BULL. AUSTRAL. MATH. SOC. 57M05, 20E26, 57M50, 57N35
VOL. 55 (1997) [261–279]

A 3-MANIFOLD WITH A NON-SUBGROUP-SEPARABLE FUNDAMENTAL GROUP

SABURO MATSUMOTO

We examine a 3-manifold Γ whose fundamental group is known to be non-subgroup-separable (non-LERF). We show that this manifold Γ is a graph manifold and that the subgroup known to be non-separable is not geometric. On the other hand, there are incompressible surfaces immersed in the manifold which do not lift to embeddings in any finite-degree covering space. We then prove that these *bad* incompressible surfaces must have non-empty boundary.

1. PRELIMINARIES

A group G is said to be *residually finite (RF)* if, for any $\gamma \in G\backslash\{1\}$, there exists a finite-index subgroup G_1 not containing γ. If S is a subgroup of G, then G is said to be S*-residually finite* $(S$*-RF)* if, for any $\gamma \in G\backslash S$, there exists a finite-index subgroup G_1 of G containing S but not γ. We say that G_1 *separates* S *from* γ and that S is *a separable subgroup of* G. G is called *locally extended residually finite (LERF)* if G is S-RF for every *finitely generated* subgroup S of G. LERF groups are sometimes referred to as *subgroup separable groups*. In this paper, the terms "LERF" and "subgroup separable" are used interchangeably.

If G is LERF, then so are any subgroups of G (this is obvious) and any finite extensions of G (Scott [**13**]). See Allenby and Gregorac [**1**], Magnus [**11**], and Scott [**13**] for various nice properties of these groups.

The following lemma, a direct corollary to a result in [**13**], provides an essential link between group theory and geometric topology.

LEMMA 1.1. *Let M be a topological 3-manifold with $\pi_1(M) = G$. If G is LERF, then any incompressible surface F immersed in M by $f : F \to M$ can be lifted to an embedding $f_1 : F \to \widehat{M}$, where $\widehat{M}$ is a finite-degree covering space for M.*

REMARK. By *incompressible surface*, we mean that the map f induces a π_1-injective map $f_*\colon F \to M$.

We say that a compact irreducible 3-manifold M is a *graph manifold* if each component of $M\backslash\mathcal{T}$ is a Seifert fibre space, where $\mathcal{T}$ is the family of tori in the canonical torus

Received 9th April, 1996
Research partially supported by Australian Research Council.

decomposition. Each Seifert fibre piece is often called a *vertex manifold*. In this class we include those manifolds that are obtained by gluing a pair (or pairs) of boundary torus components of one single Seifert fibre space.

In 1978, Scott showed that all surface groups, all Fuchsian groups, and the fundamental groups of Seifert fibre spaces are LERF [**13**] (also, see Hempel [**7**], Hall [**5**], Burns [**2**], and Karrass and Burns [**8**]). This, of course, implies that any circle bundle over a surface has a LERF fundamental group. However, Scott wrote, "I am unable to decide whether the same holds for bundles over S^1 with fibre a surface. It seems quite possible that this is false" (p.565).

As Scott speculated, almost ten years later, Burns, Karrass, and Solitar [**3**] found a non-LERF group which is the fundamental group of a surface bundle over S^1. This resulted in the discovery of more 3-manifolds with non-LERF fundamental groups (see Long and Niblo [**9**], for example). In this paper we shall study the example given in [**3**].

2. Non-LERF group K

We begin by presenting the group K proved non-LERF in [**3**]. Their paper specifies a finitely generated subgroup H of K and an element $\gamma \in K\backslash H$ which cannot be separated from H by a finite-index subgroup of K containing H. In their presentation,

$$K = \langle y, \alpha, \beta \mid y^{-1}\alpha y = \alpha\beta, : y^{-1}\beta y = \beta\rangle,$$
$$H = \langle \alpha^{-1},\ y\alpha^{-1}y^{-2}\alpha\rangle = \langle \alpha,\ y\alpha^{-1}y^{-2}\rangle,$$

and

$$\gamma = [y,\ \alpha^{-2}y\alpha^2].$$

Here, $[x_1, x_2]$ denotes the commutator $x_1^{-1}x_2^{-1}x_1x_2$. For details of their proof, see [**3**]. Note that, since $\beta = \alpha^{-1}y^{-1}\alpha y$, the group K can be presented by 2 elements and 1 relation instead.

This particular presentation of K suggests a non-closed 3-manifold Γ with $\pi_1(\Gamma) = K$, obtained as follows:

Let F be the punctured torus, and let $\pi_1(F) = \langle \alpha, \beta \rangle$, where α represents the longitude and β represents the meridian of F. Consider $F \times I$. Identify $F \times \{0\}$ and $F \times \{1\}$ with the homeomorphism of F prescribed by the two relations

$$y^{-1}\alpha y = \alpha\beta \quad \text{and} \quad y^{-1}\beta y = \beta.$$

The construction is shown in Figure 1. The gluing homeomorphism determined by these relations is a Dehn twist along the meridian curve β:

$$\alpha \longmapsto \alpha\beta, \qquad \beta \longmapsto \beta.$$

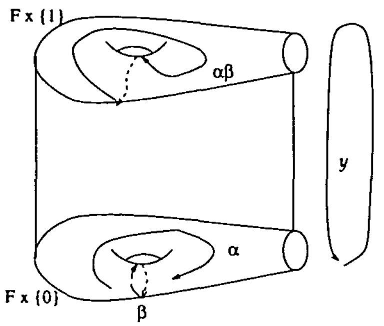

Figure 1. Non-LERF 3-manifold Γ

The resulting Γ is simply a surface bundle over S^1. Note that $\partial\Gamma$ is the torus, and if this is "capped off" in the obvious manner, we get a closed manifold Γ' having the Nil geometry [**14**].

The following lemma is useful in analysing Γ.

LEMMA 2.1. *Γ is a graph manifold consisting of one Seifert fibre space with three boundary components, two of which are identified.*

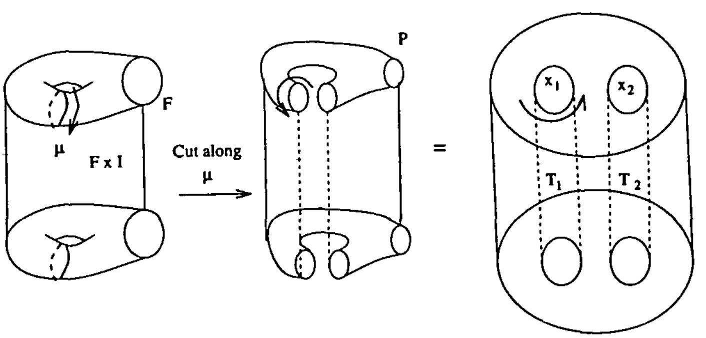

Figure 2. Γ as a graph manifold

PROOF: Let F denote the punctured torus. The description above for Γ shows that Γ is obtained from $F \times I$ by the Dehn twist along the meridian, defined by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Now, one can cut F along the meridian curve μ and get the "pair of pants" $S^2 - 3D^2$, which we denote by P. Consider $P \times S^1$, the solid torus with the interior of two disjoint

