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COMPLETE CONVERGENCE FOR ARRAYS AND THE LAW OF THE SINGLE LOGARITHM

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Abstract

The present paper is devoted to complete convergence and the strong law of large numbers under moment conditions near those of the law of the single logarithm (LSL) for independent and identically distributed arrays. More precisely, we investigate limit theorems under moment conditions which are stronger than $2p$ for any $p < 2$, in which case we know that there is almost sure convergence to 0, and weaker than $E\,X^4/(\log^+|X|)^2 < \infty$, in which case the LSL holds.

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1. Introduction and main result

Let $\{S_n,\ n \geq 1\}$ be the partial sums of independent and identically distributed (i.i.d.) random variables $\{X_k,\ k \geq 1\}$. As a complement to Kolmogorov's strong law of large numbers, Hsu and Robbins [12] introduced the concept of *complete convergence*, denoted $\overset{\text{c.c.}}{\to}$, and proved that the sequence of arithmetic means $\{S_n/n,\ n \geq 1\}$ converges completely, that is, $\sum_{n=1}^{\infty} P(|S_n| > n\varepsilon) < \infty$ for all $\varepsilon > 0$, if the mean is zero and the variance is finite. The necessity was proved by Erdős [3, 4].

If only the mean is finite this provides a nice example when the second Borel–Cantelli sum diverges, but $P(|S_n| > n\varepsilon \text{ i.o.}) = 0$ for all $\varepsilon > 0$. Here, as usual, i.o. stands for infinitely often.

For arrays the situation is different. Suppose that $\{(X_{n,k},\ 1 \leq k \leq n),\ n \geq 1\}$ is an array of i.i.d. random variables (in contrast to an array of row-wise i.i.d. random variables). Let X stand for a generic random variable, independent of, and with the same distribution as, those of the array. In this setting, it was shown in [5, Theorem 3.1] that the strong law holds if and only if complete convergence holds, and if and only if the variance is finite. For a precise statement, including the Marcinkiewicz–Zygmund version, see Theorem 1.1 below.

The 'explanation' for the discrepancy is that a necessary condition is that all random variables in the sequence/array must be small for the strong law in question to hold. For sequences this amounts to the requirement that $\sum_{n=1}^{\infty} P(|X| > n\varepsilon) < \infty$, that is, finite mean, whereas for arrays the requirement amounts to

$$\sum_{n=1}^{\infty} P\Big(\max_{1\le k\le n} |X_{n,k}| > n\varepsilon\Big) < \infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty}\sum_{k=1}^{n} P(|X_{n,k}| > n\varepsilon) = \sum_{n=1}^{\infty} n\, P(|X| > n\varepsilon) < \infty,$$

which, in turn, holds if and only if $E\,X^2 < \infty$.

The present paper is devoted to limit laws near the counterpart to the law of the iterated logarithm (LIL) for sequences, namely, the law of the single logarithm (LSL) for i.i.d. arrays. For sequences, Hartman and Wintner [10] proved the law under the assumption of finite variance. Here the 'trivial' necessary condition is, again, that all summands must be small, which yields $E\,X^2/\log^+\log^+|X| < \infty$. Here and throughout, $\log^+ x = \max\{\log x, 1\}$. However, as was proved by Strassen [16], finite variance is a necessary condition. For an overview on the LIL, see [1].

Concerning the LIL for arrays, see [17] and, more recently, [11]. For i.i.d. arrays, Qi [14] proved an LSL. As for the strong law, the necessary condition comes 'for free', in the sense that all summands being small amounts to

$$\sum_{n\ge 3}\sum_{k=1}^{n} P(|X_{n,k}| > \varepsilon\,\sqrt{n\log n}) < \infty \quad \Longleftrightarrow \quad \sum_{n\ge 3} nP(|X| > \varepsilon\,\sqrt{n\log n}) < \infty,$$

which holds if and only if $E|X|^4/(\log^+|X|)^2 < \infty$.

Next we recall the Marcinkiewicz–Zygmund strong law. From [5, Theorem 3.1], complete convergence and almost sure convergence in the strong laws coincide for i.i.d. arrays, in contrast to what is known for sequences (see [3, 4, 12] and [7, Ch. 6]).

