, s; u, v)$.

Proof. The previous lemmas show that the qG-graph only expands along its sides, and it does not grow further than the twin regions T_1 and T_2. Let I and II be the regions described in Figure 6. Then the number of elements added is

$$\#T_1 + \#T_2 + \#\mathrm{I} + \#\mathrm{II} = r^2 + (r - u)^2 + 2r(u - r) = u^2$$

exactly the number of elements in the overlap.

It remains to prove that Γ contains 1 polynomial for each 1-dimensional irreducible representation ρ_k and two pairs of polynomials for each 2-dimensional irreducible representation V_l. By the symmetry of $\widetilde{\Gamma}$ every representation not appearing in the overlap will appear twice, as required. Therefore, we need only check that the

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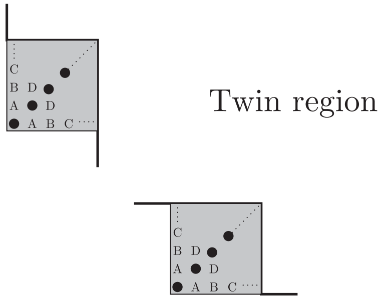
representations of the new extended blocks correspond exactly to the ones in the overlap.

Every representation contained in regions I and II is 2-dimensional, and we have one basis element coming from the overlap O and a second from the extended region. The twin relations $x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v} \in I_A$ and $x^{r+s} + (-1)^r y^{r+s} \in I_A$ give the correct number of elements in T_1 and T_2. We show in the following diagram the configuration of representations contained in the twin region T_1. The case of T_2 is analogous.

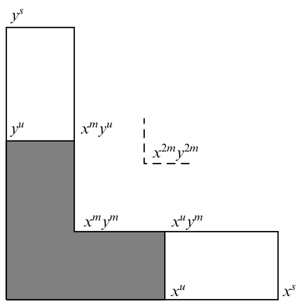


The elements on the diagonals (marked by dots) are in 1-dimensional representations. The rest (marked by letters) are in 2-dimensional representation pairs, that is a monomial $x^i y^j$ and its partner with respect to the diagonal $x^j y^i$ will form an element in some 2-dimensional representation (here we omit the sign). Elements with the same letter are in the same representation. Since twin symmetric regions count as one, the representations in the overlap are fully present in the extension. □

5.4. *G*-graphs of type *B*. Suppose that the qG-graph $\widetilde{\Gamma} = \Gamma_B(r, s; u, v)$ defined by $e_i = \frac{1}{2n}(r, s)$ and $e_{i+1} = \frac{1}{2n}(u, v)$ is of type B. Then we can define

$$m := s - v = u - r,$$

which represents the width of $\widetilde{\Gamma}$. The value of m remains constant for every qG-graph of type B. Indeed, if $e_{i+2} = \frac{1}{2n}(t, w)$ and we consider the next qG-graph $\Gamma_B(u, v; t, w)$ of type B, we have $w = 2v - s$ so that $v - w = s - v = m$. Following is a general shape for a type B qG-graph:



Notice that $(x^r, y^r), (y^s, (-1)^s x^s), (x^u y^m, (-1)^m x^m y^u) \in V_r$, and since $x^r, y^r \in \widetilde{\Gamma}$, we have $(x^r, y^r) \in \Gamma$. Thus, to pick the remaining element of V_r in Γ, we have a choice between the other two elements in V_r.
CASE 1. Suppose that $(y^s, (-1)^s x^s) \in \Gamma$, that is $x^u y^m, x^m y^u \in I$.

LEMMA 5.10. *Let $G = \mathrm{BD}_{2n}(a)$ and let $\Gamma = \Gamma_B(r, s; u, v)$ be a qG-graph of type B. Then the monomial $x^m y^m$ is H-invariant with m odd, and $x^{2m}y^{2m}$ is G-invariant.*

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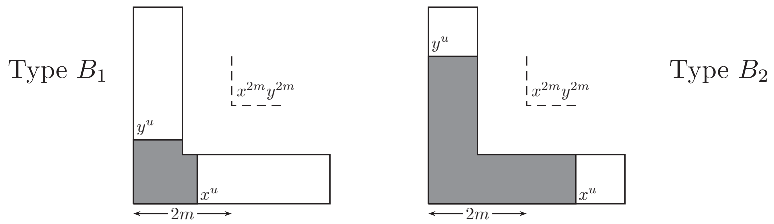


Figure 7. qG-graphs of type B_1 and B_2 according to the size of the overlap.

Proof. We have $\alpha(x^m y^m) = \varepsilon^{s-v+a(u-r)} x^m y^m = \varepsilon^{s-v+v-s} x^m y^m = x^m y^m$, so it is H-invariant. If we consider $\widetilde{\Gamma}(t, w; q, q)$ to be the last qG-graph, i.e. the one corresponding to Γ_{h-1}, we know by definition that $2m = w - t$. On the other hand, Γ_{h-1} is an H-graph, so it has $2n$ elements. Thus, $2n = q(w - t)$, and since the group is of type $\mathrm{BD}_{2n}(a)$ by Remark 3.3, we conclude that m is odd. Therefore, $\beta(x^m y^m) = -x^m y^m$ and the G-invariant monomial is $x^{2m} y^{2m}$. □

Now we split up the type B graphs into two different cases as follows:

Definition 5.11. Let $\widetilde{\Gamma} = \widetilde{\Gamma}(r, s; u, v)$ be a qG-graph of type B. We say $\widetilde{\Gamma}$ is of type B_1 if $u < 2m$, and it is of type B_2 if $u \geq 2m$. See Figure 7.

Lemma 5.12. *Let $\widetilde{\Gamma}_0, \widetilde{\Gamma}_1, \ldots, \widetilde{\Gamma}_{h-1}$ be the sequence of qG-graphs for a given group $G = \mathrm{BD}_{2n}(a)$. Let $\frac{2n}{a} = [b_1, \ldots, b_h, \ldots, b_1]$ and suppose that $b_i = 2$ for $k \leq i \leq h-1$ and $b_{k-1} \neq 2$, for some $0 \leq k \leq h-1$. Then $\widetilde{\Gamma}_k$ is always of type B_1, while the rest $\widetilde{\Gamma}_i$ for $k < i \leq h-1$ are of type B_2.*

Proof. By Corollary 5.4 the qG-graphs $\widetilde{\Gamma}_i$ are of type A for $i \leq k-1$, and of type B for $k \leq i \leq h-1$. Let $\widetilde{\Gamma}_{k-1} = \widetilde{\Gamma}_A(c, d; r, s)$ be a G-graph of type A and $\widetilde{\Gamma}_k = \widetilde{\Gamma}_B(r, s; u, v)$ a G-graph of type B. Then $b_{k+1} = 2$ and $d = 2s - v$. Now, since $\widetilde{\Gamma}_{k-1}$ is of type A, we have $r < d - s = (2s - v) - s = s - v = m$ and then $u = m + r < 2m$, so $\widetilde{\Gamma}_k$ is of type B_1.

Suppose now that $k < h-1$ so that there exists $\widetilde{\Gamma}_{k+1} = \widetilde{\Gamma}_B(u, v; w, t)$ another qG-graph of type B. Then $b_{k+2} = 2$ and $w = 2u - r$. Moreover, since m is constant for any qG-graph of type B, we have $m = w - u$. Then $2m = 2(w - u) = 2w - 2u = 2w - (w + r) = w - r$. Therefore, $w \geq 2m$, i.e. a type B_2 qG-graph. By induction the rest of qG-graphs are also of type B_2. □

Proposition 5.13. *Every free G/H-orbit of an H-cluster defined by an H-graph of type B_1 is represented by an ideal of the form*

$$I_{B_1} = \langle x^{r+s} + (-1)^r y^{r+s}, x^{m+s} y^{m-r} + (-1)^{m-r} x^{m-r} y^{m+s}, x^u y^m, x^m y^u \rangle.$$

G-graphs represented by ideals of the form I_{B_1} are said to be of type B_1 and are denoted by $\Gamma_{B_1}(r, s; u, v)$.

Proof. Since $u < 2m$ and $u = m + r$, we know that $r < m$, which implies that the monomial $x^r y^r$ is contained in the overlap O. Then $x^r y^r \in \Gamma$ and we have the relation $x^{r+s} + (-1)^r y^{r+s} \in I_{B_1}$, creating the twin region T. Using the condition $x^u y^m, x^m y^u \in I_{B_1}$ and multiplying the previous relation by x^m and y^m we get $x^{s+u}, y^{s+u} \in I_{B_1}$, which implies that T does not grow further these monomials.

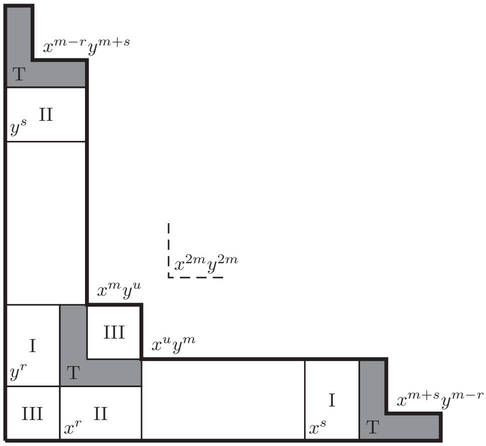


Figure 8. G-graph of type B_1.

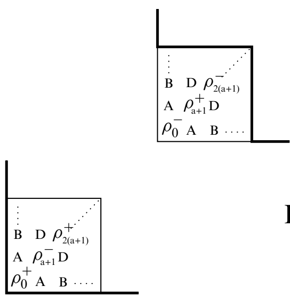
Examining the representations ρ_0^+ and ρ_0^-, we see that $x^{m+s}y^{m-r} + (-1)^{m-r}x^{m-r}y^{m+s} \in I_{B_1}$ (since always $1 \in \rho_0^+$) and $x^{m+s}y^{m-r} - (-1)^{m-r}x^{m-r}y^{m+s} \in I_{B_1}$ (otherwise $x^m y^m \in I_{B_1}$ and Γ would not have enough elements). Then $x^{m+s}y^{m-r}, x^{m-r}y^{m+s} \in I_{B_1}$. All these equations together with the G-invariant relations are enough to determine a G-graph, hence I.