https://doi.org/10.1017/S0004972700033931 Published online by Cambridge University Press

solid tori T_1 and T_2 removed (Figure 2). For $i = 1, 2$, let

$$\pi_1(T_i) = \langle x_i \rangle \oplus \langle t_i \rangle \cong \mathbb{Z} \oplus \mathbb{Z},$$

where x_i is the meridian (horizontal in $P \times S^1$) and the t_i is the longitude (vertical in $P \times S^1$). We shall identify T_1 and T_2 in such a way that the resulting manifold is homeomorphic to Γ. As one can see in Figure 2, the Dehn twist performed to make Γ corresponds to "turning" x_1 one complete rotation and gluing the two tori together. In other words, by identifying T_1 and T_2 by the homeomorphism $h : T_1 \to T_2$ inducing

$$h_*\colon \pi_1(T_1) \to \pi_1(T_2) \quad \text{defined by}$$

$$x_1 \longmapsto x_2$$

$$t_1 \longmapsto x_2 + t_2,$$

one gets a manifold homeomorphic to Γ. The space $P \times S^1$ is clearly a Seifert fibre space, so Γ is a graph manifold. $\square$

We now give yet another presentation of K.

Lemma 2.2. *The presentation*

$$K = \langle y, \alpha, \beta \mid y^{-1}\alpha y = \alpha\beta,\ y^{-1}\beta y = \beta \rangle,$$

is Tietze equivalent to (and thus isomorphic to) the presentation

$$\langle y, y_1, \alpha \mid \alpha^{-1} y \alpha = y_1,\ [y, y_1] = 1 \rangle.$$

Proof: Simply substitute $\beta = y y_1^{-1}$, and verify that

$$\begin{aligned}
K &= \langle y, \alpha, \beta \mid y^{-1}\alpha y = \alpha\beta,\ [\beta, y] = 1 \rangle \\
&= \langle y, \alpha, y y_1^{-1} \mid y^{-1}\alpha y = \alpha y y_1^{-1}, [y y_1^{-1}, y] = 1 \rangle \\
&= \langle y, \alpha, y_1 \mid y^{-1}\alpha y = \alpha y y_1^{-1}, [y, y_1] = 1 \rangle \\
&= \langle y, y_1, \alpha \mid y^{-1}\alpha y = \alpha y_1^{-1} y, [y, y_1] = 1 \rangle \\
&= \langle y, y_1, \alpha \mid \alpha^{-1} y^{-1} \alpha = y_1^{-1}, [y, y_1] = 1 \rangle \\
&= \langle y, y_1, \alpha \mid \alpha^{-1} y \alpha = y_1, [y, y_1] = 1 \rangle
\end{aligned}$$

is indeed a sequence of Tietze transformations (and obvious substitutions and manipulations). $\square$

With this new presentation of K, we can now realise Γ in yet another way. Since y and y_1 commute in G, let T be the torus whose meridian curve is represented by y and longitude curve by y_1. Start with $T \times I$, and take the regular (2-dimensional)

https://doi.org/10.1017/S0004972700033931 Published online by Cambridge University Press

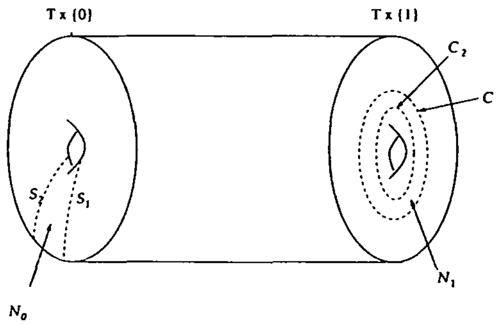

Figure 3. The two annuli to be identified

neighbourhood N_0 of the meridian curve on $T \times \{0\}$ (see Figure 3). ∂N_0 is two circles S_1 and S_2. Similarly, take the neighbourhood N_1 of the longitude curve of $T \times \{1\}$. Refer to the two boundary circles of ∂N_1 as C_1 and C_2.

Now take an annulus A with $\partial A = \partial_1 \cup \partial_2$, and construct $A \times I$. Identify $A \times \{0\}$ with N_0 and $A \times \{1\}$ with N_1 such that for $i = 1, 2$,

$$A_i = \partial_i \times I \text{ joins } S_i \text{ with } C_i.$$

Since the ∂_i are circles, the A_i are also annuli. We shall refer to these as connecting annuli (See Figure 4).

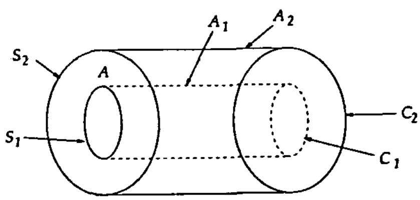

Figure 4. The connecting annuli A_1, A_2

The resulting manifold M is a 3-manifold with boundary, and $\pi_1(M) = G$, where $y, y_1 \in \pi_1(T)$, and α is a primitive loop going through the attached annuli, carrying y to y_1. Note that we may have actually constructed two distinct manifolds here, depending on how we labelled the S_i and the C_i. Indeed, Theorem 2.3 below shows that two distinct manifolds can be constructed this way and that one of them is homeomorphic to Γ. Here, we are assuming that the circles S_i are oriented in the same way, and so are the C_i. Observe that, in either way,

$$\partial M = \big((T \times \{0\}) \backslash N_0\big) \cup A_1 \cup \big((T \times \{1\}) \backslash N_1\big) \cup A_2,$$

https://doi.org/10.1017/S0004972700033931 Published online by Cambridge University Press

where A_i is the annulus connecting S_i to C_i. This is a union of four annuli making up a single torus boundary component.

THEOREM 2.3. *The above construction gives two distinct 3-manifolds, M_1 and M_2, and one of them is homeomorphic to Γ.*

PROOF: After fixing the C_i and the S_i, one gluing (taking C_1 to S_1) gives an orientable manifold while the other gluing gives a non-orientable one, so it is clear that we get two distinct manifolds. Let M_1 be the manifold constructed with the labeling given in Figure 5, and the annuli join S_1 with C_1 and S_2 with C_2. Pick a base point of M_1 as shown in Figure 5, and orient each generator curve as indicated by the arrows. We see that

$$\pi_1(M_1) = \langle \alpha, y, y_1 \mid [y, y_1] = 1, \alpha^{-1} y \alpha = y_1 \rangle \cong K,$$

as required. Consider now $\partial\Gamma \cong T^2$. $\pi_1(\partial\Gamma)$ is generated by y and $\alpha\beta\alpha^{-1}\beta^{-1}$. On the other hand, as stated before, ∂M_1 is also homeomorphic to the torus, diagrammatically shown in Figure 6. $\pi_1(\partial M_1)$ is thus generated by y and $\alpha y \alpha^{-1} y_1$.