Theorem 1.1. *Suppose that* $\{(X_{n,k},\ 1\le k\le n),\ n\ge 1\}$ *is an array of i.i.d. random variables and set* $S_n = \sum_{k=1}^{n} X_{n,k}$, $n \ge 1$. *Let* $0 < p < 2$. *The following are equivalent:*

(1) $E|X|^{2p} < \infty$, $0 < p < 2$, *and* $E\,X = 0$ *when* $E|X| < \infty$;

(2) $n^{-1/p} S_n \to 0$ *completely as* $n \to \infty$;

(3) $\sum_{n=1}^{\infty} P(|S_n| > n^{1/p}\varepsilon) < \infty$ *for all* $\varepsilon > 0$;

(4) $n^{-1/p} S_n \overset{\text{a.s.}}{\to} 0$ *as* $n \to \infty$.

Remark 1.2. If the array is only row-wise independent, then the first two statements are equivalent and they imply the last one.

Our aim is to investigate possible limit theorems under moment conditions which are stronger than $2p$ for any $p < 2$, in which case we know that there is almost sure convergence to 0, and weaker than $E|X|^4/(\log^+|X|)^2 < \infty$, in which case the LSL holds. A companion paper [8] is devoted to the analogous problem for sequences and a second one [9] on Cesàro summation involves a sequence of i.i.d. random variables and a special array of weights.

Here is our main result.

THEOREM 1.3. *Let* $\{(X_{n,k},\ 1 \le k \le n),\ n \ge 1\}$ *be an array of identically distributed, row-wise independent random variables with mean 0, and set* $S_n = \sum_{k=1}^{n} X_{n,k}$, $n \ge 1$. *Let* $\phi(x)$ *be a positive, nondecreasing and unbounded function such that* $x/\phi(x)$ *is nondecreasing at infinity and, for some* $\gamma > 0$,

$$\frac{\phi(x)}{\phi(x \log x\, \phi(x))} \ge \gamma > 0 \quad \text{as } x \to \infty. \tag{1.1}$$

If

$$\kappa := E\Big(\frac{X^4}{(\log^+ |X|\, \phi(X^2))^2}\Big) < \infty, \tag{1.2}$$

then

$$\frac{S_n}{\sqrt{n \log n\, \phi(n)}} \overset{\text{c.c.}}{\to} 0 \quad \text{as } n \to \infty. \tag{1.3}$$

Conversely, if (1.3) holds, then so does (1.2) and $E\,X = 0$.

REMARK 1.4. Condition (1.1) is needed in order for the function $x \log x\, \phi(x)$ to be invertible in a specific way. Strictly speaking we need (1.2) as well as convergence of $\sum_{n=3}^{\infty} nP(|X| > \varepsilon \sqrt{n \log n\, \phi(n)})$ for the sufficiency, and the latter for the necessity. Lemma 2.1 below shows that the two are equivalent under Condition (1.1).

The proof of the sufficiency involves truncation and estimates of sums of tail probabilities of the row sums. In particular, almost sure convergence holds under the given assumptions. In this case the converse remains true under the additional assumption that the whole array consists of i.i.d. random variables, the reason being that the second Borel–Cantelli lemma then applies (cf. the analogue for the strong laws in [5, Theorem 3.1 and Proposition 1.1]).

THEOREM 1.5. *If, in addition, the rows are independent, then almost sure convergence holds in Theorem 1.3 if and only if complete convergence holds.*

REMARK 1.6. The i.i.d. array is the most random one, the other extreme being an array in which the nth row equals the nth partial sum of a sequence of i.i.d. random variables, that is, $X_{n,k} = X_k$ for $1 \le k \le n$, $n \ge 1$. In this case it is an immediate consequence of the LIL for sequences that (1.3) holds. Of course, finite variance is sufficient.

EXAMPLE 1.7. For $\phi(x) = (\log^+ x)^{\alpha-1}$, $\alpha > 1$,

$$\frac{S_n}{\sqrt{n(\log n)^{\alpha}}} \overset{\text{c.c.}}{\to} 0 \quad \text{as } n \to \infty \quad \Longleftrightarrow \quad E\Big(\frac{X^4}{(\log^+ |X|)^{2\alpha}}\Big) < \infty.$$

For $\phi(x) = (\log^+ \log^+ x)^{\beta}$, $\beta > 0$,

$$\frac{S_n}{\sqrt{n \log n (\log \log n)^{\beta}}} \overset{\text{c.c.}}{\to} 0 \quad \text{as } n \to \infty \quad \Longleftrightarrow \quad E\Big(\frac{X^4}{(\log^+ |X|)^2 (\log^+ \log^+ |X|)^{\beta}}\Big) < \infty.$$

In particular, almost sure convergence holds in both cases.

