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The Representation Ring and the Centre of a Hopf Algebra

Sarah J. Witherspoon

Abstract. When H is a finite dimensional, semisimple, almost cocommutative Hopf algebra, we examine a table of characters which extends the notion of the character table for a finite group. We obtain a formula for the structure constants of the representation ring in terms of values in the character table, and give the example of the quantum double of a finite group. We give a basis of the centre of H which generalizes the conjugacy class sums of a finite group, and express the class equation of H in terms of this basis. We show that the representation ring and the centre of H are dual character algebras (or signed hypergroups).

1 Introduction

Let H be a finite dimensional Hopf algebra over an algebraically closed field k. Its representation ring $R(H)$ is the $\mathbb{C}$-algebra generated by finite dimensional H-modules with direct sum for addition, tensor product for multiplication, and the trivial module for the identity. If H is semisimple (that is, as an associative algebra), its representation ring (or character ring) is as well, allowing generalization of some of the theory of characters for finite groups to Hopf algebras. This has been done for example by Larson [8], Nichols and Richmond [18], and Zhu [26], and Lorenz [10] treats the nonsemisimple case in particular. Such character theory for Hopf algebras has been useful in studying the structure of the Hopf algebras themselves in work by Lorenz [11], Nichols and Richmond [17], Sommerhäuser [22], and Zhu [26].

Here we require that H be almost cocommutative as well as semisimple, and obtain some further results. Many examples of interest satisfy this hypothesis, including the quasitriangular Hopf algebras. In this case the representation ring $R(H)$ is semisimple and commutative, and so isomorphic to a direct sum of copies of $\mathbb{C}$. Each copy corresponds to a *character* of $R(H)$, that is an algebra homomorphism from $R(H)$ to $\mathbb{C}$. These characters happen to be trace functions of certain central elements defined in Section 4. We consider a *character table*, whose rows are indexed by isomorphism classes of irreducible H-modules, and whose columns are indexed by the characters of $R(H)$. This extends the notion of the character table for a finite group. We present orthogonality relations for these characters in Section 3. This leads to a formula (Theorem 3.2) for structure constants of the representation ring $R(H)$ in terms of the character values, generalizing a well known formula in the case H is a group algebra [23]. We discuss the example of the quantum double of a finite group, for which character values may be given in terms of characters of the group and its centralizer subgroups [25].

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We use these results about characters to obtain a basis for the centre $Z(H)$ in case $k = \mathbb{C}$ in Section 4; in the case of a group algebra this basis is given by the conjugacy class sums. The class equation for a finite group may be described as applying the augmentation ϵ to the sum of these basis elements. We generalize this observation in Proposition 4.3, providing a new way to view the class equation for Hopf algebras (due to Kac [7] and Zhu [26]) in the special case where H is almost cocommutative. We use Lorenz' proof of the class equation [11] for this result. We show that in case H has prime power dimension, the nontrivial central grouplike elements of Masuoka [15] are among our basis elements for $Z(H)$.

We use the basis of $Z(H)$ constructed in Section 4 to show in Theorem 5.2 that when $k = \mathbb{C}$, the representation ring $R(H)$ and the centre $Z(H)$ are *dual character algebras* (or *C-algebras*) [1], as well as *signed hypergroups* [24], providing more such examples. Such algebras generalize the duality between the character ring and the centre of a group algebra. For a history of character algebras, hypergroups, and further references, see [1], [3], [24]. The ideas in Section 5 grew out of questions raised by Terwilliger.

We refer the reader to [16] for standard facts about Hopf algebras, and to [5] for standard facts about characters of finite groups and symmetric algebras. All our modules will be finite dimensional right modules, k always denotes an algebraically closed field, and $\otimes = \otimes_k$.

2 The Representation Ring

In this section, we first review the standard notation and terminology, then collect some needed results from the literature about the representation ring.

Let H be a finite dimensional Hopf algebra over the algebraically closed field k with coproduct Δ, counit (or augmentation) ϵ, and antipode S [16]. We use *sigma notation* for Δ [16], that is, if $h \in H$, we write $\Delta(h) = \sum_{(h)} h_1 \otimes h_2$.

Let V and W be finite dimensional right H-modules. Then $V \otimes W$ is a right H-module via the pullback of the natural action of $H \otimes H$ on $V \otimes W$ from the coproduct $\Delta\colon H \to H \otimes H$. This is a right H-module since Δ is an algebra homomorphism. The field k is a right H-module via the pullback of the action of k on itself, by right multiplication, to H from the counit $\epsilon\colon H \to k$. Up to isomorphism, this *trivial* module k is a multiplicative identity with respect to tensor product of modules; this follows from the counit property of a Hopf algebra.

If V is a finite dimensional right H-module, we write V^* for the dual module $\mathrm{Hom}_k(V, k)$ with right H-action given by

$$f \cdot h(v) = f\big(v \cdot S(h)\big)$$

for all $f \in V^*$, $h \in H$, and $v \in V$. This is a right action since S reverses multiplication. If V and W are two finite dimensional H-modules, then the natural isomorphism of vector spaces $(V \otimes W)^* \cong W^* \otimes V^*$ is an isomorphism of H-modules; this follows from the fact that S reverses comultiplication.

Next we define an action of H on $\mathrm{Hom}_k(V, W)$ for any two finite dimensional right H-modules V, W, so that $\mathrm{Hom}_k(V, W)$ will be isomorphic to $V^* \otimes W$ as right H-modules: If

$f \in \mathrm{Hom}_k(V, W)$ and $h \in H$, define $f \cdot h \in \mathrm{Hom}_k(V, W)$ by

$$f \cdot h(v) = \sum_{(h)} f\big(v \cdot S(h_1)\big)h_2 \tag{2.1}$$

for all $v \in V$.