In Figure 8 we show the G-graph and the underlying relation between the representations of G for the type B_1 case. Let I, II and III be the regions described in Figure 8. Then the number of new elements are

$$\#T + \#I + \#II + \#III = m^2 - (u-m)^2 + 2r(u-r) + (u-m)^2 = u^2 - r^2,$$

which is the same number of elements in the overlap.

It remains to show that in Γ we have the right number of elements in every irreducible representation of G. In this case we only have the twin region T and, as in the proof of Proposition 5.9, the twin relation $x^{r+s} + (-1)^r y^{r+s} \in I_{B_1}$ gives the correct number of elements in the representations contained in T. Again regions I and II contain only 2-dimensional representations, where the element coming from O and the new element from the extended region make the number of elements match with the dimension of the representation. For region III, representations in the diagonal $x = y$ are 1-dimensional and the rest are 2-dimensional. As shown in the following diagram, elements located in the same place in both blocks have the same H-character but opposite G/H-character. This implies that every representation $\rho^{\pm}_{j(a+1)}$ has precisely one element in Γ.



Region III

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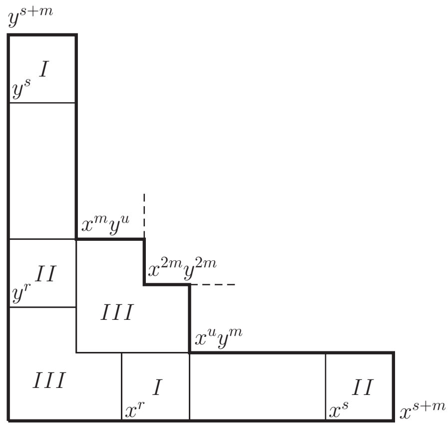

Figure 9. G-graph of type B_2.

For 2-dimensional representations we have one element in each block, which give two elements in total as required. □

PROPOSITION 5.14. *Every free G/H-orbit of an H-cluster defined by an H-graph of type B_2 is represented by an ideal of the form*

$$I_{B_2} = \langle x^{2m}y^{2m}, x^{s+m}, y^{s+m}, x^u y^m, x^m y^u \rangle.$$

G-graphs represented by ideals of the form I_{B_2} are said to be of type B_2 and are denoted by $\Gamma_{B_2}(r, s; u, v)$.

Proof. Let $e_i = \frac{1}{2n}(r, s)$ and $e_{i+1} = \frac{1}{2n}(u, v)$ be two consecutive lattice points. First observe that

$$(x^{r-m}, y^{r-m}), (y^{s+m}, (-1)^{s+m}x^{s+m}) \text{ and } (x^r y^m, -x^m y^r)$$

belong to the irreducible representation V_{2r-u}. The pair $(x^{r-m}, y^{r-m}) \in \Gamma$, since both x^{r-m} and y^{r-m} are elements in the qG-graph, thus we have two possibilities for the second element in Γ belonging to the representation V_{2r-u}.

First assume that $(x^r y^m, x^m y^r) \in \Gamma$, that is $x^{s+m}, y^{s+m} \in I_{B_2}$. Then by the assumption $x^u y^m, x^m y^u \in I_{B_2}$ and the G-invariant relation $x^{2m}y^{2m} \in I_{B_2}$ we obtain the ideal $I_{B_2} = \langle x^{2m}y^{2m}, x^{s+m}, y^{s+m}, x^u y^m, x^m y^u \rangle$ as in the statement with the representation shown in Figure 9. Note that overlap in this case is extended without twin regions, so the number of elements that we add is

$$\#I + \#II + \#III = 2m^2 + r^2 - (r-m)^2 = u^2 - r^2,$$

which is the number of elements in the overlap O.

Let $\rho \in \operatorname{Irr} G$ be a representation of G contained in O. If $\dim(\rho) = 1$, then it belongs to the region III and is located in the diagonal $x = y$. As in the proof of Proposition 5.13, the representations located at the diagonal alternate the character of β, thus we have one element in each 1-dimensional representation. If $\dim(\rho) = 2$, then in ρ we have an element coming from O and a second element coming from the extension, and we therefore have the required number of elements in each representation.

Now assume that $(y^{s+m}, x^{s+m}) \in \Gamma$, that is $x^r y^m, x^m y^r \in I_{B_2}$. Then the pairs

$$(x^{c-m}, y^{c-m}), (x^{d+m}, (-1)^{d+m} x^{d+m}), (x^c y^m, -x^m y^c)$$

are in the same representation V_{c-m}, where $c = 2r - u$ and $d = 2s - v$. By Lemma 5.12, the qG-graph defined by the previous consecutive pair e_{i-1}, e_i is of type B, which implies that $b_i = 2$. Since $e_{i-1} + e_{i+1} = b_i e_i$, we have that c and d are the entries of the lattice point $e_{i-1} = \frac{1}{2n}(c, d)$.

Following the same argument as before, the pair $(x^{c-m}, y^{c-m}) \in \Gamma$, since both x^{c-m} and y^{c-m} are elements in the qG-graph, and again we have two choices for the second basis element in V_{c-m}: either $(x^c y^m, -x^m y^c) \in \Gamma$ or $(x^{d+m}, (-1)^{d+m} x^{d+m}) \in \Gamma$.

If $(x^c y^m, -x^m y^c) \in \Gamma$ then we get the ideal $I_{B_2} = \langle x^{2m} y^{2m}, x^{d+m}, y^{d+m}, x^r y^m, x^m y^r \rangle$, which is the ideal representing the G-graph $\Gamma_{B_2}(c, d; r, s)$ obtained with the pair e_{i-1}, e_i. If $(x^{d+m}, (-1)^{d+m} x^{d+m}) \in \Gamma$ instead, we reach two new choices as before but now with the consecutive lattice points $e_{i-2} = \frac{1}{2n}(2c - r, 2d - s)$ and e_{i-1}, and we continue the process in the same way.

Therefore, at every step we obtain an ideal of the desired form $\langle x^{2m} y^{2m}, x^{s_k+m}, y^{s_k+m}, x^{u_k} y^m, x^m y^{u_k} \rangle$ where $m = s_k - v_k = u_k - r_k$, corresponding to a pair $e_k = \frac{1}{2n}(r_k, s_k)$, $e_{k-1} = \frac{1}{2n}(u_k, v_k)$ of consecutive lattice points defining a qG-graph of type B_2. The proof concludes by showing that this inductive process terminates in a finite number of steps. Indeed, by Lemma 5.12 we eventually arrive to a pair e_j, e_{j+1} for some $j < i$ defining a qG-graph of type B_1, for which the ideal is described in Proposition 5.13. $\square$

Case 2. Suppose that $(x^u y^m, (-1)^m x^m y^u) \in \Gamma$, that is $y^s, x^s \in I$. By considering the following three elements

$$(x^u, y^u), (y^v, (-1)^v x^v), (x^{2u-r} y^m, -x^m y^{2u-r}) \in V_u,$$

we can see that $x^{2u-r} y^m, x^m y^{2u-r} \in I_B$ because x^u, y^u, y^v and x^v are elements of the qG-graph $\widetilde{\Gamma} := \widetilde{\Gamma}_B(r, s; u, v)$. Since the qG-graph $\widetilde{\Gamma}$ is of type B, we have $b_i = 2$, and we can write $t = 2u - r$ and $w = 2v - s$, which coincide with the entries of the next lattice point $e_{i+1} = \frac{1}{2n}(t, w)$. We obtain in this way the ideal $I_{B_2} = \langle x^{2m} y^{2m}, x^{v+m}, y^{v+m}, x^t y^m, x^m y^t \rangle$ with $m = v - w = t - u$, which represents the G-graph $\Gamma_{B_2}(u, v; t, w)$ of type B_2. Note that even if $\widetilde{\Gamma}$ is of type B_1, by Lemma 5.12 the qG-graph $\widetilde{\Gamma}_B(u, v; t, w)$ is of type B_2. As in the proof of Proposition 5.14, the overlap $O = I \cup II \cup III$ in the qG-graph is extended without twin regions, and the total number of elements is the required count.

When $\Gamma := \Gamma_B(r, s; q, q)$ is the G-graph corresponding to the last qG-graph $\widetilde{\Gamma}_{h-1}$, we obtain the representation of Γ as shown in Figure 10, where the ideal in this case is $I = \langle x^{2m} y^{2m}, x^s, y^s \rangle$. We do not treat this case because the corresponding open set U_Γ covers the strict transformation of the curve $E_h \subset H$-Hilb$(\mathbb{C}^2)$ with two fixed points by G/H, and every point in U_Γ is already covered by the open sets corresponding to G-graphs at the blowup of the singular points in H-Hilb$(\mathbb{C}^2)/(G/H)$. These are called G-graphs of type C and are explained in the next section.