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In Section 2 we collect some preliminaries for later use. Proofs of the theorems follow in Section 3, after which Section 4 contains some remarks on dependence, and Section 5 a few words on domination and weighted averages, a special case of which is Cesàro summation, the topic of our companion paper [9].

2. Preliminaries

Here we collect some tools and observations that will be of use later. Throughout, X represents a generic random variable in cases when the random variables in the array are identically distributed. Also $\log^+ x = \max\{1, \log x\}$.

2.1. Moments and tail sums. For truncation and converses one frequently uses the connection between tail sums of the summands and existence of moments. Standard cases can be found in [7, Section 2.12]. However, here the connection is more subtle because some less elementary functions are involved.

LEMMA 2.1. *Let X be a random variable. Suppose that $\phi(\cdot)$ is a nondecreasing and unbounded function such that $x/\phi(x)$ is nondecreasing at infinity and that Condition (1.1) holds. Then*

$$E\Big(\frac{X^4}{(\log^+|X|\,\phi(X^2))^2}\Big) < \infty \quad \Longleftrightarrow \quad \sum_{n\geq 3} n\,P(|X| > \varepsilon\sqrt{n\log n\,\phi(n)}) < \infty$$

for some $\varepsilon > 0$.

PROOF. Suppose that $z \geq \varepsilon x \log x \phi(x)$. Since $h(z) = z/(\log z\,\phi(z))$ is eventually nondecreasing,

$$h(z) \geq h(\varepsilon x \log x\,\phi(x)) = \frac{\varepsilon x \log x\,\phi(x)}{\log(\varepsilon x \log x\,\phi(x))\phi(\varepsilon x \log x\,\phi(x))} \geq \varepsilon' x$$

for large x and some $\varepsilon' > 0$. Hence, if $E\,(h(X^2)^2) < \infty$, then

$$\sum_{n\geq 3} n\,P(|X| > \varepsilon\sqrt{n\log n\phi(n)}) \leq \sum_{n\geq 3} n\,P(h(X^2) > \varepsilon' n) < \infty.$$

For the inverse conclusion, put $H(x) = x\log x\,\phi(x)$, which is increasing. Then

$$H(z/(\log z\,\phi(z)) = \frac{z\log(z/(\log z\,\phi(z))\,\phi(z/(\log z\,\phi(z)))}{\log z\,\phi(z)} \leq z,$$

from which we conclude that

$$\begin{aligned}\sum_{n\geq 3} nP\Big(\frac{X^2}{\log^+|X|^2\,\phi(X^2)} > n\Big) &= \sum_{n\geq 3} nP\Big(H\Big(\frac{X^2}{\log^+|X|^2\,\phi(X^2)}\Big) > H(n)\Big)\\ &\leq \sum_{n\geq 3} nP(X^2 > n\log n\,\phi(n)) < \infty\end{aligned}$$

and, hence, the moment condition holds. □

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The following result will be needed for converses.

PROPOSITION 2.2. *Let $\{(X_{n,k},\ 1 \le k \le n),\ n \ge 1\}$ be an array of row-wise independent random variables and $\{\alpha_n,\ n \ge 1\}$ a sequence of positive reals. Then*

$$\sum_{n=1}^{\infty} P\Big(\max_{1\le k\le n} |X_{n,k}| > \alpha_n\Big) < \infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty}\sum_{k=1}^{n} P(|X_{n,k}| > \alpha_n) < \infty.$$

In particular, if the random variables in each row are i.i.d., then the above are equivalent to

$$\sum_{n=1}^{\infty} nP(|X_{n,1}| > \alpha_n) < \infty.$$

If, moreover, $\{\alpha_n = \alpha(n),\ n \ge 1\}$ with some function $\alpha(\cdot)$ having an inverse β, then the above are equivalent to

$$E((\beta(|X|))^2) < \infty.$$

The first part is [5, Theorem 2.5], the third equivalence is obvious and the last one follows via inversion.

It turns out that the sufficiency sometimes holds for arrays whether the rows are independent or not, whereas the necessity does not. The special case when the array reduces to a sequence is an illustrative example. Recall also Theorems 1.3 and 1.5.

2.2. A sharper exponential bound. Essential tools for proving LILs are the exponential bounds (see [7, Ch. 8]). Since our limits concern convergence to 0 we only need an upper bound. The basis is a Taylor expansion of the exponential function up to order two. One complication is that two truncations are necessary for the proof of the classical LIL—one to adapt to the exponential bound and one to match finite variance. The main technical task to handle is 'the middle portion'. In the present case we have higher order moments at our disposal, which permits us to improve the upper bound by a further expansion of the exponential function. As a consequence we only need to truncate the original random variables once.