We define certain *representation rings*: Let $r(H)$ be the group generated by isomorphism classes of finite dimensional H-modules with direct sum for addition. This is the Grothendieck group of the category of finite dimensional H-modules, in which the distinguished exact sequences are taken to be the *split* ones. The additive group $r(H)$ becomes a ring with tensor product for multiplication, and identity given by the isomorphism class of the trivial module. Associativity of $r(H)$ follows from coassociativity of the coproduct for H. We refer to both $r(H)$ and $R(H) = r(H) \otimes_{\mathbb{Z}} \mathbb{C}$ as representation rings. We work primarily with $R(H)$, as our main interest is in characters. If H is semisimple, then $R(H)$ is isomorphic to the *character ring* of H (over $\mathbb{C}$) via the map sending an H-module V to its trace function $\mathrm{Tr}(\cdot, V)$ (see [10], [26]). By abuse of language and notation, we shall consider H-modules to be elements of the representation rings, when we really mean their isomorphism classes.

We will need the following two propositions, due to Zhu [26, Lemmas 1 and 2]; here we translate from left to right modules. Let $\delta_{V,W} = 1$ if $V \cong W$ and 0 otherwise, and let V^H be the submodule of V on which H acts trivially:

$$V^H := \{v \in V \mid v \cdot h = \epsilon(h)v \text{ for all } h \in H\}.$$

Proposition 2.2 below technically required the characteristic of the underlying field k to be 0, however Zhu's proof holds more generally. It uses the fact that S^2 is an inner automorphism, so that in particular $(V^*)^* \cong V$ for all H-modules V. This is always the case when H is semisimple [20, Theorem 5].

Proposition 2.1 **(Zhu)** *Suppose H is a finite dimensional semisimple Hopf algebra, and V and W are finite dimensional H-modules. Then $\mathrm{Hom}_H(V, W) \cong (V^* \otimes W)^H$ as vector spaces. In particular, if V and W are irreducible, then the multiplicity of the trivial module k as a direct summand of $V^* \otimes W$ is $\delta_{V,W}$.*

Proposition 2.2 **(Zhu)** *Suppose H is a finite dimensional semisimple Hopf algebra. Then the representation ring $R(H)$ is semisimple.*

We assume from now on that H is semisimple. As in [10] and [18], we define a bilinear form on $r(H)$ by

$$(V, W) := \dim_k \mathrm{Hom}_H(V, W)$$

for all H-modules V, W. We extend it to an inner product on $R(H) = r(H) \otimes_{\mathbb{Z}} \mathbb{C}$ as follows. Let $V_1, \ldots, V_n$ be the irreducible H-modules up to isomorphism, with $V_1 = k$, and $x = \sum_{i=1}^n a_i V_i$, $y = \sum_{i=1}^n b_i V_i$ elements of $R(H)$. Define

$$(x, y) := \sum_{i,j=1}^n a_i \overline{b_j}(V_i, V_j) = \sum_{i=1}^n a_i \overline{b_i},$$

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where $\overline{b_j}$ is the complex conjugate of b_j. The *norm* of x is

$$\|x\| := \sqrt{(x,x)}.$$

Extend the dual map on modules to a conjugate linear map on $R(H)$ by defining

$$x^* := \sum_{i=1}^{n} \overline{a_i} V_i^*.$$

Then we see as in [10] or [18] that the inner product satisfies the following properties for all $x, y, z \in R(H)$:

$$(x^*, y^*) = \overline{(x,y)} = (y,x), \quad \text{and} \tag{2.2}$$

$$(xy, z) = (y, x^* z). \tag{2.3}$$

Equation (2.2) follows from the definitions, and (2.3) follows from the isomorphisms $\mathrm{Hom}_H(U \otimes V, W) \cong (V^* \otimes U^* \otimes W)^H \cong \mathrm{Hom}_H(V, U^* \otimes W)$ given by Proposition 2.1.

The remaining observations in this section were made by Zhu [26] and Nichols and Richmond [18]. Let $x \in R(H)$, and write $x = \sum_{i=1}^{n} a_i V_i$. Then

$$xx^* = \sum_{i,j=1}^{n} a_i \overline{a_j} V_i V_j^*.$$

By Proposition 2.1, the coefficient of the trivial module k in xx^* is then

$$\sum_{i=1}^{n} a_i \overline{a_i} = \| x \|^2 .$$

Thus $x = 0$ if and only if $xx^* = 0$. Letting $E_1, \ldots, E_r$ be the primitive central idempotents of the semisimple representation ring $R(H)$, we see that $E_i E_i^* \neq 0$. But E_i^* is also a primitive central idempotent, as * is an algebra anti-isomorphism. Therefore

$$E_i = E_i^*. \tag{2.4}$$

3 Orthogonality and Structure Constants

We continue under the assumption that H is a finite dimensional semisimple Hopf algebra over the algebraically closed field k, so that its representation ring $R(H)$ is semisimple by Proposition 2.2. In addition we assume that H is *almost cocommutative*, that is there exists an invertible element $R \in H \otimes H$ such that for all $h \in H$,

$$\tau\big(\Delta(h)\big) = R\Delta(h)R^{-1},$$

where τ is the *twist* map given by $\tau(a \otimes b) = b \otimes a$. In this case, $V \otimes W \cong W \otimes V$ for all H-modules V, W, the isomorphism given by the twist map followed by the natural action of R. Therefore the representation ring $R(H)$ is a finite dimensional, semisimple, *commutative* $\mathbb{C}$-algebra, and so is isomorphic to a direct sum of copies of $\mathbb{C}$. Each copy corresponds to a *character* of $R(H)$, that is an algebra homomorphism from $R(H)$ to $\mathbb{C}$. Note that the set of characters of $R(H)$ is linearly independent.

We consider a *character table* associated to H, whose rows are indexed by isomorphism classes of irreducible H-modules, and whose columns are indexed by the characters of $R(H)$. The entries are the characters evaluated on the modules (considered as elements of $R(H)$). In the case of a group algebra $\mathbb{C}G$ of a finite group G, this is precisely the usual character table of G: The characters of $R(\mathbb{C}G)$ are the trace functions $\mathrm{Tr}(g, \cdot)$ of representatives $g \in G$ of conjugacy classes (or equivalently trace functions of normalized sums of conjugacy classes). This is because group elements $g \in G$ are *grouplike* elements in the Hopf algebra $\mathbb{C}G$ (that is, $\Delta(g) = g \otimes g$). In Section 4 we will define more generally central elements z_i of H whose trace functions are precisely the characters of $R(H)$. Here we give orthogonality relations for the characters and a formula for the structure constants of the representation ring $R(H)$ in terms of the character values.