5.5. Remaining G-graphs: types C and D. For a given qG-graph $\widetilde{\Gamma}$, the corresponding G-graph Γ was constructed in previous sections by adding some suitable elements to $\widetilde{\Gamma}$. This procedure gives almost all possible G-graphs. Indeed, the involution of the middle rational curve $E_h \subset H$-Hilb$(\mathbb{C}^2)$ to itself by β gives two isolated fixed points, and therefore two more rational curves E^+ and E^- in the exceptional locus for

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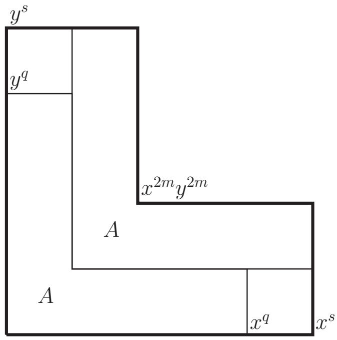
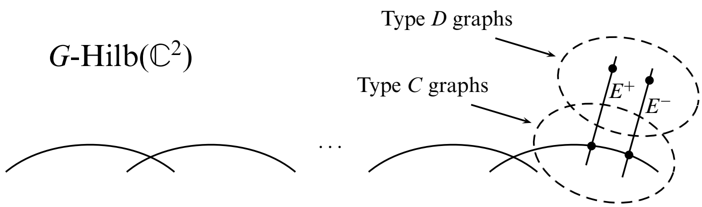


Figure 10. The last G-graph of type B.

the resolution of $\mathbb{C}^2/G$ (see the following diagram). This part of the exceptional locus cannot be recovered with qG-graphs.



In this section we construct the G-graphs that correspond to the neighbourhood of E^+ and E^-. These are called type C and type D, and each of them have two cases, C^+ and C^- (D^+ and D^-, respectively). As mentioned in Section 5.1, these G-graphs arise from the β-fixed points in H-Hilb($\mathbb{C}^2$). Since these fixed points are contained in the middle rational curve E_h, the new G-graphs must contain the qG-graph formed by the union of the two middle H-graphs Γ_{h-1} and Γ_h. Thus, to construct G-graphs of types C and D we start from the data given by $\widetilde{\Gamma}_{h-1}(r, s; q, q)$.

Let $e_h = \frac{1}{2n}(r, s)$ and $e_{h+1} = \frac{1}{2n}(q, q)$ be the lattice points giving the last qG-graph $\widetilde{\Gamma}_{h-1}(r, s; q, q)$, and define

$$m_1 := s - q \quad \text{and} \quad m_2 := q - r.$$

Note that if the last qG-graph is of type B, then $m_1 = m_2$, and if it is of type A, then $m_1 > m_2$.

Remark 5.15. The monomial $x^{s-r}y^{s-r}$ is G-invariant, and therefore is in I. Indeed, $\alpha(x^{s-r}y^{s-r}) = \beta(x^{s-r}y^{s-r}) = x^{s-r}y^{s-r}$ since r and s have the same parity, and $ar \equiv s$ and $as \equiv r \pmod{2n}$. Similarly, the pairs of monomials $x^{s-r+jq}y^{s-r-jq} + (-1)^{jq}x^{s-r-jq}y^{s-r+jq}$ are also G-invariant for every j.

The new G-graphs depend on a choice of basis between the following polynomials in irreducible representations ρ_q^+ and ρ_q^-:

$$x^q + (-i)^q y^q, \quad x^s y^{m_2} + (-1)^r i^q x^{m_2} y^s \in \rho_q^+$$
$$x^q - (-i)^q y^q, \quad x^s y^{m_2} - (-1)^r i^q x^{m_2} y^s \in \rho_q^-$$

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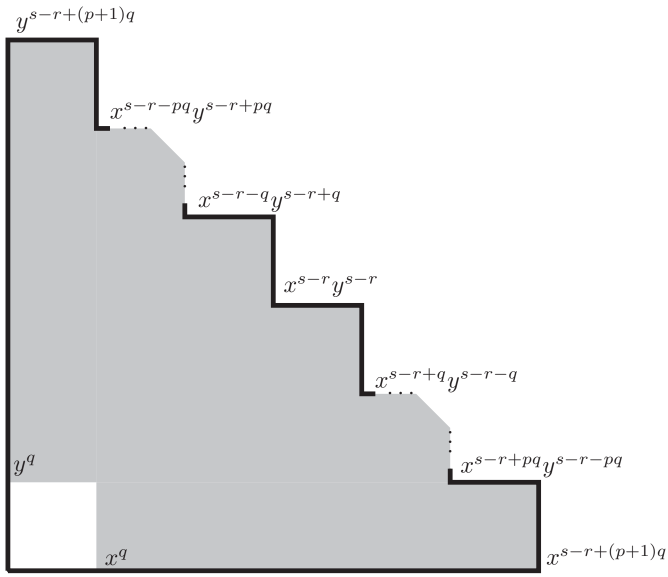

Figure 11. *G*-graph of type *D*. The monomials x^q and y^q are identified.

The choice of basis from these polynomials is sufficient, since they generate the S^G-modules $S_{\rho_q^\pm}$. In fact, $\rho_q^\pm$ are the *special* irreducible representations corresponding to curves E^+ and E^- (compare with [**7**, Theorems 6.2 and 10.1], together with Section 7).

Note that at least one of the polynomials $x^q + (-i)^q y^q$ and $x^q - (-i)^q y^q$ must be in the basis Γ of $\mathbb{C}[x, y]/I$. Indeed, if $x^q + (-i)^q y^q, x^q - (-i)^q y^q \in I$, then also $x^q, y^q \in I$, which is a contradiction as they are elements of $\widetilde{\Gamma}_{h-1}$. Thus, we have the following three possibilities: $x^q + (-i)^q y^q \in I$, $x^q - (-i)^q y^q \in I$ and both $x^q \pm (-i)^q y^q \in \Gamma$.

***G*-graphs of type *D*.** Suppose that $x^q + (-i)^q y^q \in I$. Then $x^s y^{m_2} + (-1)^r i^q x^{m_2} y^s$ must be in the *G*-graph Γ, and the basis polynomials in Γ are $x^q - (-i)^q y^q$ and $x^s y^{m_2} + (-1)^r i^q x^{m_2} y^s$. Similarly, we can choose $x^q - (-i)^q y^q \in I$, which now implies $x^q + (-i)^q y^q, x^s y^{m_2} - (-1)^r i^q x^{m_2} y^s \in \Gamma$. The first case corresponds to type D^+ and the second to type D^-.

The assumption $x^q + (-i)^q y^q \in I$ (or analogously, $x^q - (-i)^q y^q \in I$ for case D^-) identifies monomials x^q with y^q as twin elements in $\mathbb{C}[x, y]/I$, and together with $x^{s-r} y^{s-r} \in I$ characterise completely the shape of the *G*-graph Γ (see Figure 11). This gives the following proposition.

PROPOSITION 5.16. *The ideals* $I_{D^\pm} = \langle x^q \pm (-i)^q y^q, x^{s-r} y^{s-r} \rangle$ *define G-clusters. The G-graphs represented by* $I_{D^\pm}$ *are said to be of type* $D^\pm$.

Proof. We give proof for case D^-, case D^+ is almost identical. We begin by identifying a collection of monomials $Q_\mathbb{Z}$ that will prove useful. Recall that M denotes the lattice of Laurent monomials in variables x and y. Let $P \subset M \otimes \mathbb{R}$ denote the parallelogram whose vertices include the monomials $(0, 0)$, $(\frac{s-r}{2}, \frac{s-r}{2})$ and $(q, 0)$, and let P^* denotes its reflection through the line $(x = y) \subset M \otimes \mathbb{R}$. Define $Q_\mathbb{Z}$ to be a set

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of monomials in the interior of $P \cup P^*$ together with monomials

$$1, x, x^2, \dots, x^q, y, y^2, \dots, y^{q-1}, x^q(xy), \dots, x^q(xy)^{\frac{s-r}{2}-1},$$

that is $(P \cup P^*) \cap M$ with the appropriate boundary monomials removed.

Note that translating $Q_{\mathbb{Z}}$ by integer combinations of vectors $(\frac{s-r}{2}, \frac{s-r}{2})$ and $(q, -q)$ tessellates M. Also note that the Laurent monomials $(xy)^{\frac{s-r}{2}}$ and $x^q y^{-q}$ are H-invariants. So for every $\rho \in \operatorname{Irr} H$ there exists a monomial in $Q_{\mathbb{Z}}$ that belongs to ρ. Since (r, s) and (q, q) define an H-cluster, $q(r-s) = \operatorname{Area}(P \cup P^*) = 2n$. Therefore, every $\rho \in \operatorname{Irr} H$ contains exactly one monomial $\mathbf{x}^\rho$ from $Q_{\mathbb{Z}}$.

Now we use $Q_{\mathbb{Z}}$ and $(\frac{s-r}{2}, \frac{s-r}{2}) + Q_{\mathbb{Z}}$ to give a polynomial in every irreducible representation of G. Given $\rho_j \in \operatorname{Irr} H$ for $j \not\equiv aj \pmod{2n}$, take the pairs

$$(\mathbf{x}^{\rho_j}, \beta(\mathbf{x}^{\rho_j})), ((xy)^{\frac{s-r}{2}}\mathbf{x}^{\rho_j}, (xy)^{\frac{s-r}{2}}\beta(\mathbf{x}^{\rho_j})) \in V_j.$$

Take ρ_j such that $j \equiv aj \pmod{2n}$. Since $q := \frac{2n}{(a-1,2n)}$, monomials $\mathbf{x}^{\rho_j}$ are either of the form $x^q(xy)^l$ or $(xy)^l$, for some $j \in \mathbb{N}$. If $\mathbf{x}^{\rho_j} = (xy)^l$ take

$$\begin{aligned}(xy)^l &\in \rho_j^{(-1)^l}\\ (xy)^{l+\frac{s-r}{2}} &\in \rho_j^{(-1)^{l+1}}.\end{aligned}$$

If $\mathbf{x}^{\rho_j} = x^q(xy)^l$ and n and q are even, then take

$$\begin{aligned}x^q(xy)^l + (-1)^l\beta(x^q(xy)^l) &\in \rho_j^{(-1)^l}\\ x^q(xy)^{l+\frac{s-r}{2}} + (-1)^{l+1}\beta(x^q(xy)^{l+\frac{s-r}{2}}) &\in \rho_j^{(-1)^{l+1}}.\end{aligned}$$

If n and q are odd, multiply the second summand by i as in equation (5.2.1). Here we are using the fact that we are in the case D^- and the assumption that the group is small, so $\frac{s-r}{2} = k$ is odd.