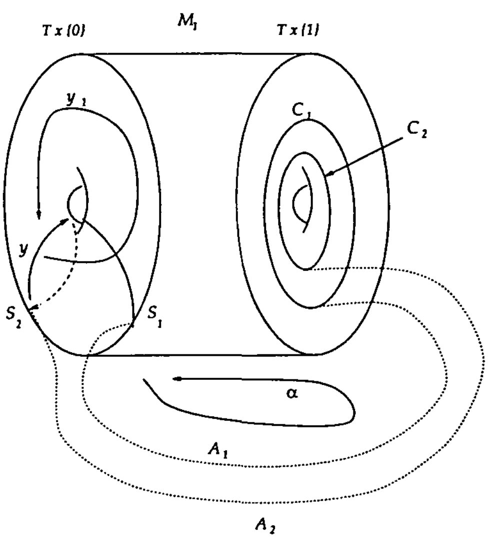

Figure 5. Γ and $\pi_1(\Gamma)$ as an HNN extension

https://doi.org/10.1017/S0004972700033931 Published online by Cambridge University Press

Using the identity $\beta = \alpha^{-1}y^{-1}\alpha y$ in K, we get

$$\begin{aligned}
\pi_1(\partial\Gamma) &= \langle y\rangle \oplus \langle \alpha\beta\alpha^{-1}\beta^{-1}\rangle \\
&= \langle y\rangle \oplus \langle \alpha\alpha^{-1}y^{-1}\alpha y\alpha^{-1}y^{-1}\alpha^{-1}y\alpha\rangle \\
&= \langle y\rangle \oplus \langle \alpha y\alpha^{-1}y^{-1}\alpha^{-1}y\alpha\rangle \\
&= \langle y\rangle \oplus \langle \alpha y\alpha^{-1}y^{-1}y_1\rangle \\
&= \langle y\rangle \oplus \langle \alpha y\alpha^{-1}y_1y^{-1}\rangle \qquad \text{since } y \text{ and } y_1 \text{ commute} \\
&= \langle y\rangle \oplus \langle \alpha y\alpha^{-1}y_1\rangle \\
&= \pi_1(\partial M_1).
\end{aligned}$$

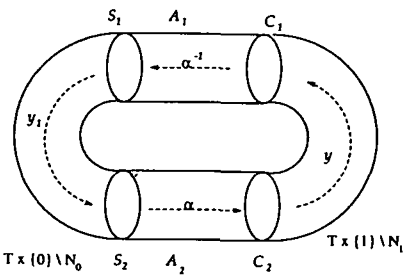

Figure 6. The boundary $\partial\Gamma$

Hence, we have a group isomorphism $\pi_1(\Gamma) \to \pi_1(M_1)$ such that it maps $\pi_1(\partial\Gamma)$ isomorphically onto $\pi_1(\partial M_1)$. By Waldhausen's Theorem, we conclude that $\Gamma \cong M_1$. ∎

3. The non-separable subgroup of K

As mentioned in Section 1, one of the implications of $\pi_1(M)$ being non-LERF is that there may be an incompressible surface S (immersed in M) not lifting to an embedding in any finite cover of M. If so, $\pi_1(S) \subset \pi_1(M)$ is non-separable. Thus, if Γ does contain such a surface S, then H is a candidate for $\pi_1(S)$. We shall see later that this is not the case (see Theorem 3.4), but before proving it, we must examine H further. The key machinery used here is the following, due to Britton [**10**].

Lemma 3.1. *Suppose G is a group with subgroups A and B, $\eta: A \to B$ is an isomorphism, and $\widehat{G}$ is an HNN extension of G defined as*

$$\widehat{G} = \langle G, t \mid t^{-1}at = \eta(a), a \in A\rangle.$$

Let $g_i \in G$ and $\varepsilon_i = \pm 1$, and suppose the word

$$w = g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n$$

(where $n \geqslant 1$) does not contain any subword $t^{-1} g_i t$, $g_i \in A$, or $t g_j t^{-1}$, $g_j \in B$. Then w is not the identity in $\widehat{G}$.

An immediate corollary is the following:

COROLLARY 3.2. *Let G and $\widehat{G}$ be defined as above, and let $g \in G$ be an element such that $g^k \notin (A \cup B)$ for all $k \neq 0$. Then, the subgroup $\langle g, t\rangle$ of $\widehat{G}$ is free of rank 2.*

PROOF: We need to show that the homomorphism

$$\Phi : \langle g, t\rangle \text{ (as a free group) } \to \langle g, t\rangle \text{ (as a subgroup of } \widehat{G}\text{)}$$

is injective. So let w be a word in the *free group* $\langle g, t\rangle$, reduced in the "free-group sense," and suppose $\Phi(w) = 1$ in $\widehat{G}$. Clearly $w \neq g^k$ $(k \neq 0)$, since $g^k \notin A \cup B$. Hence, $w = 1$ already, or w contains at least one t, so we can apply Britton's Lemma 3.1 (as $n \geqslant 1$) and conclude that w contains a subword $t^{-1} g_i t$ or $t g_j^{-1} t^{-1}$ $(g_i \in A$ or $g_j \in B)$. Now g_i, g_j must be g^k for some $k \neq 0$, since w is a reduced word in g and t. But $g^k \in A$ or B is contrary to our hypothesis. Hence, $w = 1$, and Φ is injective. ∎

COROLLARY 3.3. *H is isomorphic to the free group of rank 2.*

PROOF: Using the identity $y_1 = \alpha^{-1} y \alpha$, we see that the subgroup H is generated by α and $y\alpha^{-1} y^{-2} \alpha = y(\alpha^{-1} y \alpha)^{-2} = y y_1^{-2}$.

K is an HNN extension of the torus group generated by y and y_1. Let $G = \langle y\rangle \oplus \langle y_1\rangle$, and let $A = \langle y\rangle$, $B = \langle y_1\rangle$; define $\eta(y) = \alpha^{-1} y \alpha = y_1$. Then, $K = \widehat{G}$ in the notation of Lemma 3.1, and we can apply Corollary 3.2 since there is no $k \neq 0$ such that $(y y_1^{-2})^k \in A$ or B in G. Hence, $H = \langle y y_1^{-2}, \alpha\rangle$ is free of rank 2. ∎

Now, the classification result for compact connected surfaces gives all the surfaces S such that $\pi_1(S) \cong F_2$, the free group of rank 2. $\partial S \neq \varnothing$, so we get

$$H_0(S) = \mathbb{Z},\ \ H_1(S) = \mathbb{Z} \oplus \mathbb{Z},\ \text{ and } H_2(S) = 0,$$

and thus $\chi(S) = -1$. Hence, S is homeomorphic to the pair of pants, the punctured torus, the punctured Moebius band, or the punctured Klein bottle.

Now, we are ready to prove the main result of this section.

THEOREM 3.4. *Let Γ be the manifold described in Section 2, and present $\pi_1(\Gamma) = K$ as above. Suppose $f : S \to \Gamma$ is a proper two-sided immersion of a compact connected surface such that $f_* : \pi_1(S) \to \pi_1(\Gamma)$ is an injection. Then, $f_*(\pi_1(S))$ cannot be H.*

https://doi.org/10.1017/S0004972700033931 Published online by Cambridge University Press

REMARK. By *proper immersion*, we mean that f is a local embedding such that $f^{-1}(\partial\Gamma) = \partial S$.