We will need the following proposition due to Nichols and Richmond [18]. However as their approach involves comodules, we include a proof here for convenience.

***Proposition 3.1* (Nichols-Richmond)** *Let μ be a character of the representation ring $R(H)$, and $x \in R(H)$. Then $\mu(x^*) = \overline{\mu(x)}$, the complex conjugate of $\mu(x)$.*

Proof Let E_i be a primitive central idempotent of $R(H)$, and μ_i the corresponding character. We claim that $\mu_i(y) = (y, E_i)/\parallel E_i \parallel^2$ for all $y \in R(H)$: Write $y = \sum_{j=1}^n c_j E_j$ so that $\mu_i(y) = c_i$. On the other hand, by (2.3) and (2.4),

$$\begin{aligned}\frac{(y, E_i)}{\parallel E_i \parallel^2} &= \frac{1}{\parallel E_i \parallel^2} \sum_{j=1}^n (c_j E_j, E_i)\\ &= \frac{1}{\parallel E_i \parallel^2} \sum_{j=1}^n c_j (1, E_j^* E_i)\\ &= \frac{1}{\parallel E_i \parallel^2} c_i (E_i, E_i)\\ &= c_i.\end{aligned}$$

It follows that, by (2.2) and (2.4),

$$\begin{aligned}\mu_i(x^*) &= \frac{1}{\parallel E_i \parallel^2} (x^*, E_i)\\ &= \frac{1}{\parallel E_i \parallel^2} \overline{(x, E_i)}\\ &= \overline{\mu_i(x)}.\end{aligned}$$

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We consider a different form on $R(H)$ that is *symmetric*: Define

$$\langle V, W\rangle := \dim_k \operatorname{Hom}_H(V^*, W)$$

for all H-modules V, W. This generates a nondegenerate, bilinear, symmetric, associative form by Proposition 2.1 (see also [10, Section 3.1] and [11, Section 2.2]). Therefore $R(H)$ is a symmetric algebra with dual bases $\{V_1, \dots, V_n\}$ and $\{V_1^*, \dots, V_n^*\}$, where $V_1, \dots, V_n$ are the irreducible H-modules, as noted in [11]. We now give a formula for the primitive central idempotents of $R(H)$ and orthogonality relations for characters, as provided by [5, Section 9B] for symmetric algebras via dual bases. We point out that our characters are the *irreducible* characters of [5].

Let

$$M := \bigoplus_{i=1}^{n} (V_i^* \otimes V_i). \tag{3.1}$$

As H is semisimple, $H \cong \bigoplus_{i=1}^n \operatorname{End}_k(V_i)$ as an algebra. Using (2.1), it may be checked that M is isomorphic to the H-module H where $h \in H$ acts on $h' \in H$ by the *adjoint* action

$$h' \cdot h = \sum_{(h)} S(h_1) h' h_2. \tag{3.2}$$

Let μ be a character of $R(H)$, and note that

$$\mu(M) = \sum_{i=1}^{n} \overline{\mu(V_i)}\mu(V_i) = \sum_{i=1}^{n} \parallel \mu(V_i) \parallel^2 > 0, \tag{3.3}$$

by Proposition 3.1. This also follows from [5, Proposition 9.17 (ii)]. We have the corresponding primitive central idempotent of $R(H)$ [5, Proposition 9.17 (ii)],

$$E_\mu = \frac{1}{\mu(M)} \sum_{i=1}^{n} \mu(V_i) V_i^*. \tag{3.4}$$

Orthogonality relations are given as follows. For any character μ of $R(H)$, we write μ^* for the character defined by

$$\mu^*(V) := \mu(V^*),$$

for all H-modules V. We caution that $E_{\mu^*} \neq (E_\mu)^*$. Let $\mu_1, \dots, \mu_n$ be the characters of $R(H)$. Then we have the *column orthogonality relations* by [5, Proposition 9.19] (see also [18, Corollary 22]),

$$\sum_{\ell=1}^{n} \mu_i(V_\ell) \mu_j^*(V_\ell) = \delta_{ij} \mu_i(M). \tag{3.5}$$

In other words, the product of the transpose $\big(\mu_i(V_j)\big)$ of the character table matrix with the matrix $\big(\mu_j^*(V_i)\big)$ is a diagonal matrix with diagonal entries $\mu_i(M)$. Multiplying by the appropriate diagonal matrices, we obtain two inverse matrices. Multiplying them in the reverse order yields the *row orthogonality relations*,

$$\sum_{\ell=1}^{n} \frac{\mu_\ell^*(V_i)\mu_\ell(V_j)}{\mu_\ell(M)} = \delta_{ij}. \tag{3.6}$$

As we see next, the row orthogonality relations may be used to obtain a formula for the structure constants in $R(H)$. This is well known in the case H is a group algebra [23].

Theorem 3.2 *Let H be a finite dimensional, semisimple, almost cocommutative Hopf algebra. Let $V_1, \ldots, V_n$ be the irreducible H-modules up to isomorphism, $\mu_1, \ldots, \mu_n$ the characters of $R(H)$, and suppose that $V_i \otimes V_j \cong \bigoplus_{h=1}^{n} V_h^{\oplus N_{ij}^h}$ for $1 \le i, j \le n$. Then*

$$N_{ij}^h = \sum_{\ell=1}^{n} \frac{\mu_\ell^*(V_h)\mu_\ell(V_i)\mu_\ell(V_j)}{\mu_\ell(M)},$$

where M is the module defined in (3.1).