Observing that monomials in $Q_{\mathbb{Z}} \cup ((\frac{s-r}{2}, \frac{s-r}{2}) + Q_{\mathbb{Z}})$ form a basis of the vector space $\mathbb{C}[x, y]/I_{D^+}$ completes the proof. □

G-graphs of type C. Now assume $x^q + (-i)^q y^q$ and $x^q - (-i)^q y^q$ are basis elements for ρ_q^+ and ρ_q^-, i.e. $x^s y^{m_2} \pm (-1)^r i^q x^{m_2} y^s \in I$, which implies that $x^s y^{m_2}, x^{m_2} y^s \in I$.

The pairs (x^r, y^r), $(y^s, (-1)^s x^s)$, $(x^q y^{m_1}, (-1)^{m_1} x^{m_1} y^q) \in V_r$ (omitting the corresponding signs), and note that all monomials in them must belong to the basis (otherwise the G-graph would have less than $|G|$ elements). Therefore, there must be an identification between pairs of the same degree. We take as basic elements (x^r, y^r) and a linear combination of $(y^s, (-1)^s x^s)$ and $(x^q y^{m_1}, (-1)^{m_1} x^{m_1} y^q)$. We have two possibilities:

$$\begin{aligned}&\textbf{Case } C^+ : y^{m_1}(x^q + (-i)^q y^q), x^{m_1}(x^q + (-i)^q y^q) \in I \text{ and}\\ &\qquad (x^r, y^r), (y^{m_1}(x^q - (-i)^q y^q), x^{m_1}(x^q - (-i)^q y^q)) \in \Gamma,\\ &\textbf{Case } C^- : y^{m_1}(x^q - (-i)^q y^q), x^{m_1}(x^q - (-i)^q y^q) \in I \text{ and}\\ &\qquad (x^r, y^r), (y^{m_1}(x^q + (-i)^q y^q), x^{m_1}(x^q + (-i)^q y^q)) \in \Gamma.\end{aligned}$$

Also, note that $x^{2q} + (-1)^q y^{2q}, x^q y^q \in \rho_{2q}^+$, while we need only one element in Γ corresponding to ρ_{2q}^+. On the other hand, both of these must belong to Γ, otherwise

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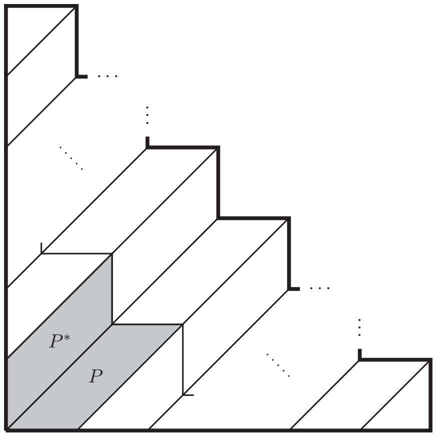

Figure 12. Parallelograms P and P^* tessellating $M \otimes \mathbb{R}$.

Γ does not contain $|G|$ elements. This implies that we have to take a combination of these as our basis for ρ^+_{2q}. We take

$$(x^q + (-i)^q y^q)^2 \in I \text{ for type } C^+, \text{ and}$$
$$(x^q - (-i)^q y^q)^2 \in I \text{ for type } C^-.$$

In both cases if the last qG-graph Γ_{h-1} is of type B, then this last equation is redundant, so it is only needed in the type A case. Indeed, substituting the value of $s = 2q - r$ into the equations for case C^+, we get $(-i)^q y^{2q-r} + x^q y^{q-r}$, $x^{2q-r} + (-i)^q x^{q-r} y^q \in I$. Now multiplying the first polynomial by $(-i)^q y^r$ and the second one by x^r, and adding these together we get $x^{2q} + (-1)^q y^{2q} + 2(-i)^q x^q y^q$ as desired.

When the last G-graph Γ_{h-1} is of type A, we do need the equation $x^{2q} + (-1)^q y^{2q} \pm 2(-i)^q x^q y^q$ together with the G-invariant $x^{m_1} y^{m_2} + (-1)^{m_2} x^{m_2} y^{m_1}$. In this case $x^{s-r} y^{s-r}$ can be obtained using the remaining identities. In other words, $x^{s-r} y^{s-r} \in \langle x^{2q} + (-1)^q y^{2q} + 2(-i)^q x^q y^q, x^s y^{m_2} - (-i)^q x^{m_2} y^s, x^{m_1} y^{m_2} + (-1)^{m_2} x^{m_2} y^{m_1} \rangle$. Indeed, using the second and third generators of the ideal we have

$$x^{s-r-q} y^{s-r+q} = (-i)^q x^{2(s-q)} y^{2(q-r)} = -(-1)^{q-r} (-i)^q x^{r-s} y^{r-s}, \text{ and}$$
$$x^{s-r+q} y^{s-r-q} = (-i)^q x^{2(q-r)} y^{2(s-q)} = -(-1)^{q-r} (-i)^q x^{r-s} y^{r-s}.$$

Multiplying the first generator by $x^{s-r-q} y^{s-r-q}$, we get the polynomial $-(-1)^{q-r} x^{s-r} y^{s-r} - (-1)^r x^{s-r} y^{s-r} + 2x^{s-r} y^{s-r}$ (note that $s < 2(s-q)$). The only possibility for this to be identically zero is only if both q and r are even at the same time. But this is impossible because it would imply that every boundary lattice point in the lattice L is even, a contradiction. Therefore, the sum above is not identically zero, which implies that $x^{s-r} y^{s-r} \in I$, and we obtain the following proposition.

Proposition 5.17. *The ideals*

$$I_{C^\pm_A} = \langle (x^q \pm (-i)^q y^q)^2, x^s y^{m_2} \pm (-1)^r i^q x^{m_2} y^s, x^{m_1} y^{m_2} + (-1)^{m_2} x^{m_2} y^{m_1} \rangle$$

if Γ_{h-1} is of type A, *or*

$$I_{C^\pm_B} = \langle y^m (x^q \pm (-i)^q y^q), x^m (x^q \pm (-i)^q y^q), x^{s-r} y^{s-r}, x^s y^m, x^m y^s \rangle \text{ if } \Gamma_{h-1} \text{ is of type } B.$$

define G-clusters. The G-graphs represented by $I_{C^\pm}$ are said to be of type $C^\pm$.

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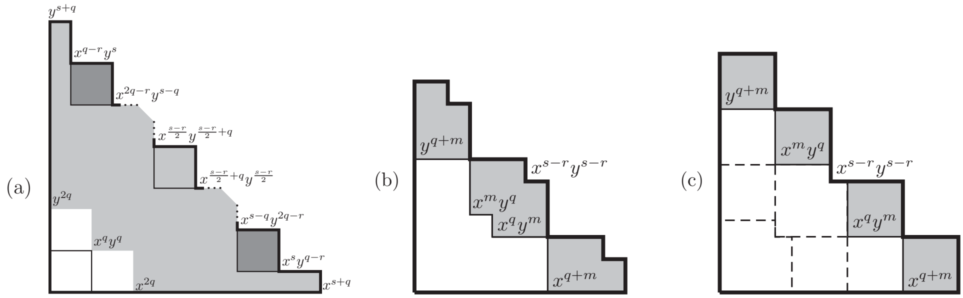

Figure 13. G-graph of type C when Γ_{h-1} is of (a) type A, (b) type B_1 and (c) type B_2.

The shape of the G-graph is the same in both cases, and as in the type D case, it has a stair shape. In this case, the conditions $x^s y^{m_2}, x^{m_2} y^s \in I$ make the stair smaller than in type D. See Figure 13.

Proof. We treat the case when Γ_{h-1} is of type A, and the type B case is similar. Consider parallelograms P and P^*, and a set of monomials $Q_{\mathbb{Z}}$ tessellating $M \otimes \mathbb{R}$ (see Figure 12) as in the proof of Proposition 5.16. Denote by $L_0 : (x = y) \subset M \otimes \mathbb{R}$ the diagonal, and by L_c the parallel line to L_0 passing through the point $(cq, 0)$ where $c \in \mathbb{Z}$. Let also $v_c \in \operatorname{Aut}(M \otimes \mathbb{R})$ be the reflection through line L_c. Note that $P^* = v_0(P)$.

Let C_0 be the square in $M \otimes \mathbb{R}$ of vertices $(0, 0)$, $(q - 1, 0)$, $(q - 1, q - 1)$ and $(0, q - 1)$, and define C to be the square $(\frac{s-r}{2}, \frac{s-r}{2}) + C_0$. Note that reflecting the polygon $P \cup P^* \cup C$ along the line L_c and taking only monomials in the positive quadrant we obtain the representation of Γ (see Figure 14).

Consider the region $\mathcal{R} := (P \cup P^* \cup v_1(P) \cup v_{-1}(P^*)) \cap \square$, where $\square$ denotes the positive quadrant, and define $\widetilde{Q}_{\mathbb{Z}}$ to be a set of monomials in the interior of $\mathcal{R}$ together with every monomial in C and

$$1, x, \ldots, x^{2q}, y, \ldots, y^{2q-1}, x^{2q}(xy), \ldots, x^{2q}(xy)^{\frac{s-r}{2}-q-1}.$$

As in the proof of Proposition 5.16, we have that $\widetilde{Q}_{\mathbb{Z}}$ tessellates M by translating it this time with vectors $(2q, -2q)$ and $(\frac{s-r}{2}, \frac{s-r}{2})$. Also, note that monomials $x^{2q}y^{-2q}$ and $x^{\frac{s-r}{2}}y^{\frac{s-r}{2}}$ are H-invariants. Moreover, since $Q_{\mathbb{Z}}$ contains one monomial $\mathbf{x}^{\rho}$ in each $\rho \in \operatorname{Irr} H$ and $Q_{\mathbb{Z}}$ tessellates M, then $\widetilde{Q}_{\mathbb{Z}}$ contains exactly two monomials $\mathbf{x}^{\rho}$ and $\mathbf{y}^{\rho}$ in each $\rho \in \operatorname{Irr} H$.