By modifying the procedure of [14, page 221], we first note that, for $x > 0$,

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + x^5 \sum_{k=5}^{\infty} \frac{1}{k!}x^{k-5} \\ &\le 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{x^5}{5!}e^x. \end{aligned} \tag{2.1}$$

Now suppose that Y is a symmetric random variable (which implies that odd moments vanish) and let $t > 0$. It then follows from (2.1) that

$$\begin{aligned} E(e^{tY}) &\le 1 + 0 + \frac{t^2}{2}E\,Y^2 + 0 + \frac{t^4}{4!}E\,Y^4 + \frac{t^5}{5!}E(|Y|^5 e^{t|Y|}) \\ &\le \exp\Big\{\frac{t^2}{2}E\,Y^2 + \frac{t^4}{4!}E\,Y^4 + \frac{t^5}{5!}E(|Y|^5 e^{t|Y|})\Big\}. \end{aligned}$$

The following upper exponential bound is now immediate.

Lemma 2.3. *Let* $\{X_k,\ k \geq 1\}$ *be i.i.d. symmetric random variables with variance* 1. *For* n *fixed, set* $Y_{n,k} = X_k \mathbb{I}\{|X_k| \leq b_n\}$ *for* $k = 1, 2, \ldots, n$ *and some* $b_n > 0$. *Let* $t > 0$. *Then, for* $x > 0$,

$$P\bigg(\bigg|\sum_{k=1}^{n} Y_{n,k}\bigg| > x\bigg) \leq 2\exp\bigg\{-tx + \frac{nt^2}{2} + \frac{nt^4}{4!} E\, Y_{n,1}^4 + \frac{nt^5}{5!} E(|Y_{n,1}|^5 e^{t|Y_{n,1}|})\bigg\}.$$

3. Proofs of Theorems 1.3 and 1.5

Since symmetrisation and desymmetrisation follow standard procedures we assume without loss of generality that all random variables are symmetric. Moreover, since variance is a matter of scaling, we assume, without restriction, that $\operatorname{Var} X = 1$ throughout the proof.

3.1. Theorem 1.3: sufficiency.

Proof. Set $\kappa = E\,(X^4/(\log^+|X|\,\phi(X^2))^2)$, which, by assumption, is finite. For $n \geq 1$ fixed and $1 \leq k \leq n$, define $Y_{n,k} = X_{n,k} \mathbb{I}\{|X_{n,k}| \leq b_n\}$ and $Z_{n,k} = X_{n,k} \mathbb{I}\{|X_{n,k}| > b_n\}$, and set $S'_n = \sum_{k=1}^n Y_{n,k}$ and $S''_n = \sum_{k=1}^n Z_{n,k}$.

In Lemma 2.3, take $b_n = (\varepsilon/4)\sqrt{n \log n\, \phi(n)}$, $x = \varepsilon\sqrt{n \log n\, \phi(n)}$ and $t = (2/\varepsilon)\sqrt{\log n / n\phi(n)}$ (so that, in particular, $tx = 2\log n$ and $tb_n = \frac{1}{2}\log n$) and set $A(n) = \sqrt{n \log n\, \phi(n)}$. Then, for n large,