Proof For each pair i, j, consider the n equations

$$\mu_\ell(V_i)\mu_\ell(V_j) = \sum_{h=1}^{n} N_{ij}^h \mu_\ell(V_h).$$

Solving these systems of equations for the N_{ij}^h by using the row orthogonality relations (3.6), we obtain the desired result. ■

Example: The Quantum Double of a Finite Group Let G be a finite group. The *quantum double* (or *Drinfel'd double*) $D(G)$ is a smash product of the group algebra kG with its Hopf algebra dual $(kG)^*$. Specifically, the space $(kG)^* \otimes kG$ is given the structure of a Hopf algebra as follows. If $\{\phi_g\}_{g\in G}$ is the basis of $(kG)^*$ dual to $\{g\}_{g\in G}$, then $D(G)$ has as a basis all elements $\phi_g \otimes h$, which we write more simply as $\phi_g h$, for $g, h \in G$. On this basis, the product is defined by $\phi_g h \phi_{g'} h' = \phi_g \phi_{hg'h^{-1}} hh' = \delta_{g,hg'h^{-1}} \phi_g hh'$. The identity is $1_{D(G)} = \sum_{g\in G} \phi_g 1$, where 1 is the identity for G. The coproduct is given by $\Delta(\phi_g h) = \sum_{x\in G} \phi_x h \otimes \phi_{x^{-1}g} h$, the counit by $\epsilon(\phi_g h) = \delta_{1,g}$, and the coinverse by $S(\phi_g h) = \phi_{h^{-1}g^{-1}h} h^{-1}$. The Hopf algebra $D(G)$ is almost cocommutative with $R = \sum_{g\in G} \phi_g \otimes g$ (in fact, it is *quasitriangular* [16, 10.1.5]). Maschke's Theorem for Hopf algebras [16, Theorem 2.2.1] implies that $D(G)$ is semisimple if and only if the characteristic of k does not divide the order of G [25, Proposition 1.2]. Thus we will restrict ourselves to that case.

It is well known that the irreducible $D(G)$-modules are indexed by pairs (g, V), where g is a representative of a conjugacy class of G, and V is an irreducible $kC(g)$-module (here $C(g) = \{h \in G \mid gh = hg\}$ is the centralizer of g in G). The resulting $D(G)$-modules are induced from these $kC(g)$-modules. Different approaches to this result appear in [13], [25]; see also [6] for the special case $k = \mathbb{C}$.

The characters of $R(D(G))$ are given explicitly in [25, Theorem 3.4], and for the case $k = \mathbb{C}$, also in [12], in terms of characters of G and its centralizer subgroups. The characters in the case $k = \mathbb{C}$ are indexed by pairs (g, ρ), where g is a representative of a conjugacy class of G, and ρ is an irreducible character of $C(g)$. The corresponding character $\mu_{g,\rho}$ of $R\big(D(G)\big)$ sends a $D(G)$-module V to [25, p. 316]

$$\mu_{g,\rho}(V) = \frac{1}{\deg \rho} \sum_{h \in C(g)} \rho(h) \operatorname{Tr}(\phi_h g, V).$$

Let $V_1, \ldots, V_n$ be the irreducible $D(G)$-modules over $\mathbb{C}$ up to isomorphism, and suppose $V_i \otimes V_j \cong \bigoplus_{h=1}^n V_h^{\oplus N_{ij}^h}$. Then by Theorem 3.2,

$$N_{ij}^h = \sum_{(g,\rho)} \frac{\mu^*_{g,\rho}(V_h)\mu_{g,\rho}(V_i)\mu_{g,\rho}(V_j)}{\mu_{g,\rho}(M)}, \tag{3.7}$$

the sum ranging over the pairs (g, ρ). The $D(G)$-module M is the space $D(G)$ with right action

$$\phi_x y \cdot \phi_g h = \delta_{g,x^{-1}y^{-1}xy} \phi_{h^{-1}xh} h^{-1} y h.$$

We note that the values of the $\mu_{g,\rho}$ are sums of products of values from the character tables of G and its centralizer subgroups, and thus the structure constants may be calculated from the values in such character tables. Indeed, $\operatorname{Tr}(\phi_h g, V) = \operatorname{Tr}(g, V_h)$ where $V_h = V\phi_h$ may be considered to be a $\mathbb{C}C(h)$-module, as is discussed in [25, Section 2]. A different approach to characters for this example is given in [2], and an apparently simpler formula than (3.7) for the structure constants is given in [2], [6].

4 The Centre and the Class Equation

In this section, we construct two bases for the centre $Z(H)$ of H, and give a new presentation of the class equation of Kac [7] and Zhu [26] using work of Lorenz [11]. We keep our assumptions that H is finite dimensional, semisimple, and almost cocommutative. In addition, we take the field k to be $\mathbb{C}$ here.

As before, let $V_1, \ldots, V_n$ be the irreducible H-modules up to isomorphism (with $V_1 = \mathbb{C}$ the trivial module), $\mu_1, \ldots, \mu_n$ the characters of the representation ring $R(H)$ (with μ_1 the dimension homomorphism, $\mu_1(V) = \dim(V)$ for all H-modules V), and $E_1, \ldots, E_n$ the corresponding primitive central idempotents of $R(H)$ as given by (3.4). As H is semisimple, we have $H \cong \bigoplus_{i=1}^n \operatorname{End}_{\mathbb{C}}(V_i)$ as an algebra. Let e_i be the primitive central idempotent of H corresponding to V_i, that is e_i arises from the identity transformation of $\operatorname{End}_{\mathbb{C}}(V_i)$ in the above isomorphism. Define the elements z_i of $Z(H)$ by

$$z_i := \sum_{j=1}^n \frac{\mu_i(V_j)}{\dim(V_j)} e_j. \tag{4.1}$$

As $\big(\mu_i(V_j)\big)$ is nonsingular by the orthogonality relations (3.5) and (3.6), the elements z_i are linearly independent, and thus form a basis of $Z(H)$. Notice that $\mu_i = \mathrm{Tr}(z_i, \cdot)$, so in fact we may label the columns of the character table in Section 3 with the elements z_i rather than μ_i. In the case H is a group algebra, we may use the formula [5, Proposition 9.21 (ii)] for the primitive central idempotents e_i to see that each of these elements z_i is a normalized sum of elements in a conjugacy class.