PROOF: Of the four surfaces S whose fundamental group is free of rank 2, the two non-orientable ones cannot be immersed in a two-sided way. Hence, we need to consider only the following two cases.

CASE 1. First, suppose S is the pair of pants. Since f is a proper map, ∂S, which is three disjoint circles, must be mapped into the boundary torus of Γ. Let

$$f_*(a) = \overline{a}, \quad f_*(b) = \overline{b},$$

where a, b are shown in Figure 7, and $\overline{a}$, $\overline{b}$ are their images in $\pi_1(\Gamma)$. The base point of $\pi_1(S)$ is on the boundary component associated with a, so $\overline{a} \in \pi_1(\partial\Gamma)$. As for $\overline{b}$, since a conjugate of b represents a boundary component of S, $\overline{b}$ is conjugate to $\overline{c} \in \pi_1(\partial\Gamma)$ by some $g \in \pi_1(\Gamma)$. Hence, $\overline{b} = g^{-1}\overline{c}g$. Note that g may not be in $\pi_1(\partial\Gamma)$. However, we know that $[\overline{a}, \overline{c}] = 1$ since $\overline{a}, \overline{c} \in \pi_1(\partial\Gamma)$, an Abelian group. Therefore, we have $f_*(\pi_1(S)) = \langle \overline{a},\ g^{-1}\overline{c}g\rangle$, a subgroup of K.

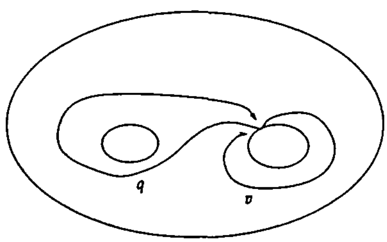

Figure 7. S, the "pair of pants"

Now, let $A\colon K \to K/K'$ be the Abelianisation map. Then, $K/K' \cong \mathbb{Z}^2$ is generated by the images of y and α, and $A(\beta) = 1$. Note that

$$A(\pi_1(\partial\Gamma)) = A\big(\langle \alpha\beta\alpha^{-1}\beta^{-1}\rangle \oplus \langle y\rangle\big) = \langle y\rangle \cong \mathbb{Z}.$$

Now, consider the image of the subgroup $f_*(\pi_1(S))$ under A in K/K'.

$$A(f_*(\pi_1(S))) = A\big(\langle \overline{a},\ g^{-1}\overline{c}g\rangle\big) = A(\langle \overline{a}, \overline{c}\rangle) \subset A(\pi_1(\partial\Gamma)) = \langle y\rangle.$$

On the other hand,

$$A(H) = A\big(\langle \alpha,\ y\alpha^{-1}y^{-2}\rangle\big) = A(\langle \alpha, y\rangle) = \langle \alpha\rangle \oplus \langle y\rangle.$$

Hence, $f_*(\pi_1(S)) \neq H$ as their images under A are not even isomorphic.

CASE 2. Next, suppose S is the punctured torus. Let $\pi_1(S) = \langle a,\ b\rangle$, and so $\pi_1(\partial S) = \langle aba^{-1}b^{-1}\rangle$. Since $\pi_1(\partial S)$ lies in the commutator subgroup of $\pi_1(S)$, its image $f_*(\pi_1(\partial S)) \subset \pi_1(\partial\Gamma) \cap K' = \langle \alpha\beta\alpha^{-1}\beta^{-1}\rangle$.

Now, consider a new (closed) 3-manifold N obtained by "capping off" the boundary torus of Γ in such a way that $[\alpha, \beta]$ gets "killed" in $\pi_1(N)$. In other words, we attach a solid torus on $\partial\Gamma$ so that

$$\pi_1(N) = \pi_1(\Gamma)/\langle[\alpha, \beta] = 1\rangle.$$

N is a torus bundle over S^1, obtained by filling in the "puncture" of the fibre of Γ. Now,

$$\pi_1(N) = \langle y, \alpha, \beta \mid y^{-1}\alpha y = \alpha\beta,\ y^{-1}\beta y = \beta,\ [\alpha, \beta] = 1\rangle,$$

so β commutes with everything. As S is mapped properly to Γ, this "capping off" process will close up $f(S)$, and $\pi_1(S)$ will become $\pi_1(S \cup D^2) = \pi_1(T^2)$. As the result, we have the following commutative diagram:

$$\begin{tikzcd}
\pi_1(S) \arrow[r, "f_*"] \arrow[d, "c_1"'] & \pi_1(\Gamma) \arrow[d, "c_2"] \\
\pi_1(T^2) \arrow[r, "j_*"] & \pi_1(N)\ (\mathrm{Nil})
\end{tikzcd}$$

where j_* is induced by f_*, and c_1, c_2 are induced by the capping operation. Note that c_1, c_2 are surjective. Now, $\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$, so $j_* \cdot c_1(\pi_1(S))$ is an Abelian subgroup of $\pi_1(N)$. Thus, $c_2 \cdot f_*(\pi_1(S))$ is also Abelian.

Now, suppose $H = \langle \alpha,\ y\alpha^{-1}y^{-2}\rangle = f_*(\pi_1(S))$ in $\pi_1(\Gamma)$. Then, clearly $c_2(H)$ must be Abelian as well, that is, $[\alpha, y\alpha^{-1}y^{-2}] = 1$ in $\pi_1(N)$ with the additional relation.

Observe that the question is now reduced to the algebraic problem of H : whether or not H becomes Abelian when the relation $[\alpha, \beta] = 1$ is added. Denote $\pi_1(N)$ by $\overline{K}$. If $H = f_*(\pi_1(S))$, then $c_2(H)$ is Abelian, and we have

$$[\alpha,\ y\alpha^{-1}y^{-2}] = \alpha^{-1}y^2\alpha y^{-1}\alpha y\alpha^{-1}y^{-2} = 1$$

in $\overline{K}$. Recall that in $\overline{K}$, β commutes with both α and y. So we get

$$\begin{aligned}
\beta = \beta \cdot 1 &= (\beta)\alpha^{-1}y^2\alpha y^{-1}\alpha y\alpha^{-1}y^{-2} \\
&= \alpha^{-1}y^2\alpha(\beta)y^{-1}\alpha y\alpha^{-1}y^{-2} \\
&= \alpha^{-1}y^2\alpha(\alpha^{-1}y^{-1}\alpha y)y^{-1}\alpha y\alpha^{-1}y^{-2} \\
&= \alpha^{-1}y\alpha^2 y\alpha^{-1}y^{-2}.
\end{aligned}$$

https://doi.org/10.1017/S0004972700033931 Published online by Cambridge University Press

Now, since $\beta^{-1} = y^{-1}\alpha^{-1}y\alpha$, commuting with α and y, we have

$$\begin{aligned} 1 = \beta\beta^{-1} &= \alpha^{-1}y\alpha^2 y(\beta^{-1})\alpha^{-1}y^{-2} \\ &= \alpha^{-1}y\alpha^2 y(y^{-1}\alpha^{-1}y\alpha)\alpha^{-1}y^{-2} \\ &= \alpha^{-1}y\alpha y^{-1}. \end{aligned}$$

By taking the inverse, we get $1 = y\alpha^{-1}y^{-1}\alpha$, and thus $y^{-1}\alpha y = \alpha$. But $y^{-1}\alpha y = \alpha\beta$, so these two equations imply that $\beta = 1$, which is impossible since β is the meridian of the fibre $T^2 - D^2$ in Γ, which is not capped off by the process. Hence, H cannot be the subgroup $f_*(\pi_1(S))$ in $\pi_1(\Gamma)$ when S is the punctured torus. This completes the proof of Theorem 3.4. ☐

Actually the argument of Case 1 can be modified for the punctured Moebius band case, also. However, we are not very much concerned with this case since the punctured Moebius band would be one-sided.