$$\begin{aligned}
&P(|S'_n| > \varepsilon\sqrt{n \log n\, \phi(n)}) \\
&\quad \leq 2\exp\bigg\{-2\log n + \frac{2\log n}{\varepsilon^2\,\phi(n)} + \frac{16(\log n)^2 E(Y_{n,1})^4}{n\varepsilon^4(\phi(n))^2\, 4!} + \frac{32(\log n)^{5/2} E|Y_{n,1}|^5 e^{t|Y_{n,1}|}}{n^{3/2}\varepsilon^5(\phi(n))^{5/2}\, 5!}\bigg\} \\
&\quad \leq 2\exp\bigg\{-2\log n + \frac{2\log n}{\varepsilon^2\,\phi(n)} + \frac{2(\log n)^2\kappa}{3n\varepsilon^4(\phi(n))^2} \cdot \Big(\log\Big(\frac{\varepsilon}{4}A(n)\Big)\Big)^2 \cdot \Big(\phi\Big(\frac{\varepsilon^2}{16}\,A(n)^2\Big)\Big)^2 \\
&\qquad + \frac{4(\log n)^{5/2}\kappa\, e^{1/2\log n}}{15n^{3/2}\varepsilon^5(\phi(n))^{5/2}} \cdot \frac{\varepsilon}{4}A(n) \cdot \Big(\log\Big(\frac{\varepsilon}{4}A(n)\Big)\Big)^2 \cdot \Big(\phi\Big(\frac{\varepsilon^2}{16}\,A(n)^2\Big)\Big)^2\bigg\} \\
&\quad \leq 2\exp\bigg\{-2\log n + \frac{2\log n}{\varepsilon^2\,\phi(n)} + C_1\frac{(\log n)^2}{n\varepsilon^4(\phi(n))^2} \cdot (\log n + \log(\phi(n)))^2 \cdot (\phi(n)/\gamma)^2 \\
&\qquad + C_2\frac{(\log n)^{5/2} e^{1/2\log n}}{n^{3/2}\varepsilon^4(\phi(n))^{5/2}} \cdot A(n) \cdot (\log n + \log(\phi(n)))^2 \cdot (\phi(n)/\gamma)^2\bigg\} \\
&\quad \leq 2\exp\bigg\{-2\log n\Big(1 - \frac{2}{\varepsilon^2\,\phi(n)}\Big) + C_3\frac{(\log n)^4}{n\varepsilon^4} + C_4\frac{(\log n)^5}{\sqrt{n}\,\varepsilon^4}\bigg\} \\
&\quad = 2\exp\bigg\{-2\log n\Big(1 - \frac{2}{\varepsilon^2\,\phi(n)}\Big) + O\Big(\frac{(\log n)^5}{\sqrt{n}}\Big)\bigg\} \leq 2\exp\{-3/2\log n\},
\end{aligned}$$

which tells us that

$$\sum_{n\geq 3} P(|S'_n| > \varepsilon\sqrt{n \log n\, \phi(n)}) < \infty \quad \text{for all } \varepsilon > 0. \tag{3.1}$$

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As for the other sum,

$$P(|S''_n| > \varepsilon\sqrt{n\log n\,\phi(n)}) \le nP(|Z_{n,1}| \ne 0) = nP\Big(|X| > \frac{\varepsilon}{4}\sqrt{n\log n\,\phi(n)}\Big),$$

so that

$$\sum_{n\ge 3} P(|S''_n| > \varepsilon\sqrt{n\log n\,\phi(n)}) \le \sum_{n\ge 3} nP\Big(|X| > \frac{\varepsilon}{4}\sqrt{n\log n\,\phi(n)}\Big) < \infty \tag{3.2}$$

for all $\varepsilon > 0$, in view of Lemma 2.1.

Finally, combining (3.1) and (3.2) shows that

$$\frac{S_n}{\sqrt{n\log n\,\phi(n)}} \overset{\text{c.c.}}{\to} 0 \quad \text{as } n \to \infty.$$

□

3.2. Theorem 1.3: necessity.

Proof. We first observe that, if (1.3) holds, then the Kolmogorov strong law holds, which, via its converse, shows that the mean equals zero. It follows that the Marcinkiewicz–Zygmund law holds and so $E|X|^r < \infty$ for any $r \in (0, 2)$.

In order to verify (1.2), we assume, without loss of generality, that X is symmetric. Following the arguments of [5], we first note that $|X_{n,k}| \le |S_k| + |S_{k-1}|$, so that $\max_{1\le k\le n} |X_{n,k}| \le 2\max_{1\le k\le n} |S_k|$. If (1.3) holds, then, via an application of the Lévy inequalities (see [7, Theorem 3.7.1]), it follows that

$$\sum_{n\ge 3} P\Big(\max_{1\le k\le n} |X_{n,k}| > \varepsilon\sqrt{n\log n\,\phi(n)}\Big) < \infty \quad \text{for all } \varepsilon > 0,$$

which, in turn, in view of Proposition 2.2, is equivalent to

$$\sum_{n\ge 3} nP(|X| > \varepsilon\sqrt{n\log n\,\phi(n)}) < \infty \quad \text{for all } \varepsilon > 0.$$

A concluding appeal to Lemma 2.1 establishes the desired result. □

3.3. Theorem 1.5.

Proof. Since complete convergence always implies almost sure convergence, it follows that almost sure convergence always holds in Theorem 1.3. The converse holds if, in addition, the rows are independent, since then almost sure convergence holds if and only if the corresponding Borel–Cantelli sum converges. This completes the remaining piece in the proof of Theorem 1.5. □

4. Dependence between rows

In this section we discuss analogues when there is special dependence between the rows, such as partly overlapping rows and m-dependence between rows.