Example When $H = D(G)$ is the quantum double of the finite group G (Section 3), the basis elements z_i are indexed by pairs (g, ρ), where g is a representative of a conjugacy class of G, and ρ an irreducible character of $C(g)$. They are given by

$$z_{g,\rho} = \frac{1}{|G| \deg(\rho)} \sum_{h \in C(g), x \in G} \rho(h) \phi_{xhx^{-1}} xgx^{-1}.$$

This follows from the observations that these elements $z_{g,\rho}$ are central, and in general the central element z_i is determined uniquely by the fact that $\mu_i = \mathrm{Tr}(z_i, \cdot)$.

We will need the following two lemmas. The first generalizes the formula [5, Proposition 9.21 (ii)] for the primitive central idempotents e_i in the case H is a group algebra.

Lemma 4.1 $e_i = \sum_{j=1}^{n} \frac{\mu_j^*(V_i) \dim(V_i)}{\mu_j(M)} z_j$.

Proof By the definition (4.1) of z_i, we may express the z_i in terms of the e_j by means of the matrix equation

$$\left(\frac{\mu_i(V_j)}{\dim(V_j)}\right)(e_j) = (z_i).$$

By the column orthogonality relations (3.5), the inverse of the coefficient matrix is

$$\left(\frac{\mu_j^*(V_i) \dim(V_i)}{\mu_j(M)}\right). \qquad \blacksquare$$

Lemma 4.2 *If $z_i z_j = \sum_{h=1}^{n} m_{ij}^h z_h$, then $m_{ij}^1 = \delta_{ij^*} \frac{\mu_i(M)}{\dim(H)}$.*

Proof Using the definition (4.1) of z_i and Lemma 4.1, we have

$$\begin{aligned} z_i z_j &= \sum_{\ell=1}^{n} \frac{\mu_i(V_\ell)\mu_j(V_\ell)}{\dim(V_\ell)^2} e_\ell \\ &= \sum_{h,\ell=1}^{n} \frac{\mu_i(V_\ell)\mu_j(V_\ell)\mu_h^*(V_\ell)}{\dim(V_\ell)\mu_h(M)} z_h. \end{aligned}$$

Therefore by the column orthogonality relations (3.5), as $\mu_1^*(V_\ell) = \mu_1(V_\ell) = \dim(V_\ell)$ and $\mu_1(M) = \dim(H)$,

$$\begin{aligned} m_{ij}^1 &= \frac{1}{\dim(H)} \sum_{\ell=1}^{n} \mu_i(V_\ell)\mu_j(V_\ell) \\ &= \delta_{ij^*} \frac{\mu_i(M)}{\dim(H)}. \end{aligned}$$

■

We modify the basis $\{z_i\}$ of $Z(H)$ slightly. Let

$$\zeta_i := \frac{\dim(H)}{\mu_i(M)} z_i, \tag{4.2}$$

where $M = \bigoplus_{i=1}^n (V_i^* \otimes V_i)$ as before. In the case H is a group algebra, it may be checked, using (3.2), that each of these elements ζ_i is the sum of the elements in a conjugacy class. In this case, note too that the *class equation* may be described as applying the augmentation ϵ to the equation $|G|e_1 = \sum_{i=1}^n \zeta_i$, as $\epsilon(\zeta_i)$ is the number of elements in the corresponding conjugacy class, and $\epsilon(e_1) = 1$. We generalize this observation in the next proposition, providing a new way to view the class equation for Hopf algebras (due to Kac [7] and Zhu [26]) in the special case where H is almost cocommutative.

Let H^* denote the Hopf algebra dual to H [16, Example 1.5.5]. Identify the representation ring $R(H)$ with a subalgebra of H^* by identifying an H-module V with the trace function $\mathrm{Tr}(\cdot, V)$ (this is the *character ring* as a subalgebra of H^*). In this way the primitive central idempotents $E_1, \ldots, E_n$ of $R(H)$ may be considered to be elements of H^*.

Proposition 4.3 **(Class Equation)** *If H is a finite dimensional, semi-simple, almost cocommutative Hopf algebra over $\mathbb{C}$, then*

$$\dim(H) = \sum_{i=1}^n \epsilon(\zeta_i).$$

Further, for all $1 \le i \le n$, $\epsilon(\zeta_i) = \dim(H)/\mu_i(M) = \dim(E_i H^*)$ is an integer dividing $\dim(H)$.

Proof By Lemma 4.1, the definitions of ζ_i (4.2) and z_i (4.1), and as $\mu_i^*(V_1) = 1$ for all i,

$$\dim(H)e_1 = \sum_{i=1}^n \zeta_i.$$

Note that $\epsilon(e_i) = \delta_{1,i}$, as elements of H act on the trivial module $V_1 = k$ via ϵ. Therefore, by applying ϵ to the above equation, we obtain

$$\dim(H) = \sum_{i=1}^n \epsilon(\zeta_i).$$

Again by the definitions of ζ_i and z_i, we have $\epsilon(z_i) = \mu_i(k)/\dim(k) = 1$, and $\epsilon(\zeta_i) = \dim(H)/\mu_i(M)$.

By Lorenz' proof of the class equation [11, Section 3], under the hypothesis that H is almost cocommutative (and so $R(H)$ is commutative), we have

$$\frac{\dim(H)}{\dim(E_i H^*)} = \mu_i(M),$$

and is an integer. Therefore $\dim(H)/\mu_i(M) = \dim(E_iH^*)$ is an integer dividing $\dim(H)$ for $1 \le i \le n$. ∎

For each index $i \in \{1,\dots,n\}$ of the characters $\mu_1,\dots,\mu_n$, let $i^* \in \{1,\dots,n\}$ be the index satisfying $\mu_{i^*} = \mu_i^*$, where $\mu_i^*(V) := \mu_i(V^*)$ for all H-modules V. In particular then, $\mu_{i^*} = \mathrm{Tr}(z_{i^*},\cdot)$.