4. Separability of closed surfaces in Γ

A natural question to ask next is the following: does every incompressible immersion of a surface in Γ lift to an embedding in some finite cover of Γ? We begin with the following definition.

Definition 4.1: Let $f\colon S \to M$ be a proper, π_1-injective immersion of a connected surface S into a 3-manifold M. S is called *separable* if there is a finite cover $\widehat{M}$ of M such that f lifts to $\hat{f}\colon S \to \widehat{M}$ as an embedding.

We shall assume that all such immersions are least-area maps in their homotopy classes. Let us first consider closed surfaces. We begin with the following, somewhat surprising lemma.

Lemma 4.2. *Suppose $f\colon S \to \Gamma$ is a π_1-injective proper immersion of a connected closed orientable surface into Γ, and S is not S^2. Then, S must be the torus.*

Proof: We use the graph-manifold description of Γ (Lemma 2.1 and Figure 2). Cut open Γ along the glued torus T to get $\Gamma' = P \times S^1$, where P is the pair of pants. Define $f'\colon S' \to \Gamma'$, where $S' = S - f^{-1}(T)$, possibly disconnected. Each component of $f'(S')$ is π_1-injective (by least area) in Γ', so assume it is either horizontal or vertical in Γ'. But if a component is horizontal, it must be transverse to every fibre of Γ', meeting every component of $\partial\Gamma'$. This contradicts the hypothesis that S is a proper immersion of a *closed* surface. Hence, each component is vertical. This implies that $f(S)$ is disjoint from T since the gluing does not preserve the vertical fibres. Hence, $S = S'$ and is immersed in Γ', and it is a union of Seifert fibres; therefore, it must be the torus. ☐

https://doi.org/10.1017/S0004972700033931 Published online by Cambridge University Press

This lemma means that $f(S)$ is a product of some closed (but not necessarily simple) curve c on the fibre P with an n-degree covering of S^1. (c represents the meridian of S). We now state one of the main theorems about surfaces in Γ.

THEOREM 4.3. *Suppose $f: S \to \Gamma$ is a π_1-injective proper immersion of an orientable closed connected surface S into Γ. Then, S is separable.*

PROOF: The result is trivial if $S = S^2$, so by Lemma 4.2, we may suppose that S is the torus. This time, use the HNN-extension model of Γ (Theorem 2.3 and Figure 5). Denote $T \times \{0\}$ by T_0 and $T \times \{1\}$ by T_1. With this description (Figure 3 and Figure 4), we use the term "connecting annulus" A to refer to the thickened annulus $A \times I$ connecting the 2-dimensional neighbourhoods N_0 of μ in T_0 and N_1 of λ in T_1. Let us refer to $T \times I$ as Γ'. This is obtained after Γ is cut along the middle of the connecting annulus A. We have the following two cases.

CASE 1. $S \cap A = \varnothing$. Here, $S \subset \Gamma'$. In this case, S can be homotoped to cover the torus $T \times \{1/2\}$ in Γ'. Let $\rho: S \to T \times \{1/2\}$ be this covering. $\rho_*(\pi_1(S))$ is some $\mathbb{Z} \oplus \mathbb{Z}$ subgroup in $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$. Then, since $\mathbb{Z} \oplus \mathbb{Z}$ is LERF, there exists a finite cover $\widehat{\Gamma'}$ of $\Gamma' = T \times I$ in which ρ lifts to $\widehat{\rho}: S \to \widehat{\Gamma'}$ as an embedding. Let $\partial\widehat{\Gamma'} = T_0' \sqcup T_1'$. There are some m_0 and l_0, positive integers, such that on T_0' we see m_0 copies of the meridian curves $\mu_1', \mu_2', \ldots, \mu_{m_0}'$, each covering the original meridian $\mu \subset T_0$ with some degree l_0. On T_1, then, there are l_1 copies of the longitude curves $\lambda_1', \lambda_2', \ldots, \lambda_{l_1}'$, each covering the original $\lambda \subset T_1$ with degree m_1 (and we have $m_1 l_1 = m_0 l_0$). Next, we create the "dual space" $\widehat{\Gamma^*}$, a covering space of Γ' just like $\widehat{\Gamma'}$ except the meridian and the longitude are reversed. Let $\partial\widehat{\Gamma^*} = T_0^* \sqcup T_1^*$. As before, T_0^* has l_1 copies of the meridian curves $\mu_1^*, \ldots, \mu_{l_1}^*$, each covering μ with degree m_1, and T_1^* has m_0 copies of the longitude curves $\lambda_1^*, \ldots, \lambda_{m_0}^*$, each covering λ with degree l_0. These degrees on $\partial\widehat{\Gamma'}$ and $\partial\widehat{\Gamma^*}$ match up, so we can join the μ_i' with the λ_i^* (m_0 of these), and the λ_i' with the μ_i^* (l_1 of these). The result, $\widehat{\Gamma}$, is a finite covering space for Γ in which S is embedded.

This procedure will be used in Case 2 as well. In fact, it is highly useful to describe this process using graphs with edges and half-edges. $\widehat{\Gamma'}$ can be represented by a finite graph with one vertex v (for $\widehat{\Gamma'}$), m_0 incoming half-edges labelled l_0 (for $\mu_1', \mu_2', \ldots, \mu_{m_0}'$), and l_1 outgoing half-edges labelled m_1 (for $\lambda_1', \ldots \lambda_{l_1}'$). Note that the incoming half-edges represent the curves on $\partial\widehat{\Gamma'}$ covering the meridian of $\partial\Gamma'$ with the labelled degrees, and the outgoing half-edges represent the curves on $\partial\widehat{\Gamma'}$ covering the longitude of $\partial\Gamma'$ with the labelled degrees. In this way, each covering space $\widehat{\Gamma'}$ of Γ' can be represented by a vertex with an appropriate number of labelled, directed half-edges (see Figure 8 for $\widehat{\Gamma'}$ when $m_0 = m_1 = 3$, $l_0 = l_1 = 2$).