Now using $\widetilde{Q}_{\mathbb{Z}}$ we produce the correct number of polynomials in $\operatorname{Irr} G$. We consider the case C^-, the case C^+ is almost identical. Let $\rho_j \in \operatorname{Irr} H$ and suppose that $j \equiv aj \pmod{2n}$. Then we take

$$(\mathbf{x}^{\rho_j}, \beta(\mathbf{x}^{\rho_j})), (\mathbf{y}^{\rho_j}, \beta(\mathbf{y}^{\rho_j})) \in V_j.$$

Note that if $\mathbf{x}^{\rho_j} \in C_0$, then $\mathbf{y}^{\rho_j} = (xy)^{\frac{s-r}{2}}\mathbf{x}^{\rho_j}$, and if $\mathbf{x}^{\rho_j} \notin C_0 \cup C$, then either $\mathbf{x}^{\rho_j} \in P$ and $\mathbf{y}^{\rho_j} \in v_{-1}(P^*)$, or $\mathbf{x}^{\rho_j} \in P^*$ and $\mathbf{y}^{\rho_j} \in v_1(P)$.

If $j \not\equiv aj \pmod{2n}$, then $\mathbf{x}^{\rho_j}$ is either of the form $(xy)^l$, $x^q(xy)^l$ or $x^{2q}(xy)^l$ for some $l \in \mathbb{N}$. If $\mathbf{x}^{\rho_j} \in C_0$, then $\mathbf{x}^{\rho_j} = (xy)^l$, so take

$$(xy)^l \in \rho_j^{(-1)^l}$$
$$(xy)^{l+\frac{s-r}{2}} \in \rho_j^{(-1)^{+1}l}$$

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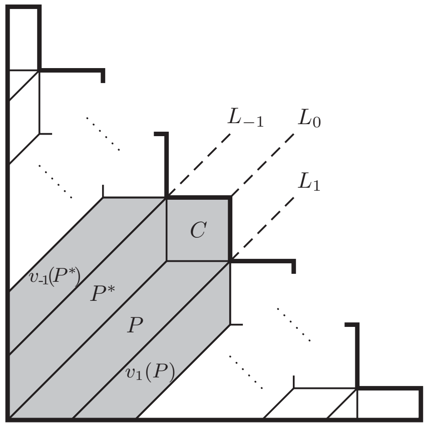

Figure 14. Region $\mathcal{R} \cup C$ containing the set of monomials $\widetilde{Q}_{\mathbb{Z}}$.

If $\mathbf{x}^{\rho_j} \notin C_0$, then we have the following two cases:

(i) $\mathbf{x}^{\rho_j} = (xy)^l$ and $\mathbf{y}^{\rho_j} = x^{2q}(xy)^{l-q}$. Then take

$$(xy)^l \in \rho_j^{(-1)^l}$$
$$(xy)^{l-q}(x^{2q} - (-1)^q \beta(x^{2q})) \in \rho_j^{(-1)^{+1}l}$$

(ii) $\mathbf{x}^{\rho_j} = x^q(xy)^l$ and $\mathbf{y}^{\rho_j} = \beta(\mathbf{x}^{\rho_j})$, in which case we take the same elements as in case D, which are

$$x^q(xy)^l + (-1)^l \beta(x^q(xy)^l) \in \rho_j^{(-1)^l}$$
$$x^q(xy)^{l+\frac{s-r}{2}} + (-1)^{l+1}\beta(x^q(xy)^{l+\frac{s-r}{2}}) \in \rho_j^{(-1)^{l+1}}$$

if n and q are even, and multiply the second summand by i as in equation (5.2.1) if n and q are odd.

Observing that the monomials in $\widetilde{Q}_{\mathbb{Z}}$ form a basis of the vector space $\mathbb{C}[x, y]/I_{C^-}$ completes the proof. □

Example 5.18. Consider the group $\mathrm{BD}_{42}(13)$. We have $\frac{42}{13} = [4, 2, 2, 2, 4]$ and the lattice points in the Newton polygon that we need to consider are $e_0 = \frac{1}{42}(0, 42)$, $e_1 = \frac{1}{42}(1, 13)$, $e_2 = \frac{1}{42}(4, 10)$ and $e_3 = \frac{1}{42}(7, 7)$. Therefore, we have seven distinguished $\mathrm{BD}_{42}(13)$-graphs as shown in Figure 15 together with their corresponding ideals.

6. Walking along the exceptional divisor. In this section we prove that every G-cluster in G-Hilb($\mathbb{C}^2$) corresponds to an ideal with G-graph of either type A, B, C or D. We start by giving a one-parameter family of ideals, which connects any two consecutive G-graphs, obtaining every G-cluster at the exceptional divisor $E \subset G$-Hilb($\mathbb{C}^2$) by passing through all G-graphs for a given $\mathrm{BD}_{2n}(a)$ group.

Theorem 6.1. *Let $G = \mathrm{BD}_{2n}(a)$ and let $\Gamma_0, \Gamma_1, \ldots, \Gamma_{h-1}, C^+, C^-, D^+, D^-$ be the sequence of G-graphs with $I_{\Gamma_0}, I_{\Gamma_1}, \ldots, I_{\Gamma_{h-1}}, I_{C^+}, I_{C^-}, I_{D^+}, I_{D^-}$ the corresponding defining ideals. Then,*

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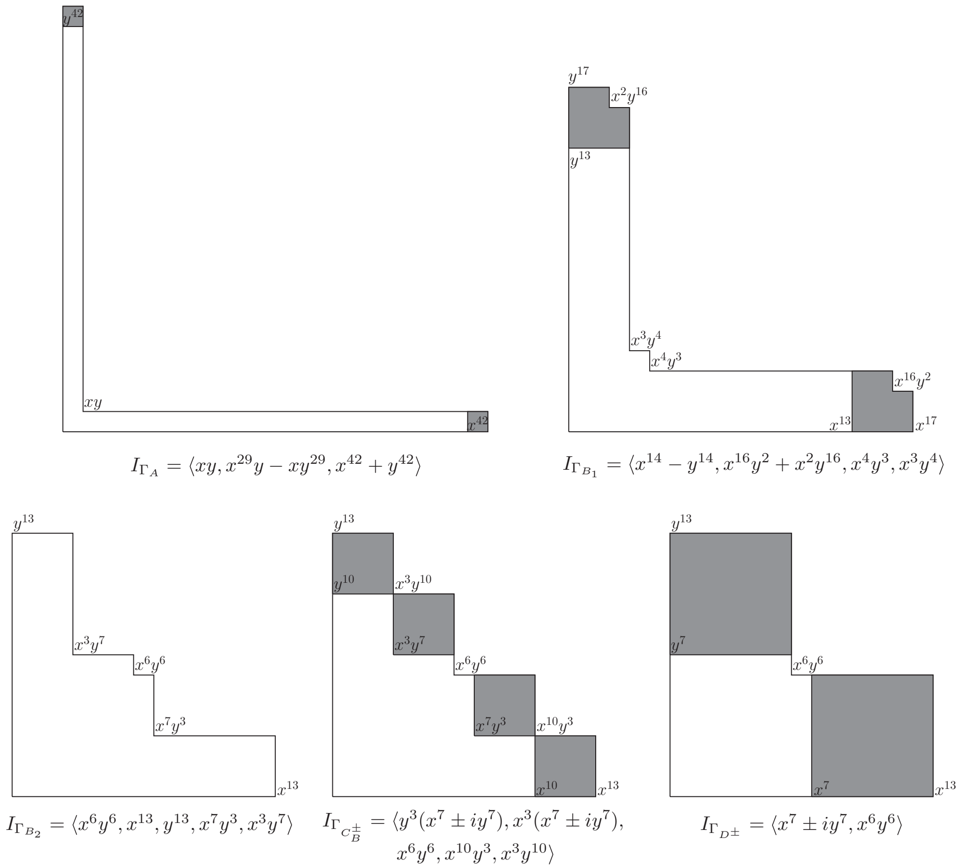

Figure 15. G-graphs for the group $\mathrm{BD}_{42}(13)$. The shaded areas represent twin regions.

(i) *for any two consecutive G-graphs, Γ_i and Γ_{i+1}, there exists a family of ideals $J_{(\xi_i,\eta_i)}$ with $(\xi_i : \eta_i) \in \mathbb{P}^1$ such that $J_{(0:1)} = I_{\Gamma_i}$, $J_{(1:0)} = I_{\Gamma_{i+1}}$ and $J_{(\xi_i,\eta_i)}$ defines a G-cluster;*

(ii) *there exists a family of ideals $J^+_{(\gamma_+,\delta_+)}$ (respectively $J^-_{(\gamma_-,\delta_-)}$) with $(\gamma_+, \delta_+) \in \mathbb{P}^1$ such that $J^+_{(0:1)} = I_{C^+}$, $J^+_{(1:0)} = I_{D^+}$ and $J^+_{(\gamma_+,\delta_+)}$ defines a G-cluster (similarly for $J^-_{(\gamma_-,\delta_-)}$);*

(iii) *there exists a family of ideals $J^C_{(\tau,\mu)}$ with $(\tau, \mu) \in \mathbb{P}^1$ such that $J^C_{(0:1)} = I_{C^+}$, $J^C_{(1:0)} = I_{C^-}$, $J^C_{(1:1)} = I_{\Gamma_{h-1}}$ and $J^C_{(\tau,\mu)}$ defines a G-cluster.*

Proof. Let $\Gamma_1 := \Gamma(r, s; u, v)$ and $\Gamma_2 := \Gamma(u, v; t, w)$ be two consecutive G-graphs with I_{Γ_1} and I_{Γ_2} ideals representing Γ_1 and Γ_2 from Propositions 5.9, 5.13, 5.14, 5.16 and 5.17. The ideals I_{Γ_1} and I_{Γ_2} define G-clusters corresponding to the intersection points of two consecutive exceptional curves in G-Hilb$(\mathbb{C}^2)$ (see Remark 5.5).