Suppose $\dim(H) = p^n$, with p a prime. For example, the quantum double (or Drinfel'd double) of any of Masuoka's semisimple Hopf algebras of dimension p^3 [14] is a semisimple [19] and almost cocommutative (in fact, quasitriangular by [4, Proposition 4.2.12]) Hopf algebra of dimension p^6. We obtain central grouplike elements of H via the class equation in the following way. By Proposition 4.3, as $\zeta_1 = 1$, there is some $i \ne 1$ such that $1 = \epsilon(\zeta_i) = \dim(H)/\mu_i(M) = \dim(E_iH^*)$. From the assumption $\dim(E_iH^*) = 1$, Masuoka proved that H has a nontrivial central grouplike element [15]. In our case, we use Schneider's formulation of Masuoka's result [21] to show that ζ_{i^*} is the corresponding (central) grouplike element: First note that as $M^* \cong M$ and $\epsilon(\zeta_{i^*}) = \dim(H)/\mu_i^*(M)$, we have $\epsilon(\zeta_{i^*}) = \epsilon(\zeta_i) = 1$. Now let $\lambda \in H^*$ be a nonzero integral (that is, λ is invariant under left and right multiplication in H^*) such that $\lambda(1) = 1$. By (4.1), (4.2), the proof and statement of [21, Proposition 4.5], and (3.4), ζ_{i^*} is the *unique* element $h \in Z(H)$ such that $h \rightharpoonup \lambda = E_i$. (Here $h \rightharpoonup \lambda := \sum_{(\lambda)} \lambda_2(h)\lambda_1$.) By [21, Lemma 4.14 (2)], there exists a nontrivial central grouplike element g such that E_i is a scalar multiple of $g \rightharpoonup \lambda$. By uniqueness, and as $\epsilon(\zeta_{i^*}) = 1$, ζ_{i^*} is forced to be grouplike.

5 Dual Character Algebras

Let H be a finite dimensional, semisimple, almost cocommutative Hopf algebra over $\mathbb{C}$. We use the results of the previous sections to show that the representation ring $R(H)$ and the centre $Z(H)$ are dual character algebras.

First we recall the definition from [1]. A *character algebra* (or *C-algebra*) is a finite dimensional commutative algebra A over $\mathbb{C}$ together with a distinguished basis $X_1,\dots,X_n$ such that $X_1 = 1$ is the multiplicative identity of A and:

(1) There is an involution $i \mapsto i^*$ of $\{1,\dots,n\}$ such that the linear map from A to A sending X_i to X_{i^*} is a $\mathbb{C}$-algebra isomorphism.
(2) If $X_iX_j = \sum_{h=1}^n p_{ij}^h X_h$, then $p_{ij}^h \in \mathbb{R}$ $(1 \le h,i,j \le n)$.
(3) There are positive real numbers $k_1,\dots,k_n$ such that $p_{ij}^1 = \delta_{ij^*}k_i$ $(1 \le i,j \le n)$.
(4) The linear map from A to $\mathbb{C}$ sending X_i to k_i is a $\mathbb{C}$-algebra homomorphism.

We point out that if we take instead the normalized basis $\{X_i/k_i\}$ of A, we have a *signed hypergroup* [24], as in this case the sum over h of the structure constants p_{ij}^h (for fixed i and j) is 1 by property (4).

Let $V_1,\dots,V_n$ be the irreducible H-modules up to isomorphism (with $V_1 = \mathbb{C}$ the trivial module), and

$$X_i := \dim(V_i)V_i \quad (1 \le i \le n) \tag{5.1}$$

as elements of $R(H)$. Let $\mu_1,\dots,\mu_n$ be the characters of $R(H)$, with μ_1 the dimension homomorphism $\mu_1(V) = \dim(V)$ for all H-modules V. Let $\zeta_1,\dots,\zeta_n$ be the central elements of H defined in (4.2), so that $\zeta_1 = 1$ is the multiplicative identity.

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Theorem 5.1 *Let H be a finite dimensional, semisimple, almost cocommutative Hopf algebra over $\mathbb{C}$. Then:*

(i) The representation ring $R(H)$ is a character algebra with basis $X_1, \ldots, X_n$.
(ii) The centre $Z(H)$ is a character algebra with basis $\zeta_1, \ldots, \zeta_n$.

Proof (i) Let i^* be the element such that $X_{i^*} = X_i^* = \dim(V_i^*)V_i^*$. Then the linear map sending X_i to X_{i^*} is simply the map taking any element to its "dual", and is a $\mathbb{C}$-algebra isomorphism. As $V_iV_j = \sum_{h=1}^n N_{ij}^h V_h$ in $R(H)$ with N_{ij}^h positive integers, we have

$$X_iX_j = \sum_{h=1}^n p_{ij}^h X_h$$

with $p_{ij}^h = \dim(V_i)\dim(V_j)N_{ij}^h/\dim(V_h)$ rational numbers. We also then have $p_{ij}^1 = \dim(V_i)\dim(V_j)N_{ij}^1$. By Proposition 2.1, $N_{ij}^1 = \delta_{ij^*}$, so $p_{ij}^1 = \delta_{ij^*}k_i$, with $k_i = \big(\dim(V_i)\big)^2$ a positive integer. Finally, the linear map from $R(H)$ to $\mathbb{C}$ sending X_i to k_i is just the dimension homomorphism μ_1. Therefore $R(H)$ is a character algebra.

(ii) First note that S is an involution on the basis $\zeta_1, \ldots, \zeta_n$: Recall that e_i is the identity map on V_i. Therefore $S(e_i)$ is the identity map on $V_i^* = V_{i^*}$, that is $S(e_i) = e_{i^*}$, as S^2 is an inner automorphism [20, Theorem 5] and e_i is central in H. By (4.1) and (4.2) then,

$$\begin{aligned} S(\zeta_i) &= \sum_{j=1}^n \frac{\dim(H)\mu_i(V_j)}{\mu_i(M)\dim(V_j)} e_{j^*} \\ &= \sum_{j=1}^n \frac{\dim(H)\mu_i(V_j^*)}{\mu_i(M)\dim(V_j^*)} e_j \\ &= \sum_{j=1}^n \frac{\dim(H)\mu_i^*(V_j)}{\mu_i^*(M)\dim(V_j)} e_j, \end{aligned}$$

as $\mu_i(M) = \mu_i^*(M)$ by the definition (3.1) of M. Thus $S(\zeta_i)$ is another element of the basis $\zeta_1, \ldots, \zeta_n$, that corresponding to $\mu_i^* = \mu_{i^*}$. Therefore $S(\zeta_i) = \zeta_{i^*}$, and * is an involution on the indices $\{1, \ldots, n\}$. As the coinverse S is an algebra antihomomorphism on H, it is an algebra homomorphism on $Z(H)$. This shows that property (1) in the definition of character algebra holds for $Z(H)$.