Note that the dual space $\widehat{\Gamma^*}$ has the graph representation where all the edges are

https://doi.org/10.1017/S0004972700033931 Published online by Cambridge University Press

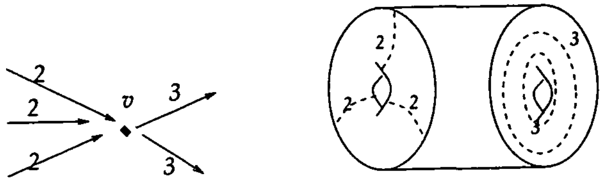

Figure 8. $\widehat{\Gamma'}$ when $m_0 = m_1 = 3$, $l_0 = l_1 = 2$

reversed; therefore, we can connect the half-edges to "close up" the space (see Figure 9) and obtain a covering space $\widehat{\Gamma}$ of Γ. Note that the surface S is embedded entirely in $\widehat{\Gamma'}$, and the sole purpose for constructing $\widehat{\Gamma^*}$ is to close up the space $\widehat{\Gamma'}$ so that $\widehat{\Gamma}$ becomes a covering space of Γ. The covering projection $\widehat{\Gamma} \to \Gamma$ is an extension of the covering $\widehat{\Gamma'} \to \Gamma'$; the covering $\widehat{\Gamma^*} \to \Gamma'$ is defined naturally as the dual of $\widehat{\Gamma'}$, and the degree-m_i (and -l_i) maps of the annuli joining $\widehat{\Gamma'}$ and $\widehat{\Gamma^*}$ cover the connecting annuli A_i (see Figure 4), making the map $\widehat{\Gamma} \to \Gamma$ a covering projection.

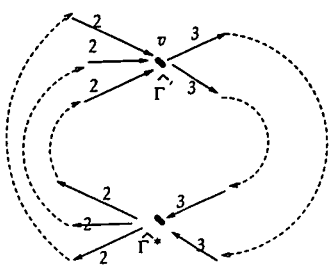

Figure 9. Covering Space $\widehat{\Gamma}$ of Γ

CASE 2. $S \cap A \neq \varnothing$. Assume that f is transverse to A. Let $S' = S - f^{-1}(A)$, and cut Γ open along A to get $f'\colon S' \to \Gamma'$. Observe that we can now identify A as $N_0 \subset T_0$ and $N_1 \subset T_1$ as in Section 2. Now, $f^{-1}(A)$ is a 1-manifold on S, so by making both A and f least area in their homotopy classes (so none of the circles are trivial), we may assume $f^{-1}(A)$ is a disjoint union of essential circles of the torus S. Each component C of S' is immersed properly and π_1-injectively in Γ', and $\partial C \subset A$. All these circles are parallel in the torus S, implying that each C is an annulus. Denote the two boundary components of the annulus C by $\partial_0 C$ and $\partial_1 C$. $f(\partial_0 C)$ and $f(\partial_1 C)$ are in the same component of $\partial\Gamma'$ since otherwise C would be an annulus in Γ' joining λ to μ in $\pi_1(\Gamma')$, which is impossible. Suppose for now that $f(\partial_i C)$ are both homotopic to $\mu^d \in \pi_1(T_0)$ for some d. Again, by least area, C cannot be homotopic rel ∂C into $A = N_0$. So C must go around T_0 at least once in the longitude direction. Hence, C is an annulus "beginning" at $\partial_0 C$ on $N_0 \subset T_0$, going "around" T (in $T \times I = \Gamma'$) a number of times, and "ending" at $\partial_1 C$, also on $N_0 \subset T_0$. Clearly, a similar condition holds if the boundary curves of C are in $N_1 \subset T_1$. In fact, as S is a torus, these must match together to close up. So $S' = S \backslash f^{-1}(A)$ consists of an even number of annuli, where the annuli have their boundary circles in T_0 and T_1 alternatingly. Let us refer to those annuli with boundary in N_0 ("the left-hand side" of A) as the "meridian annuli," and the other ones on N_1 as the "longitude annuli." We can then write $S' = \bigcup C_i$, where an odd index indicates a meridian annulus and an even index indicates a longitude annulus.

We can describe each component C_i of S' by a pair (m_i, l_i) as follows. Suppose C_i is a meridian annulus. Then, m_i is the integer such that each component of $f(\partial C_i)$ covers the meridian of the fibre torus T of Γ' by the degree-m_i cover, and l_i indicates how many times the I-factor of $C_i \cong S^1 \times I$ goes around the longitude of T. If C_i is a longitude annulus, one can similarly describe it by (m_i, l_i); each component of $f(\partial C_i)$ covers the longitude of T by the degree-l_i cover, and the I-factor of C_i goes around the meridian m_i times.

We now return to the construction of a finite-degree cover $\widehat{\Gamma}$ of Γ. The procedure is similar to that of Case 1. Say there are $2k$ components of $S \backslash f^{-1}(A)$. Since $C_1, C_3, \ldots$ are the meridian annuli while $C_2, C_4, \ldots$ are the longitude ones, we must have $m_1 = l_2 = m_3 = l_4 = \cdots = l_{2k}$. Let this number be $d \in \mathbb{Z}^+$ (for "degree").

Now, define the (x, y)-cover of T to mean the covering torus T' defined by the map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_x \oplus \mathbb{Z}_y$, sending each generator to the corresponding generator. For C_i, described by (m_i, l_i), where i is odd, take Γ'_i to be $T'_i \times I$, where T'_i is the $(d, l_i + 1)$-cover of T. We use $l_i + 1$ so that we can be sure that C_i is embedded in Γ'_i. C_i enters Γ'_i at one component of the pre-image of μ and leaves Γ'_i at another, which will later enable us to join the embedded C_i together. For C_i, i even, let $\Gamma'_i = T'_i \times I$, where T'_i is the $(m_i + 1, d)$-cover of T. Again, C_i embeds in Γ'_i. We obtain $2k$ distinct covering spaces of Γ', possibly of different degrees. But the embedded C_i can be connected by appropriate covers A_i of A because $d = m_1 = l_2 = \cdots = l_{2k}$. So, the $2k$ covering spaces Γ'_i of Γ', joined together by the A_i, contain an embedded copy of S.

We now need to make this object into a covering space for Γ by "closing up" those pre-images of N_0 and N_1. Again, we reverse the longitude and the meridian of Γ'_i to create the covering spaces Γ^*_i of Γ', dual to Γ'_i. Now, $\bigcup_{i=1}^{2k} \Gamma'_i$ can be represented by the graph seen in Figure 10, where the vertex v_i represents Γ'_i, the cover of Γ' having the right number of incoming and outgoing edges of appropriate degrees and two of the edges are connected to the adjacent vertices v_{i-1} and v_{i+1} just as the A_i connect the C_i. The fact that the A_i join the C_i to complete the embedded torus S is reflected by the fact that there is a complete circle connecting the v_i by edges labelled d. We