Depending on types Γ_1 and Γ_2, there are eight possible cases. In every case we give a family of ideals $J_{(a:b)}$ parametrized by $\mathbb{P}^1$ such that $J_{(1:0)} = I_{\Gamma_1}$, $J_{(0:1)} = I_{\Gamma_2}$, and for any $(a : b) \in \mathbb{P}^1$ the ideal $J_{(a:b)}$ defines a G-cluster. In other words, we give by definition a family of G-clusters parametrized by $\mathbb{P}^1_{(a:b)}$ connecting Z_1 and Z_2. Since G-Hilb$(\mathbb{C}^2)$ is a fine moduli space and it is a minimal resolution, the target of map $\mathbb{P}^1 \to G$-Hilb$(\mathbb{C}^2)$

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is contained in exceptional divisor $E \subset G\text{-Hilb}(\mathbb{C}^2)$. Therefore, by considering all G-graphs for a given group G we cover the whole of the exceptional divisor E in this way.

Note that every generator of I_{Γ_1} is contained in a representation of subset $\{\rho, \rho', \rho_0\} \subset \operatorname{Irr} G$, and every generator of I_{Γ_2} is contained in a representation of subset $\{\rho, \rho'', \rho_0\} \subset \operatorname{Irr} G$, for some $\rho, \rho', \rho'' \in \operatorname{Irr} G$ and ρ_0 denotes trivial representation.

Let $f_\rho, g_\rho \in \rho$ be the given generators of I_{Γ_1} and I_{Γ_2}, respectively. Note that $f_\rho \notin I_{\Gamma_2}$ and $g_\rho \notin I_{\Gamma_1}$. The proposed family $J_{(a:b)}$ is given by the union of the two sets of generators with f_ρ and g_ρ replaced by $\sigma_\rho := af_\rho - bg_\rho$, for $a, b \in \mathbb{C}$. The subset of irreducible representations of G involved in $J_{(a:b)}$ is therefore $\{\rho, \rho', \rho'', \rho_0\} \subset \operatorname{Irr} G$.

We now prove the theorem case by case using the following argument. Let us denote by $I_{\Gamma_1} := \langle f_\rho, f_{\rho'}, f_{\rho_0} \rangle$ and $I_{\Gamma_2} := \langle g_\rho, g_{\rho''}, g_{\rho_0} \rangle$ the ideals defining Γ_1 and Γ_2. Since we can always choose $1 \in \rho_0$ to be in the G-graph, note that $f_{\rho_0} \in I_{\Gamma_2}$ and $g_{\rho_0} \in I_{\Gamma_1}$. Thus, in the case $a = 0$, to show that $J_{(0:1)} = I_{\Gamma_2}$ it suffices to show that $f_{\rho'} \in I_{\Gamma_2}$. Similarly when $b = 0$, to show that $J_{(1:0)} = I_{\Gamma_1}$ it suffices to show that $g_{\rho''} \in I_{\Gamma_1}$. For the rest of the values if, for instance, we take $a \neq 0$ (or equivalently $a = 1$), we show that $\mathbb{C}[x, y]/J_{(1:b)}$ admits Γ_1 as basis for any value of $b \in \mathbb{C}$. Since $f_\rho \equiv bg_\rho \bmod J_{(1:b)}$, we can take $g_\rho \in \Gamma_1$ in the basis of $\mathbb{C}[x, y]/J_{(1:b)}$, and it follows that every polynomial in Γ_1 is not in $J_{(1:b)}$. It remains to show that every polynomial not contained in the representation of Γ_1 can we written in terms of elements in Γ_1 modulo $J_{(1:b)}$. The same argument applies if we instead take $b = 1$.

Case 1. $\Gamma_1 := \Gamma_A(r, s; u, v) \to \Gamma_2 := \Gamma_A(u, v; t, w)$: Then the family of ideals $J_{(a:b)}$ given by polynomials

$$\begin{aligned} &\sigma_\rho := ax^u y^u - b(x^{u+v} + (-1)^u y^{u+v}), \\ &f_{\rho'} := x^{r+s} + (-1)^r y^{r+s}, \qquad g_{\rho''} := x^t y^t, \\ &f_{\rho_0} := x^{s-v} y^{u-r} + (-1)^{u-r} x^{u-r} y^{s-v}, \qquad g_{\rho_0} := x^{v-w} y^{t-u} + (-1)^{t-u} x^{t-u} y^{v-w}, \end{aligned}$$

which defines a 1-parameter family of G-clusters parametrised by a $\mathbb{P}^1$ with coordinates a and b. Recall that we have $r < u < t \leq w < v < s$ together with the type A conditions $u < s - v$ and $t < v - w$.

Let $a = 0$. Then by the proof of Lemma 5.8 we have $x^{t+v}, y^{t+v} \in I_{\Gamma_2}$, which implies that $x^{r+s}, y^{r+s} \in I_{\Gamma_2}$ since $t + v < r + s$. Thus, $J_{(0:1)} = I_{\Gamma_2}$. When $b = 0$, since $u < t$, we have $g_{\rho''} \in I_{\Gamma_1}$, thus $J_{(1:0)} = I_{\Gamma_1}$.

If $a = 1$, we show that $\mathbb{C}[x, y]/J_{(1:b)}$ admits Γ_1 as basis for any value of $b \in \mathbb{C}$. To show that every polynomial contained in some representation ρ but not in Γ_1 can be written in terms of elements in Γ_1 modulo $J_{(1:b)}$, it is enough to check it for monomials $x^j y^u, x^u y^j, x^l y^{u-r}, x^{u-r} y^l$ and x^{s+u}, y^{s+u} with $u \leq j < r + s - v$ and $r + s - v \leq l < s + u$ (recall Figure 6).

Indeed, first note that combining f_{ρ_0} and $g_{\rho''}$ we see that $x^{s-v+t} y^{u-r}, x^{u-r} y^{s-v+t} \in I_{(1:b)}$. In addition, using $f_{\rho'}$ we have $x^{s+u}, y^{s+u} \in I_{(1:b)}$. For monomials $x^l y^{u-r}$, by the use of f_{ρ_0} it suffices to show that $x^u y^{s-v}$ can be written in terms of elements in Γ_1 (similarly for $x^{u-r} y^l$). In this case

$$x^u y^{s-v} \equiv -b(x^{u+v} y^{s-u-v} + (-1)^u y^s) \equiv b(bx^{u+2v} y^{s-2u-v} - (-1)^u y^s) \bmod J_{(1:b)}.$$

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By induction, there exists $c > 0$ such that $u + cv \geq s - v + t$, thus $x^{u+2v}y^{s-2u-v}$ is either in $J_{(1:b)}$ or in Γ_1 mod $J_{(1:b)}$.

In the case of monomials $x^j y^u$, by the use of σ_ρ it suffices to show that $x^u y^{u+v}$ can be written in terms of elements in Γ_1 (similarly for $x^u y^j$). In this case $x^u y^{u+v} \equiv -(-1)^u b y^{u+2v}$, which is either in Γ_1 if $2v < s$, or in $J_{(1:b)}$ otherwise, and we are done.

CASE 2. $\Gamma_1 := \Gamma_A(r, s; u, v) \to \Gamma_2 := \Gamma_{B_1}(u, v; t, w)$: In this case the family of G-clusters is defined by the ideals $J_{(a:b)}$ generated by polynomials

$$\begin{aligned} &\sigma_\rho := ax^u y^u - b(x^{u+v} + (-1)^u y^{u+v}), && g_{1,\rho''} := x^t y^m, && f_{\rho_0} := x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v},\\ &f_{\rho'} := x^{r+s} + (-1)y^{r+s}, && g_{2,\rho''} := x^m y^t, && g_{\rho_0} := x^{m+v}y^{m-u} + (-1)^{m-u} x^{m-u}y^{m+v}. \end{aligned}$$

Note that $\rho'' \in \operatorname{Irr} G$ is 2-dimensional, so I_{Γ_2} contains two polynomials $g_{1,\rho''}$ and $g_{2,\rho''}$ in its set of generators. Recall that in this case $u < s - v$ and $m = v - w = t - u$.

As in the previous case, if $a = 0$ then $x^{t+v}, y^{t+v} \in I_{\Gamma_2}$ and it follows that $f_{\rho'} \in I_{\Gamma_2}$, and when $b = 0$, the fact that $u < m$ gives us $g_{1,\rho''}, g_{2,\rho''} \in I_{\Gamma_1}$. Then $J_{(0:1)} = I_{\Gamma_2}$ and $J_{(1:0)} = I_{\Gamma_1}$.

The case $a = 1$ is identical to Case 1 since $m < t$ so that this time $g_{1,\rho''}, g_{2,\rho''}$ impose stronger conditions.

CASE 3. $\Gamma_1 := \Gamma_{B_1}(r, s; u, v) \to \Gamma_2 := \Gamma_{B_2}(u, v; t, w)$: In this case the family of ideals $J_{(a:b)}$ is given by:

$$\begin{aligned} &\sigma_{1,\rho} := ax^u y^m - by^s, && g_{1,\rho''} := x^t y^m, && f_{\rho_0} := x^{m+s}y^{m-r} + (-1)^{m-r}x^{m-r}y^{m+s},\\ &\sigma_{2,\rho} := ax^m y^u + (-1)^v bx^s, && g_{2,\rho''} := x^m y^t, && g_{\rho_0} := x^{2m}y^{2m},\\ &f_{\rho'} := x^{r+s} + (-1)y^{r+s}. \end{aligned}$$

In this case ρ and ρ'' are 2-dimensional, and for inequalities we have $m = s - v = u - r = v - w = t - u$ and $u < 2m \leq t$. If $a = 0$ then clearly $f_{\rho'} \in I_{\Gamma_2}$, and if $b = 0$ then $g_{1,\rho''}, g_{2,\rho''} \in I_{\Gamma_1}$, as desired.