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4.1. Examples with special dependence schemes between rows. In this subsection we consider a special class of row-wise dependent random variables via a certain class of weighted sums. Let $X, X_1, X_2, \dots,$ be i.i.d. random variables with partial sums S_n, $n \geq 1$, and define

$$\widetilde{S}_n = \sum_{k\geq 1} a_{n,k} X_k \quad \text{for } n \geq 1,$$

with weights

$$a_{n,k} = \begin{cases} 1 & \text{for } \alpha_n < k \leq \alpha_n + n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha_n = [c\, n^{\alpha}]$ for some $c \geq 1/2$ and $\alpha \geq 0$, so that $\widetilde{S}_n = S_{\alpha_n + n} - S_{\alpha_n}$.

The following result on complete convergence follows from [5].

Theorem 4.1. *Let α_n be any nondecreasing sequence and $0 < p < 2$ and let $E\,X = 0$. If $p \geq 1/2$, then*

$$\sum_{n\geq 1} P(|\widetilde{S}_n| > \varepsilon n^{1/p}) < \infty \quad \textit{for some } \varepsilon > 0 \quad \Longleftrightarrow \quad E\,|X|^{2p} < \infty.$$

Here the dependence structure between the rows does not matter. However, when we look at the strong law it does.

Theorem 4.2. *Let $0 < p < 2$ and $E\,X = 0$ whenever the first moment exists.*

(i) *If $\alpha_n \equiv 1$, then*

$$\frac{\widetilde{S}_n}{n^{1/p}} \overset{\text{a.s.}}{\to} 0 \quad \textit{as } n \to \infty \quad \Longleftrightarrow \quad E\,|X|^{p} < \infty.$$

(ii) *If $c = 1$ and $\alpha_n = n^2$, then*

$$\frac{\widetilde{S}_n}{n^{1/p}} \overset{\text{a.s.}}{\to} 0 \quad \textit{as } n \to \infty \quad \Longleftrightarrow \quad E\,|X|^{2p} < \infty.$$

Proof. The first part of the result follows from the Marcinkiewicz–Zygmund theorem [13], whereas the second, once again, follows from [5], since, in this case, the rows are independent, so that complete convergence and almost sure convergence are equivalent. □

The next natural question is: what happens in between these cases?

Theorem 4.3. *Let $0 < p < 2$ and $E\,X = 0$ whenever the first moment exists. Define*

$$\alpha^* = \mathbb{I}\{\alpha \leq 1\} + \alpha\,\mathbb{I}\{1 < \alpha < 2\} + 2\,\mathbb{I}\{\alpha \geq 2\}.$$

Then

$$\frac{\widetilde{S}_n}{n^{1/p}} \overset{\text{a.s.}}{\to} 0 \quad \textit{as } n \to \infty \quad \Longleftrightarrow \quad E\,|X|^{\alpha^* p} < \infty.$$

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PROOF. For $\alpha \geq 2$, the rows are independent and the argument from Section 3 applies. Since $\alpha_{n_{k+1}} - \alpha_{n_k} \geq n_k$, the subsequence $\{\widetilde{S}_{n_k}, k \geq 1\}$ with $n_k = ((\alpha - 1/\alpha\, c)k)^{1/(\alpha-1)}$ is a sequence of independent random variables. Hence,

$$\frac{\widetilde{S}_{n_k}}{n_k^{1/p}} \xrightarrow{\text{a.s.}} 0 \quad \text{as } n \to \infty \iff \sum_{k=1}^{\infty} P(|\widetilde{S}_{n_k}| > \varepsilon(n_k)^{1/p}) < \infty$$

$$\iff \sum_{k=1}^{\infty} P(|S_{n_k}| > \varepsilon(n_k)^{1/p}) < \infty.$$

The latter sum converges if and only if $\sum_{k=1}^{\infty} k^{1/(\alpha-1)} P(|X|^p > \varepsilon k^{1/(\alpha-1)})$ converges. But

$$\sum_{k=1}^{\infty} k^{1/(\alpha-1)} P\,(|X|^p > \varepsilon k^{1/(\alpha-1)}) = \sum_{k=1}^{\infty} k^{1/(\alpha-1)} P(|X|^{p(\alpha-1)} > \varepsilon^{1/(\alpha-1)}\, k) < \infty$$
$$\iff \quad E|X|^{p\alpha} < \infty.$$