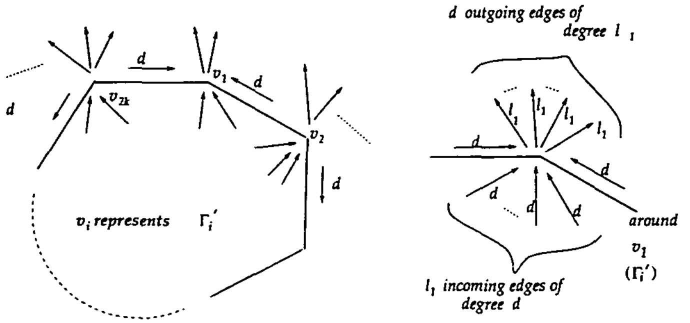

Figure 10. Space containing S as an embedded surface

may have many half-edges now. But once again, the dual spaces Γ_i^* are represented by the same graph except that the orientations of the edges are reversed. Join the Γ_i^* $(1 \leqslant i \leqslant 2k)$ exactly as the Γ_i' so that the Γ_i^* also makes a circle. We can now pair up the "open" half-edges of Γ_i' with those of Γ_i^* to create $\widehat{\Gamma}$, a covering space of Γ, since the degrees, orientations, and the number of the half-edges match up by construction. S is embedded in $\bigcup \Gamma_i' \subset \widehat{\Gamma}$, so $\widehat{\Gamma}$ is a desired finite cover of Γ. Incidentally, the degree of $\widehat{\Gamma}$ over Γ is $2d\Big(2k + \sum\limits_{i\,\mathrm{odd}}^{2k} l_i + \sum\limits_{i\,\mathrm{even}}^{2k} m_i\Big)$. ∎

5. Separability of non-closed surfaces in Γ

The next question is for properly immersed incompressible surfaces with boundary. If $f\colon S \to M$ is a proper immersion of a compact orientable surface into a graph manifold $M = \bigcup M_i$, we say S is *horizontal* if $f(S) \cap M_i$ is horizontal or empty for all i. Of course, many immersed surfaces are neither horizontal nor vertical, but we can assume that each non-empty intersection $f(S) \cap M_i$ is either horizontal or vertical (up to homotopy) by the theory of least-area maps (see Hass and Scott [6] and Freedman, Hass, and Scott [4]). We now need one more definition to state the crucial criterion proved by Rubinstein and Wang [**12**].

Definition 5.1: Suppose $f\colon S \to M$ is a π_1-injective horizontal proper immersion into a graph manifold M, where $M = \bigcup M_i$, glued along a family $\mathcal{T}$ of tori. Assume transversality so that $f^{-1}(\mathcal{T})$ is a family $\mathcal{C}$ of disjoint simple closed curves c on S. Let γ be an oriented simple closed curve on S, parameterised by $t \in [0, 1]$. As t increases, γ can meet $f^{-1}(\mathcal{T})$ at $c \in \mathcal{C}$. Say at c, $f(\gamma)$ goes from M_i to M_j, where $f_i(c) = f(c)$ on ∂M_i and $f_j(c) = f(c)$ on ∂M_j. Then, define

$$\rho_c = \frac{|t_i \cap f_i(c)|}{|t_j \cap f_j(c)|},$$

https://doi.org/10.1017/S0004972700033931 Published online by Cambridge University Press

where t_i (and t_j) is a regular fibre of M_i (and of M_j). If γ intersects $\mathcal{C}$ at $c_1, c_2, \ldots, c_k$ in that order, then define

$$s_\gamma = \prod_{i=1}^{k} \rho_{c_i}.$$

Hence, for any $c \in \mathcal{C}$ intersecting γ, ρ_c is a ratio: the numerator measures how many times $f_i(c)$ intersects a regular fibre of ∂M_i and the denominator similarly measures how many times $f_j(c)$ intersects a regular fibre of ∂M_j, the "other side." For γ not intersecting $\mathcal{C}$, define $s_\gamma = 1$. For every oriented simple closed curve $\gamma \subset S$, then, s_γ is well-defined.

As γ goes from M_i to M_j at $c \in \mathcal{C}$, $f_i(c)$ is often represented (up to homotopy) by a pair of coordinates $(\alpha,\ \beta)$ on ∂M_i. The corresponding coordinates $(\alpha',\ \beta')$ for $f_j(c)$ on ∂M_j can be obtained by

$$(1) \qquad \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

where the 2×2 matrix represents the gluing. If we use the convention that the first coordinate is the meridian (horizontal), then $|t_i \cap f_i(c)| = \alpha$, and $|t_j \cap f_j(c)| = \alpha'$, making $\rho_c = \alpha/\alpha'$. Then, we can write

$$s_\gamma = \prod_{i=1}^{k} \frac{\alpha_i}{\alpha_i'},$$

and the β_i and β_i' are irrelevant.

Now, we are ready to state Rubinstein and Wang's criterion [**12**]:

Theorem 5.2. *Suppose $f\colon S \to M$ is a π_1-injective proper horizontal immersion of a compact connected surface S into a graph manifold M. Then, S is separable if and only if $s_\gamma = 1$ for each oriented simple closed curve $\gamma \subset S$.*

We shall soon use this criterion to construct a surfaces not separable in Γ; however, the construction requires some more definitions.

Definition 5.3: Let $f\colon S \to M$ be a proper map of a connected surface (not D^2) into some 3-manifold. S is called *arc-incompressible* if the following holds: suppose there is a disk D in M with the boundary consisting of a proper arc δ in $f(S)$ and an arc γ in ∂M sharing the same endpoints such that $D \cap f(S) = \delta$. Then, there is a path δ' in $f(\partial S) \subset \partial M$ such that $\delta \cup \delta'$ bounds a disc in S.

If S is not arc-incompressible, S is said to be *arc-compressible*, and the disc D such that $\partial D = \delta \cup \gamma$ is called a *compressing disc.*

We shall use this to prove the following.

https://doi.org/10.1017/S0004972700033931 Published online by Cambridge University Press

Lemma 5.4. *Suppose $M = \bigcup_{\mathcal{T}} M_i$ is a graph manifold. Let $f: S \to M$ be a proper immersion of a connected compact surface. If each component of $S - f^{-1}(\mathcal{T})$ is a π_1-injective horizontal surface in the corresponding M_i, then S is also a π_1-injective horizontal surface in M.*

Proof: By definition, S is horizontal. Suppose S fails to be π_1-injective in M. Then, there is a compressing disc D in M. But D cannot be disjoint from all the tori in $\mathcal{T}$ since that would make a component of $S \backslash f^{-1}(\mathcal{T})$ compressible in M_i, contrary to the hypothesis. So the compressing disc D intersects some torus $T \in \mathcal{T}$. In D, $D \cap (\bigcup \mathcal{T})$ is a disjoint collection of proper arcs and circles. But by least area of the map and the tori in $\mathcal{T}$, there are no circles and that any outermost arc represents a compressing disc D_i. Now, a horizontal surface C_i lifts to an embedding $e_i: C_i \to C_i \times S^1 = \widehat{M_i}$ in a finite cover $\widehat{M_i}$ of M_i, and D_i also lifts to a disc $\widehat{D_i}$ in $C_i \times S^1$. But this implies that C_i is arc-compressible in $C_i \times S^1$, which is absurd. □

To give an explicit description of a surface which is not separable in Γ, we shall need the following construction. Let $M = F \times S^1$, where F is some compact surface. Suppose $f: S \to M$ is a proper immersion such that $f(S) \subset F \times \{0\}$ ("really horizontal"), and let $\gamma \subset f(S)$ be a simple closed curve or a simple proper arc in $f(S)$. Then, by the *Dehn twist along γ of degree d* ($d \in \mathbb{Z}$), we mean the following modification of f to obtain $f': S \to M$.