Next note that the function $f_\ell \colon Z(H) \to \mathbb{C}$ defined by

$$f_\ell(z) := \big(1/\dim(V_\ell)\big)\operatorname{Tr}(z, V_\ell)$$

is an algebra homomorphism: If $z, z' \in Z(H)$, write $z = \sum_{i=1}^n c_ie_i$ and $z' = \sum_{i=1}^n c_i'e_i$. Then $f_\ell(z) = c_\ell$, $f_\ell(z') = c_\ell'$, and

$$f_\ell(zz') = \frac{1}{\dim(V_\ell)}\operatorname{Tr}\Big(\sum_{i=1}^n c_ic_i'e_i, V_\ell\Big) = c_\ell c_\ell' = f_\ell(z)f_\ell(z').$$

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We will first work with the basis $z_1, \ldots, z_n$ of $Z(H)$. Suppose that

$$z_i z_j = \sum_{h=1}^{n} m_{ij}^h z_h.$$

Applying f_ℓ to this equation we obtain

$$f_\ell(z_i) f_\ell(z_j) = \sum_{h=1}^{n} m_{ij}^h f_\ell(z_h). \tag{5.2}$$

Similarly, applying f_{ℓ^*} we obtain

$$f_{\ell^*}(z_i) f_{\ell^*}(z_j) = \sum_{h=1}^{n} m_{ij}^h f_{\ell^*}(z_h).$$

As $f_{\ell^*}(z_i) = \bigl(1/\dim(V_\ell)\bigr) \operatorname{Tr}(z_i, V_\ell^*)$, and $\operatorname{Tr}(z_i, \cdot) = \mu_i$ is a character of $R(H)$, Proposition 3.1 implies that the latter equation may be rewritten as

$$\overline{f_\ell(z_i)} \cdot \overline{f_\ell(z_j)} = \sum_{h=1}^{n} m_{ij}^h \cdot \overline{f_\ell(z_h)}. \tag{5.3}$$

On the other hand, taking the complex conjugate of (5.2), we have

$$\overline{f_\ell(z_i)} \cdot \overline{f_\ell(z_j)} = \sum_{h=1}^{n} \overline{m_{ij}^h} \cdot \overline{f_\ell(z_h)}.$$

We obtain $\sum_{h=1}^{n} m_{ij}^h \overline{\operatorname{Tr}(z_h, V_\ell)} = \sum_{h=1}^{n} \overline{m_{ij}^h} \cdot \overline{\operatorname{Tr}(z_h, V_\ell)}$ by comparing to (5.3). As the V_ℓ form a basis for $R(H)$, we have

$$\sum_{h=1}^{n} m_{ij}^h \cdot \overline{\operatorname{Tr}(z_h, \cdot)} = \sum_{h=1}^{n} \overline{m_{ij}^h} \cdot \overline{\operatorname{Tr}(z_h, \cdot)}.$$

But the functions $\operatorname{Tr}(z_h, \cdot) = \mu_h$ are linearly independent, so $m_{ij}^h = \overline{m_{ij}^h}$. Therefore m_{ij}^h is a real number for all h, i, j.

Now let $\zeta_i \zeta_j = \sum_{h=1}^{n} p_{ij}^h \zeta_h$. As $\zeta_i = \bigl(\dim(H)/\mu_i(M)\bigr) z_i$, we have

$$p_{ij}^h = \frac{\mu_h(M) \dim(H)}{\mu_i(M) \mu_j(M)} m_{ij}^h.$$

By (3.3), $\mu_i(M)$ is a real number. As a result, p_{ij}^h is a real number, and $Z(H)$ satisfies property (2) of the definition of a character algebra.

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By Lemma 4.2, the above paragraph, and as μ_1 is the dimension homomorphism, we have

$$p_{ij}^1 = \delta_{ij^*}\frac{\mu_1(M)\dim(H)\mu_i(M)}{\mu_i(M)\mu_j(M)\dim(H)} = \delta_{ij^*}\frac{\dim(H)}{\mu_j(M)}.$$

By Proposition 4.3, $k_i := \dim(H)/\mu_i(M)$ is a positive integer. Therefore $Z(H)$ satisfies property (3) of the definition of a character algebra.

Finally, consider the linear map from $Z(H)$ to $\mathbb{C}$ sending ζ_i to $k_i = \dim(H)/\mu_i(M)$. By Proposition 4.3, this is just the counit ϵ. It follows that $Z(H)$ satisfies property (4) of the definition of character algebra. ■

Next we recall the definition of dual character algebras and show that $R(H)$ and $Z(H)$ are dual. If A is a character algebra with basis $X_1, \ldots, X_n$, then A is semisimple [1, Proposition 2.5.4]. Let $E_1, \ldots, E_n$ be a basis of primitive central idempotents of A. Let $P = (P_{ij})$ be the $n \times n$ matrix such that

$$X_i = \sum_{j=1}^{n} P_{ji}E_j \quad (1 \le i \le n),$$

called the *matrix of eigenvalues* of A.

Let A and A^* be character algebras with bases $X_1, \ldots, X_n$ and $X_1^*, \ldots, X_n^*$, respectively. Let $E_1, \ldots, E_n$ and $E_1^*, \ldots, E_n^*$ be bases of primitive central idempotents, and P and P^* the matrices of eigenvalues of A and A^*, respectively. Then A and A^* are *dual* if PP^* is a multiple of the identity matrix.