In order to fill the gaps, we consider symmetric random variables and use the Lévy inequalities and the fact that $n_k/n_{k+1} \geq 1/2$ to see that

$$\begin{aligned}
&P\Big(\max_{n_k \leq n \leq n_{k+1}} \frac{|\widetilde{S}_n|}{n^{1/p}} > \varepsilon\Big) \\
&\leq P\Big(\max_{n_k \leq n \leq n_{k+1}} \frac{|S_{\alpha_n+n} - S_{\alpha_{n_k}}|}{n_k^{1/p}} > \varepsilon/2\Big) + P\Big(\max_{n_k \leq n \leq n_{k+1}} \frac{|S_{\alpha_n} - S_{\alpha_{n_k}}|}{n_k^{1/p}} > \varepsilon/2\Big) \\
&\leq P\Big(\max_{1 \leq j \leq \alpha_{n_{k+1}} + n_{k+1} - \alpha_{n_k}} \frac{|S_j|}{n_k^{1/p}} > \varepsilon/2\Big) + P\Big(\max_{1 \leq j \leq \alpha_{n_{k+1}} - \alpha_{n_k}} \frac{|S_j|}{n_k^{1/p}} > \varepsilon/2\Big) \\
&\leq P\Big(\max_{1 \leq j \leq 2\, n_{k+1}} \frac{|S_j|}{n_k^{1/p}} > \varepsilon/2\Big) + P\Big(\max_{1 \leq j \leq n_k} \frac{|S_j}{n_k^{1/p}} > \varepsilon/2\Big) \\
&\leq 2\, P\Big(\frac{|S_{2\, n_{k+1}}|}{(2\, n_{k+1})^{1/p}} > \varepsilon/8\Big) + 2\, P\Big(\frac{|S_{n_k}|}{n_k^{1/p}} > \varepsilon/2\Big).
\end{aligned}$$

This and the subsequence considerations from above imply the sufficiency of the moment condition. The necessity follows via the above subsequence argument.

In case $c = \alpha = 1$, we have $\widetilde{S}_n = S_{n+n} - S_n$ and, under the moment condition $E|X|^{1/p} < \infty$, both sums normalised with $n^{1/p}$ converge to zero and, hence, so does the difference. Conversely, if $P(|\widetilde{S}_n| > \varepsilon n^{1/p} \text{ i.o.}) = 0$, then

$$P(|X_{2(n+1)} + X_{2n+1} - X_{n+1}| > 2\varepsilon n^{1/p} \text{ i.o.}) = P(|\widetilde{S}_{n+1} - \widetilde{S}_n| > 2\varepsilon n^{1/p} \text{ i.o.}) = 0,$$

so that $P(|X_{n+1}| > (2/3)\varepsilon(n+1)^{1/p} \text{ i.o.}) = 0$, from which we recover the desired moment. The proof for $\alpha \leq 1$ and different values of c works similarly. □

REMARK 4.4. Suppose that $1 < \alpha < 2$. Then $\widetilde{S}_n/n^{1/p}$ is a subsequence of

$$S_{m,m^{1/\alpha}}/(m/c)^{1/(p\alpha)} = (S_{m+(m/c)^{1/\alpha}} - S_m)/(m/c)^{1/(p\alpha)}$$

and the sufficiency is contained in [2, Theorem 1].

REMARK 4.5. Note that, for $n \le m$,

$$\operatorname{corr}(\widetilde{S}_n, \widetilde{S}_m) = \begin{cases} \dfrac{cn^\alpha + n - cm^\alpha}{\sqrt{mn}} & \text{if } cm^\alpha \le cn^\alpha + n, \\ 0 & \text{if } cm^\alpha > cn^\alpha + n. \end{cases}$$

4.2. *m*-dependent rows. Here we consider a triangular array of row-wise i.i.d. random variables $\{(X_{n,k},\ 1 \le k \le n),\ n \ge 1\}$, but assume a more special dependence structure on the row sums.

THEOREM 4.6. *If the row sums S_n are m-dependent for some $m \in \mathbb{N}$, then*

$$\frac{S_n}{n^{1/p}} \overset{\text{a.s.}}{\to} 0 \quad \text{as } n \to \infty \quad \Longleftrightarrow \quad E|X|^p < \infty.$$

PROOF. For each $\ell = 1, \dots, m-1$, we consider the independent subsequences $\{S_{mk+\ell},\ k \ge 1\}$ for which the law holds if and only if the moment condition is satisfied. But then the full sequence converges as well. □

5. Domination

The sufficiency part of most (almost all) limit theorems for sums of i.i.d. random variables can be extended to limit theorems for sums of independent, not necessarily identically distributed, random variables, under suitable additional conditions. The same goes for arrays. Let us make some elementary remarks to this end.