Take a regular neighbourhood $\gamma \times I$ of γ, with two boundary components γ_0 and γ_1. Fix a point x of γ, and let δ_x be the path $x \times I$ from $(x, 0)$ to $(x, 1)$ in $F \times \{0\}$. Now, replace δ_x with δ'_x, the path from $(x, 0)$, traveling in the S^1-direction d times and terminating at $(x, 1)$. This is also referred to as a "d-floor staircase construction" for an obvious reason. Now, replace $\gamma \times I \subset F \times \{0\}$ with $\bigcup_{x \in \gamma} \delta'_x$ (homeomorphic to $\gamma \times I$), and take this new image to be $f'(S)$. This resulting map f' is still horizontal because the image of the immersed surface is still transverse to the fibres. Being horizontal, f' is still π_1-injective.

Theorem 5.5. *There is a proper immersion $f: S \to \Gamma$, π_1-injective, where S is a connected orientable surface and is not separable.*

Proof: $S = S_1 \cup S_2$ is constructed as follows. Let P denote the pair of pants described as a disk with two holes labelled T_x and T_y, corresponding to the generators x and y of $\pi_1(P)$. Let S_1 be the double cover of P defined by the homomorphism $\Theta: \pi_1(P) = \langle x, y \rangle \to \mathbb{Z}_2$ sending $x \longmapsto 0$ and $y \longmapsto 1$. This defines $p_1: S_1 \to P$ such that $p_1^{-1}(x)$ is two disjoint circles each covering x by degree 1, and $p_1^{-1}(y)$ is one circle covering y by degree 2. This map p_1 describes a really horizontal surface in $\Gamma' = P \times S^1$ (as described in Figure 2) so these pre-image circles have coordinates $(1,0), (1,0)$ for $p_1^{-1}(x)$ and $(2,0)$ for $p_1^{-1}(y)$. Now, do the Dehn twist of degree 1 as shown (Figure

11) along two paths of S_1. Now the coordinates are $(1,0),(1,1)$, and $(2,1)$, where the first one is the circle left untouched in Figure 11. For S_2, reverse x and y. Define $p_2: S_2 \to P$ by $\Theta': \pi_1(P) \to \mathbb{Z}_2$ sending $x \longmapsto 1$, and $y \longmapsto 0$. Do the Dehn twist as shown (Figure 12) of degree -1 to get the coordinates $(2,-1)$ for $p_2^{-1}(x)$ and $(1,0)$ and $(1,-1)$ for $p_2^{-1}(y)$. Now, in $P \times S^1$, we see two surfaces S_i, $i = 1,2$ with coordinates shown (Figure 13).

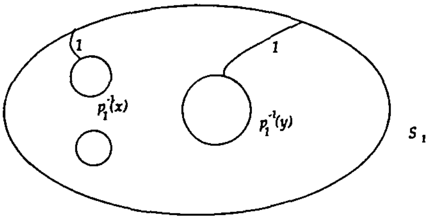

Figure 11. S_1 in the construction

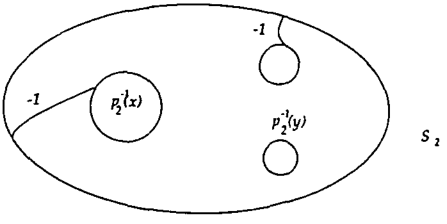

Figure 12. S_2 in the construction

Immerse $S = S_1 \cup S_2$ such that the circle $(1,1)$ is glued (correctly) to $(2,1)$ by $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $(1,0)$ is glued to $(1,0)$, and $(2,-1)$ is glued to $(1,-1)$. Note that this surface is the connected sum of two punctured tori properly immersed in Γ. This immersion is π_1-injective by Lemma 5.2 above.

Consider γ shown in Figure 13. $s_\gamma = 1/2$ (or 2, depending on the orientation of γ). By Rubinstein and Wang's criterion (Theorem 5.2), S is not separable. $\square$

It is clear that the criterion gives many other horizontal surfaces that are not separable in Γ. All one needs to do is to construct surfaces similar to the one in Theorem 5.5 with a loop γ such that $s_\gamma \neq 1$. This suggests that graph manifolds have many immersed surfaces that are not separable in them.

References

[1] R.B.J.T. Allenby and R.J. Gregorac, 'On locally extended residually finite groups', in

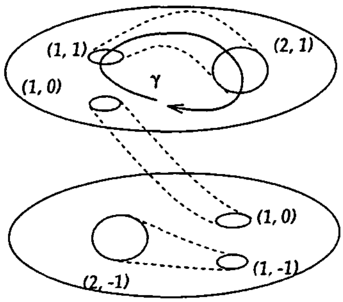

Figure 13. Non-separable surface in Γ

Conference on group theory, Lecture Notes in Mathematics **319** (Springer-Verlag, Berlin, Heidelberg, New York, 1973), pp. 9–17.

[2] R.G. Burns, 'A note on free groups', *Proc. Amer. Math. Soc.* **23** (1969), 14–17.

[3] R.G. Burns, A. Karrass and D. Solitar, 'A note on groups with separable finitely generated subgroups', *Bull. Austral. Math. Soc.* **36** (1987), 153–160.

[4] M. Freedman, J. Hass and P. Scott, 'Least area incompressible surfaces in 3-Manifolds', *Invent. Math.* **71** (1983), 609–642.

[5] M. Hall, 'Coset representations in free groups', *Trans. Amer. Math. Soc.* **67** (1949), 421–432.

[6] J. Hass and P. Scott, 'The existence of least area surfaces in 3-Manifolds', *Trans. Amer. Math. Soc.* **310** (1988), 87–114.

[7] J. Hempel, 'Residual finiteness of surface groups', *Proc. Amer. Math. Soc.* **32** (1972), 323.

[8] A. Karrass and D. Solitar, 'On finitely generated subgroups of a free group', *Proc. Amer. Math. Soc.* **22** (1969), 209–213.

[9] D. Long and G.A. Niblo, 'Subgroup separability and 3-manifold groups', *Math. Z.* **207** (1991), 209–215.

[10] R. Lyndon and P. Schupp, *Combinatorial group theory* (Springer-Verlag, Berlin, Heidelberg, New York, 1977).

[11] W. Magnus, 'Residually finite groups', *Bull. Amer. Math. Soc.* **75** (1969), 305–315.

[12] J.H. Rubinstein and S. Wang, 'On π_1-injective surfaces in graph manifolds', (preprint).

[13] P. Scott, 'Subgroups of surface groups are almost geometric', *J. London Math. Soc. (2)* **17** (1978), 555–565.

[14] P. Scott, 'The geometries of 3-manifolds', *Bull. London Math. Soc.* **15** (1983), 401–487.

Department of Mathematics
University of Melbourne
Parkville Vic 3052
Australia
e-mail: saburo@maths.mu.oz.au