For the rest of the points we consider this time the case $b = 1$, and show that we can always take Γ_2 as basis for $\mathbb{C}[x, y]/J_{(a:1)}$. Since $x^t y^m, x^m y^t \in J_{(a:1)}$, it is enough to check that $x^s y^j$, $x^j y^s$ for $0 \leq j < m$ can be written in terms of elements in the representation of Γ_2 (recall Figure 9). But this is obvious since $x^s \equiv (-1)^v ax^m y^u$ and $y^s \equiv ax^u y^m$ mod $J_{(a:1)}$, thus $x^s y^j \equiv (-1)^v ax^m y^{u+j}$ mod $J_{(a:1)}$ for $1 \leq j < m$, where $x^m y^{u+j}$ is contained in the representation of Γ_2 (similarly for $x^s y^j$ for $1 \leq j < m$), and we are done.

CASE 4. $\Gamma_1 := \Gamma_{B_2}(r, s; u, v) \to \Gamma_2 := \Gamma_{B_2}(u, v; t, w)$: The family of ideals $J_{(a:b)}$ is given by generators

$$\begin{aligned} &\sigma_{1,\rho} := ax^u y^m - by^s, && f_{1,\rho'} := y^{s+m}, && g_{1,\rho''} := x^t y^m, && f_{\rho_0} := x^{2m}y^{2m},\\ &\sigma_{2,\rho} := ax^m y^u + (-1)^v bx^s, && f_{2,\rho'} := x^{s+m}, && g_{2,\rho''} := x^m y^t. \end{aligned}$$

In this case, ρ, ρ' and ρ'' are 2-dimensional, $f_{\rho_0} = g_{\rho_0}$ and m is as in the previous case but $u, t \geq 2m$. It is immediate to see in this case that if $a = 0$ then $f_{1,\rho'}, f_{2,\rho'} \in I_{\Gamma_2}$, and if $b = 0$ then $g_{1,\rho''}, g_{2,\rho''} \in I_{\Gamma_1}$, as desired. The case $b = 1$ is analogous to Case 3.

From now on suppose that the last G-graph is $\Gamma_{h-1} := \Gamma(r, s; q, q)$. Depending on the type of Γ_{h-1}, we have a different G-graph of type C, which we will denote by a subindex A or B.

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CASE 5. $\Gamma_1 := \Gamma_{C_A^-}(r,s;q,q) \to \Gamma_2 := \Gamma_{C_A^+}(r,s;q,q)$: In this case the family of ideals $J_{(a:b)}$ is given by generators

$$\sigma_\rho := a(x^q - (-i)^q y^q)^2 - b(x^q + (-i)^q y^q)^2, \quad g_{\rho''} := x^s y^{m_2} + (-1)^r i^q x^{m_2} y^s,$$
$$f_{\rho'} := x^s y^{m_2} - (-1)^r i^q x^{m_2} y^s, \quad f_{\rho_0} := x^{m_1} y^{m_2} + (-1)^{m_2} x^{m_2} y^{m_1}.$$

Note that $f_{\rho_0} = g_{\rho_0}$, and recall that $r < q < s$ and $b_h q = r + s$.

Suppose that $a = 0$ so that $\sigma_\rho = x^{2q} + 2(-i)^q x^q y^q + (-1)^q y^{2q}$. We show that $f_{\rho'} \in I_{\Gamma_2}$. Define polynomials $F_j := x^{s-jq} y^{(j+1)q-r} + c x^{(j+1)q-r} y^{s-jq}$ for $j = 0, \dots, b_h - 1$ and $c := -(-1)^{r+jq} i^q$. Note that $f_{\rho'} = F_0$ (recall Figure 13 (a)). Note that $F_j + 2(-i)^q F_{j+1} + (-1)^q F_{j+2} \in \langle \sigma_\rho \rangle$ and F_j and F_{b_h-1-j} are same up to a constant. Then $f_{\rho'} \in I_{\Gamma_2}$ if and only if $F_{\frac{b_h}{2}-2}, F_{\frac{b_h}{2}-1} \in I_{\Gamma_2}$, which is true since $F_{\frac{b_h}{2}-2} + (2(-i)^q + c) F_{\frac{b_h}{2}-1}, (1 + 2(-i)^q) c F_{\frac{b_h}{2}-1} + (-1)^q F_{\frac{b_h}{2}-2} \in I_{\Gamma_2}$. The case when $b = 0$ is done identically up to a change of sign.

Now suppose $a = b = 1$. Then $\sigma_\rho = x^q y^q$ and $f_{\rho'}, g_{\rho''} \in \langle \sigma_\rho, f_{\rho_0} \rangle$, which imply that $J_{(1:1)}$ represents the G-graph Γ_{h-1} of type A. It follows that using the same argument as in Case 1, and fixing the value $a = 1$ we obtain that Γ_{h-1} can be chosen to be the basis for $\mathbb{C}[x,y]/J_{(1:b)}$ for every $b \in \mathbb{C}$, and we are done.

CASE 6. $\Gamma_1 := \Gamma_{C_B^+}(r,s;q,q) \to \Gamma_2 := \Gamma_{C_B^-}(r,s;q,q)$: If the last G-graph is of type B_1 then the family is given by the ideal $J_{(a:b)}$ generated by

$$\sigma_{1,\rho} := k_-(x^s + (-i)^q x^m y^q) - k_+(x^s - (-i)^q x^m y^q), \quad f_{1,\rho'} := x^s y^m, \quad f_{\rho_0} := x^{2m} y^{2m},$$
$$\sigma_{2,\rho} := k_-(y^s + i^q x^q y^m) - k_+(y^s + i^q x^q y^m), \quad f_{2,\rho'} := x^m y^s.$$

In this case ρ, ρ' and ρ'' are 2-dimensional. By the equalities $f_{1,\rho'} = g_{1,\rho''}, f_{2,\rho'} = g_{2,\rho''}$ and $f_{\rho_0} = g_{\rho_0}$ it is clear that $J_{(0:1)} = I_{\Gamma_2}$ and $J_{(1:0)} = I_{\Gamma_1}$.

As in the previous case, the ideal $J_{(1:1)}$ represents the G-graph Γ_{h-1} of type B, and using the same argument as in Case 3, it follows that if $b = 1$, we can always find a basis for $\mathbb{C}[x,y]/J_{(a:1)}$, which is a G-graph of type B.

The last two cases correspond to the families at the two 'horns' of exceptional divisor.

CASE 7. $\Gamma_1 := \Gamma_{C_A^\pm}(r,s;q,q) \to \Gamma_2 := \Gamma_{D^\pm}(r,s;q,q)$: In this case the family of ideals $J_{(a_\pm:b_\pm)}$ is given by

$$\sigma_\rho := a_\pm(x^s y^{m_2} \pm (-1)^r i^q x^{m_2} y^s) - b_\pm(x^q \pm (-i)^q y^q), \quad g_{\rho_0} := x^{s-r} y^{s-r},$$
$$f_{\rho'} := (x^q \pm (-i)^q y^q)^2, \quad f_{\rho_0} := x^{m_1} y^{m_2} + (-1)^{m_2} x^{m_2} y^{m_1}.$$

It is straightforward to see that $J_{(0:1)} = I_{\Gamma_2}$ and $J_{(1:0)} = I_{\Gamma_1}$ corresponding to the cases $a = 0$ and $b = 0$, respectively.

Now suppose that $a_+ = 1$. The case $a_- = 1$ is done similarly. First note that using f_{ρ_0} we have $x^{2q-2r} y^{2s-2q}, x^{2s-2q} y^{2q-2r} \in J_{(1:b_+)}$. Then $x^{s-r+q} y^{s-r-q} \equiv \frac{w_+}{z_+} x^{2s-r} y^{s-2r} \equiv 0$ mod $J_{(1:b_+)}$, where the last equivalence is obtained because $(b_h - 1)q < s$ and $b_h \neq 2$, since the last G-graph is of type A. By the same process we see that for $0 \le j \le k+1$ we have $x^{s-r+jq} y^{s-r-jq}, x^{s-r-jq} y^{s-r+jq} \in J_{(1:b_+)}$, thus every monomial not contained in the representation of G-graph of type D^+ belongs to the ideal $J_{(1:b_+)}$, and we are done.

CASE 8. $\Gamma_{C_B^\pm} \to \Gamma_{D^\pm}$: In this case the family of ideals $J_{(a_\pm:b_\pm)}$ is given by

$$\sigma_\rho := a_\pm(x^s y^m \pm (-1)^r i^q x^m y^s) - b_\pm(x^q \pm (-i)^q y^q), \quad f_{1,\rho'} := x^m(x^q \pm (-i)^q y^q),$$
$$f_{\rho_0} := x^{s-r} y^{s-r}, \quad f_{2,\rho'} := y^m(x^q \pm (-i)^q y^q),$$

where $m = s - q = q - r$. In this case ρ' is 2-dimensional and $f_{\rho_0} = g_{\rho_0}$.