Theorem 5.2 *Let H be a finite dimensional, semisimple, almost cocommutative Hopf algebra over $\mathbb{C}$. Then the representation ring $R(H)$ and the centre $Z(H)$ are dual character algebras.*

Proof Consider the basis elements $X_1, \ldots, X_n$ of $R(H)$ as defined in (5.1), and primitive idempotents $E_{1^*}, \ldots, E_{n^*}$ (in that order) corresponding to the characters $\mu_i^* = \mu_{i^*}$. Letting

$$X_i = \sum_{j=1}^{n} P_{ji}E_{j^*}$$

and applying μ_ℓ^*, we have $\mu_\ell^*(X_i) = P_{\ell i}$ as $\mu_\ell^*(E_{j^*}) = \delta_{\ell j}$. On the other hand, $\mu_\ell^*(X_i) = \dim(V_i)\mu_\ell^*(V_i)$. So $P_{ij} = \dim(V_j)\mu_i^*(V_j)$.

Consider the basis elements $\zeta_1, \ldots, \zeta_n$ of $Z(H)$ as defined in (4.2), and primitive central idempotents $e_1, \ldots, e_n$ corresponding to the irreducible H-modules $V_1, \ldots, V_n$ as before. By the definitions (4.1) and (4.2), we have

$$\zeta_i = \sum_{j=1}^{n} \frac{\dim(H)\mu_i(V_j)}{\mu_i(M)\dim(V_j)} e_j.$$

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Thus if P^* is the matrix of eigenvalues of $Z(H)$, we have

$$P^*_{ij} = \frac{\dim(H)\mu_j(V_i)}{\mu_j(M)\dim(V_i)}.$$

By the column orthogonality relations (3.5), we have

$$\begin{aligned}(PP^*)_{ij} &= \sum_{\ell=1}^{n} \frac{\mu_i^*(V_\ell)\dim(H)\mu_j(V_\ell)}{\mu_j(M)} \\ &= \frac{\dim(H)}{\mu_j(M)} \sum_{\ell=1}^{n} \mu_i^*(V_\ell)\mu_j(V_\ell) \\ &= \delta_{ij}\dim(H),\end{aligned}$$

so PP^* is a multiple of the identity matrix. Therefore $R(H)$ and $Z(H)$ are dual character algebras. ■

References

[1] E. Bannai and T. Ito, *Algebraic Combinatorics I. Association Schemes.* Benjamin/Cummings, 1984.
[2] P. Bantay, *Orbifolds, Hopf algebras, and the moonshine.* Lett. Math. Phys. **22**(1991), 187–194.
[3] H. I. Blau, *Quotient structures in C-algebras.* J. Algebra **175**(1995), 24–64.
[4] V. Chari and A. Pressley, *A Guide to Quantum Groups*. Cambridge University Press, 1994.
[5] C. W. Curtis and I. Reiner, *Methods of Representation Theory with Applications to Finite Groups and Orders, Volume I*. Wiley, 1981.
[6] R. Dijkgraaf, V. Pasquier, and P. Roche, *Quasi Hopf algebras, group cohomology and orbifold models.* Nuclear Physics B (Proc. Suppl.) **18B**(1990), 60–72.
[7] G. I. Kac, *Certain arithmetic properties of ring groups.* Funct. Anal. Appl. **6**(1972), 158–160.
[8] R. G. Larson, *Characters of Hopf algebras.* J. Algebra **17**(1971), 352–368.
[9] R. G. Larson and M. E. Sweedler, *An associative orthogonal bilinear form for Hopf algebras.* Amer. J. Math. **91**(1969), 75–94.
[10] M. Lorenz, *Representations of finite-dimensional Hopf algebras.* J. Algebra **188**(1997), 476–505.
[11] ———, *On the class equation for Hopf algebras.* Proc. Amer. Math. Soc. (10) **126**(1998), 2841–2844.
[12] G. Lusztig, *Leading coefficients of character values of Hecke algebras.* Proc. Sympos. Pure Math. **47**(1987), 235–262.
[13] G. E. Mason, *The quantum double of a finite group and its role in conformal field theory.* In: Groups '93, Galway/St. Andrews, London Math. Soc. Lecture Note Ser. **212**, Cambridge University Press, 1995, 405–417.
[14] A. Masuoka, *Selfdual Hopf algebras of dimension p^3 obtained by extension.* J. Algebra **178**(1995), 791–806.
[15] ———, *The p^n-theorem for semisimple Hopf algebras.* Proc. Amer. Math. Soc. **124**(1996), 735–737.
[16] S. Montgomery, *Hopf Algebras and Their Actions on Rings.* CBMS Regional Conference Series in Mathematics **82**, Amer. Math. Soc., 1993.
[17] W. D. Nichols and M. B. Richmond, *The Grothendieck group of a Hopf algebra.* J. Pure Appl. Algebra **106**(1996), 297–306.
[18] ———, *The Grothendieck algebra of a Hopf algebra, I.* Comm. Algebra (4) **26**(1998), 1081–1095.
[19] D. E. Radford, *Minimal quasitriangular Hopf algebras.* J. Algebra **157**(1993), 281–315.
[20] ———, *The trace function and Hopf algebras.* J. Algebra **163**(1994), 583–622.
[21] H.-J. Schneider, *Lectures on Hopf Algebras.* Lecture Notes, Universidad Nacional de Cordoba, 1995.
[22] Y. Sommerhäuser, *On Kaplansky's fifth conjecture.* J. Algebra (1) **204**(1998), 202–224.
[23] van der Waerden, *Moderne Algebra II.* Springer-Verlag, 1931.
[24] N. J. Wildberger, *Finite commutative hypergroups and applications from group theory to conformal field theory.* In: Applications of Hypergroups and Related Measure Algebras, Contemp. Math. **183**, Amer. Math. Soc., 1995, 413–434.

[25] S. J. Witherspoon, *The representation ring of the quantum double of a finite group.* J. Algebra **179**(1996), 305–329.

[26] Y. Zhu, *Hopf algebras of prime dimension.* Internat. Math. Res. Notices **1**(1994), 53–59.

Department of Mathematics
University of Toronto
Toronto, Ontario
M5S 3G3
email: sjw@math.toronto.edu

Current address:
Department of Mathematics
University of Wisconsin
Madison, Wisconsin 53706
USA
email: withersp@math.wisc.edu