The simplest case is when an array $\{(X_{n,k},\ 1 \le k \le n),\ n \ge 1\}$ is *dominated* by a positive random variable X^*, that is, $|X_{n,k}| \le X^*$ for all n and $k = 1, 2, \dots, n$. Then, trivially, if $E|X^*|^r < \infty$ for some $r > 0$, then $E|X_{n,k}|^r < \infty$ for all n and $k = 1, 2, \dots, n$, and standard moment inequalities can be applied. A weaker concept is *weak domination*, which means that tail probabilities are dominated. Even weaker is *weak mean domination (WMD)*. The array $\{(X_{n,k},\ 1 \le k \le n),\ n \ge 1\}$ is weakly mean dominated by the positive random variable X^* if, for some $\theta > 0$,

$$\frac{1}{n}\sum_{k=1}^{n} P(|X_{n,k}| > x) \le \theta P(X^* > x) \quad \text{for all } x > 0 \text{ and all } n. \tag{5.1}$$

An application of the fact that $E\,U^r = \int_0^\infty \overline{F}_U(u)\,d(u^r)$ for a positive random variable with finite moment of order r shows immediately that $(1/n)\sum_{k=1}^n E|X_{n,k}|^r \le \theta E(X^*)^r$ for all n; for more on this, we refer to [5, 6]. The same applies to $E(h(U))$ for a nonnegative nondecreasing function $h(\cdot)$.

THEOREM 5.1. *Let $\{(X_{n,k},\ 1 \le k \le n),\ n \ge 1\}$ be an array of row-wise independent random variables with mean 0 satisfying Condition (5.1), and set $S_n = \sum_{k=1}^n X_{n,k}$, $n \ge 1$. Further, let $\phi(x)$ be a positive, nondecreasing and unbounded function such that $x/\phi(x)$ is nondecreasing at infinity and Condition (1.1) is satisfied. If the moment condition (1.2) is satisfied, then*

$$\frac{S_n}{\sqrt{n \log n\, \phi(n)}} \overset{\text{c.c.}}{\to} 0 \quad \text{as } n \to \infty.$$

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PROOF. We follow the proof of Theorem 1.3. An inspection of the proofs of the sufficiency parts in Section 3 shows that all estimates are based on tail sums and (truncated) moments. In order to estimate the moments in the exponential inequality, we have to keep in mind that

$$|Y_{n,k}| \leq |X_{n,k}|\mathbb{I}\{|X_{n,k}| \leq b_n\} + b_n \mathbb{I}\{|X_{n,k}| > b_n\},$$

from which it follows, via [5, Lemma 2.1.c], that

$$\sum_{k=1}^{n} E\,|Y_{n,k}|^p \leq \theta n E\,(X^*(b_n))^p \quad \text{for } p \geq 1,$$

where $X^*(b_n) = X^*\,\mathbb{I}\{X^* \leq b_n\} + b_n\,\mathbb{I}\{X^* > b_n\}$. The bounds for the moments of $X^*(b_n)$ are the same as those for $Y_{n,k}$ in Section 3. In view of the previous remarks, this shows that the sufficiency part of Theorem 1.3 remains valid for weakly mean dominated arrays. □

5.1. Weighted sums. A special example is weighted sums (as we have already seen in Section 4). Let $Y, Y_1, Y_2, \ldots,$ be i.i.d. random variables, let $\{(a_{n,k},\ 1 \leq k \leq n),\ n \geq 1\}$ be an array of real numbers and set $S_n = \sum_{k=1}^{n} a_{n,k} Y_k$, $n \geq 1$. A particular case of interest is when the weights are uniformly bounded; $|a_{n,k}| \leq a$ for all k, n and some $a > 0$. Then $a_{n,k}Y_k$ is dominated by the random variable $a|Y|$.

This is the case in Cesàro summation for $0 < \alpha < 1$. There one investigates the sums $S_n = \sum_{j=0}^{n} A_{n-j}^{\alpha-1} Y_k$, where the weights $A_{n-j}^{\alpha-1} \in (0,1)$ for all n and $j = 0, 1, \ldots, n$. This is, however, a kind of 'in between case' in the sense that one considers *sequences* of random variables, $Y_1, Y_2, \ldots,$ with weights $\{(a_{n,k},\ 1 \leq k \leq n),\ n \geq 1\}$ that constitute an array. This means that the object under investigation is the sequence $S_n = \sum_{k=1}^{n} a_{n,k} Y_k$, $n \geq 1$ and, at the same time, $S_{n+1} - S_n$ is *not* just a 'last term', so in that sense one has an array at hand. Our companion paper [9] using [15] is devoted to this problem.

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