This case is analogous as the previous one. Again it is straightforward to see that $J_{(0:1)} = I_{\Gamma_2}$ and $J_{(1:0)} = I_{\Gamma_1}$ corresponding to the cases $a = 0$ and $b = 0$, respectively. Finally, suppose $a_+ = 1$ (the case $a_- = 1$ is done similarly). Then $J_{(1,b_+)} = \langle \sigma_\rho, f_{\rho_0} \rangle$, and in the same way as in the previous case we have $x^{s-r+jq}y^{s-r-jq}, x^{s-r-jq}y^{s-r+jq} \in J_{(1:b_+)}$ for $0 \leq j \leq k+1$, and we are done. □

THEOREM 6.2. *Let $G = \mathrm{BD}_{2n}(a)$ be small and let $P \in G$-Hilb$(\mathbb{C}^2)$ be defined by the ideal I. Then we can always choose a basis for $\mathbb{C}[x, y]/I$ from one of the following list:*

$$\Gamma_A, \Gamma_B, \Gamma_{C^+}, \Gamma_{C^-}, \Gamma_{D^+}, \Gamma_{D^-}.$$

Proof. By construction, every point in G-Hilb$(\mathbb{C}^2)$ away from the 'horns' corresponds to a pair of H-clusters in H-Hilb$(\mathbb{C}^2)$. Therefore, we can choose for these H-clusters the H-graphs $\widetilde{\Gamma}(r, s; u, v)$ and $\widetilde{\Gamma}(v, u; s, r)$ identified by β for some (r, s) and (u, v) boundary lattice points in the Newton polygon for $\frac{1}{2n}(1, a)$, and we can take $\Gamma(r, s; u, v)$ to be the G-graph (of type A or B) for our G-cluster.

For the clusters in the exceptional 'horns' E^+ and E^-, we know by Theorem 6.1 that these exceptional curves are covered by the ideals $J^+_{(\gamma_+,\delta_+)}$ and $J^-_{(\gamma_-,\delta_-)}$, which correspond to G-graphs of type $C^\pm$ and $D^\pm$, and we are done. □

Let U_Γ be an open set in G-Hilb$(\mathbb{C}^2)$, which consists of all G-clusters $\mathcal{Z}$ such that Γ is a basis of $\mathcal{O}_\mathcal{Z}$. As a corollary of previous theorem we have that G-graphs for a $\mathrm{BD}_{2n}(a)$ group gives us an open set for the covering of G-Hilb$(\mathbb{C}^2)$ that we are looking for.

COROLLARY 6.3. *Let $G = \mathrm{BD}_{2n}(a)$ be a small binary dihedral group and let $\Gamma_0, \Gamma_1, \ldots, \Gamma_{h-1}, \Gamma_{C^+}, \Gamma_{C^-}, \Gamma_{D^+}, \Gamma_{D^-}$ be the list of G-graphs. Then*

$$U_{\Gamma_0}, U_{\Gamma_1}, \ldots, U_{\Gamma_{h-1}}, U_{\Gamma_{C^+}}, U_{\Gamma_{C^-}}, U_{\Gamma_{D^+}}, U_{\Gamma_{D^-}}$$

form an open cover of G-Hilb$(\mathbb{C}^2)$.

REMARK 6.4. By deforming the G-graph Γ_i located at the origin, we can calculate the explicit equation of the open set U_{Γ_i} as it is done in **[9]** for binary dihedral subgroups in $\mathrm{SL}(2, \mathbb{C})$. Extending the groups to $\mathrm{GL}(2, \mathbb{C})$ increases the amount of choices for generators of these ideals, which makes this approach much harder in practice. Fortunately, one can associate to any G-graph an open set of the moduli space $\mathcal{M}_\theta(Q, R)$ of θ-stable quiver representations of the McKay quiver with relations (Q, R), which coincides with G-Hilb$(\mathbb{C}^2)$, and where the calculation of the open set turns out to be much easier, See **[13]**.

7. Special representations. For a finite small subgroup $G \subset \mathrm{GL}(2, \mathbb{C})$, the special McKay correspondence states that there is a one-to-one correspondence between exceptional divisors E_i in the minimal resolution of $\mathbb{C}^2/G$ and the *special* irreducible representations ρ_i of G. Ishii in **[4]** proves that the minimal resolution is in fact G-Hilb$(\mathbb{C}^2)$.

THEOREM 7.1 **[4]**, *Section 7.1. Let $G \subset \mathrm{GL}(2, \mathbb{C})$ be small and denote by I_y the ideal corresponding to $y \in G$-Hilb$(\mathbb{C}^2)$ and by $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{\mathbb{C}^2}$ corresponding to*

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the origin 0. If y is in the exceptional locus, then we have an isomorphism

$$I_y/\mathfrak{m}I_y \cong \begin{cases} \rho_i \oplus \rho_0 & \text{if } y \in E_i, \text{ and } y \notin E_j \text{ for } j \neq i \\ \rho_i \oplus \rho_j \oplus \rho_0 & \text{if } y \in E_i \cap E_j \end{cases}$$

as representations of G, where ρ_i is the special representation associated with the irreducible exceptional curve E_i.

In other words, for any point in the exceptional divisor of G-Hilb($\mathbb{C}^2$), only the trivial and the special representations corresponding to the curves where the point lies upon are involved in the ideal defining the G-cluster. In our case, following is the explicit description of these ideals.

PROPOSITION 7.2. *Let $G = \mathrm{BD}_{2n}(a)$ be small and $y \in G$-Hilb($\mathbb{C}^2$) be a point in the exceptional locus. Denote by I_y the ideal defining y and by $\Gamma_y = \Gamma(r, s; u, v)$ the corresponding G-graph. Then*

$$I_y/\mathfrak{m}I_y \cong \begin{cases} \rho_{r+s}^{(-1)^r} \oplus \rho_{u+v}^{(-1)^u} \oplus \rho_0^+ & \text{if } \Gamma_y \text{ is of type } A \\ \rho_{r+s}^{(-1)^r} \oplus V_r \oplus \rho_0^+ & \text{if } \Gamma_y \text{ is of type } B_1 \\ V_{2r-u} \oplus V_r \oplus \rho_0^+ & \text{if } \Gamma_y \text{ is of type } B_2 \\ \rho_{2q}^{(-1)^q} \oplus \rho_q^\pm \oplus \rho_0^+ & \text{if } \Gamma_y \text{ is of type } C^\pm \text{ and } \Gamma_{h-1} \text{ is of type } A \\ V_r \oplus \rho_q^\pm \oplus \rho_0^+ & \text{if } \Gamma_y \text{ is of type } C^\pm \text{ and } \Gamma_{h-1} \text{ is of type } B \\ \rho_q^\pm \oplus \rho_0^+ & \text{if } \Gamma_y \text{ is of type } D^\pm \end{cases},$$

where $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{\mathbb{C}^2}$ corresponding to origin 0.

Proof. Reformulating Propositions 5.9, 5.13, 5.14, 5.16 and 5.17 in the language of Theorem 7.1, we see that the representations involved in the generators of each of the ideals are the ones presented above. By Theorem 6.1 every point in the exceptional divisor $E \subset G$-Hilb($\mathbb{C}^2$) is defined by one of those ideals, so the result follows. □

As a consequence of both Theorem 7.1 and Proposition 7.2 we obtain the special representations for any group $\mathrm{BD}_{2n}(a)$ in terms of the continued fraction $\frac{2n}{a}$, which we list in the following theorem.

THEOREM 7.3. *Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_{h-1}$ the sequence of qG-graphs given by $e_0 = \frac{1}{2n}(0, 2n)$, $e_1 = \frac{1}{2n}(1, a)$, $e_2 = \frac{1}{2n}(c_1, d_1), \ldots, e_{m-1} = \frac{1}{2n}(c_{m-2}, d_{m-2})$, $e_m = \frac{1}{2n}(q, q)$, where*

$$\begin{aligned} \Gamma_0, \ldots, \Gamma_i &\text{ are of type } A, \\ \Gamma_{i+1} &\text{ is of type } B_1, \\ \Gamma_{i+2}, \ldots, \Gamma_{h-1} &\text{ are of type } B_2. \end{aligned}$$

Then the special representations are

$$\begin{aligned} \rho_{1+a}^-, \rho_{c_1+d_1}^{(-1)^{c_1}}, \rho_{c_2+d_2}^{(-1)^{c_2}} \ldots, \rho_{c_{i+1}+d_{i+1}}^{(-1)^{c_{i+1}}} &\text{ from type } A, \\ V_{c_i} &\text{ from type } B_1, \\ V_{c_{i+1}}, \ldots, V_{c_{h-2}} &\text{ form type } B_2 \text{ and} \\ \rho_q^+, \rho_q^- &\text{ from types } C \text{ and } D \end{aligned}$$

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REMARK 7.4. We want to note that the same result holds for groups of the form $\mathrm{BD}_{2n}(a, q)$, since the G-graphs are not affected by change in generator β. In other words, Theorem 7.3 is also valid for any small binary dihedral subgroup $G \subset \mathrm{GL}(2, \mathbb{C})$ with maximal cyclic subgroup $H = \left\langle \frac{1}{2n}(1, a) \right\rangle$.

There is also a relation between G-graphs of types A and B, and the dimension of the corresponding irreducible special representations. Let $E = \bigcup E_i \subset \mathrm{BD}_{2n}(a)$-Hilb$(\mathbb{C}^2)$ be the exceptional locus. By **[20]**, the dimension of the special representation ρ_i corresponding to E_i is equal to the coefficient of E_i in the fundamental cycle Z_{fund} (the smallest effective divisor such that $Z_{\mathrm{fund}} \cdot E_i \leq 0$). Let $-2, -2, -a_m, \ldots, -a_2, -a_1$ be the self-intersections along the minimal resolution of $G/\,\mathrm{BD}_{2n}(a)$, where the first two $-2s$ correspond to the 'horns' of the Dynkin diagram.

COROLLARY 7.5. *The special irreducible representations of* $\mathrm{BD}_{2n}(a)$ *are all 1-dimensional if and only if the middle entry* b_h *in the continued fraction* $\frac{2n}{a} = [b_1, \ldots, b_h, \ldots, b_1]$ *is not* 2.

Proof. The result follows from Corollary 5.4 (ii) and Theorem 7.3. □

Thus, we have a one-to-one correspondence between qG-graphs of type A and 1-dimensional special representations (except the two corresponding to the 'horns' that are also 1-dimensional and covered by the G-graphs of type C and D), and another correspondence between qG-graphs of type B and 2-dimensional special representations.

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