be nonplussed by chapter three, which gives a rather dry account of the etymologies of the six standard trigonometric functions, while anyone trusting the author's assertion that the first nine chapters only require basic algebra and trigonometry would be surprised by chapter four, which discusses topics such as De Moivre's theorem, partial differential equations and Fourier series with little help offered to the uninitiated. I'm sure that even the most obscure historical content will be of great interest to some, and both complex numbers and Fourier analysis are revisited more thoroughly later in the book in a manner that could be followed by a keen reader with a decent grasp of calculus. However, the first quarter of the book in particular is heavy on historical details and light on mathematical ones, meaning that it will present a barrier to entry for many readers. School students will find it especially tough going at times, despite being half the intended audience, and even the "basic trigonometry" often requires a surprising level of fluency with compound angle and sum-to-product formulae.

Despite these flaws, persistence will certainly be rewarded. Highlights such as an exploration of Lissajous figures, the intriguing story of the so-called Witch of Agnesi and some rather ingenious geometric constructions of infinite series are all waiting in the latter half of the book, while a particularly satisfying description of how to teach the trigonometric functions and laws using projections doesn't appear until the appendix. The lack of handholding has its advantages as well, as those with the ability to keep up will be impressed by the elegance and efficiency of much of the mathematics. The chapter on epicycloids and hypocycloids springs to mind, as the curves are defined, parametrised and fully explored over the course of seven pages, and similar levels of mathematical dexterity are on display in numerous other sections. The story of trigonometry is also made more engaging by the inclusion of historical excerpts and diagrams which clearly show how the subject has evolved alongside mathematics, and any reader will leave the book with a renewed appreciation for trigonometry's lasting importance.

Overall, the book was rewarding and enjoyable to read, but I had to work my way through the first few chapters before I really started to feel engaged. I would suggest that the book is not particularly well suited to the author's intended audience of school and undergraduate students, and that it would be better appreciated by an experienced mathematician with a keen interest in history. A motivated teacher would then be able to pick and choose highlights to share with pupils, supplemented with some extra scaffolding and stripped of some of the unnecessary historical detail. In the right hands, the book could definitely help to dispel the notion that trigonometry is a dry and dull branch of mathematics. In my opinion, however, the book would struggle to do so on its own.

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**The doctrine of triangles** by Glen Van Brummelen, pp. 376, £25 (hard), ISBN 978-0-69117-941-4, Princeton University Press (2021)

The word 'Trigonometrie' first appeared in the English language in 1614, which was the year of the first edition of the book *Trigonometrie: or The Doctrine of Triangles*, translated by Raphe Handson. The third edition (1642) has the following title page:

> Trigonometrie: or the doctrine of triangles. First written in Latin, by Bartholomew Pitiscus of Grunberg in Silesia, and now translated into English by Raphe Handson. Whereunto is added (for the mariners use)

certain nautical questions, together with the finding of the variation of the compass. All performed arithmetically, without map, sphere, globe or astrolabe, by the said Raphe Handson … [*Bound with, as issued*] A canon of triangles: or, the tables, of sines, tangents and secants, the radius assumed to be 100000.

Hence 'The doctrine of triangles' is now the title of this book by Glen Van Brummelen. Its subtitle, 'A history of modern trigonometry', contrasts it with his earlier work, *The mathematics of the heavens and the earth: The early history of trigonometry* [1] and takes the history of trigonometry from 1550 to the early 1900s. As suggested by Handson's title page, this history is closely tied to that of astronomy, navigation, surveying and cartography, which are recurring themes throughout the book.

Trigonometry began with Greek astronomers, such as Hipparchus of Rhodes, who constructed geometric models of the motions of the sun and moon. Based upon the conversion of magnitudes of circular arcs into lengths of line segments, it gave rise to the formulation of the 'chord function' within a circle of given magnitude. For example, due to the absence of decimal notation, Pitiscus (via Handson as above) specifies a radius of 100 000. This method yielded rudimentary tables of trigonometric data (expressed in integers), known as 'chord tables'.

Until the sixteenth century, approaches to trigonometry were based upon such circle-based geometry, and triangles did not emerge as the basic objects of study until the arrival of Regiomontanus's *De Triangulus omnimodus* (1533). This stage in the evolution of modern trigonometry is discussed by Van Brummelen in the first chapter, 'European Trigonometry Comes of Age'. Also portrayed are the contributions of Georg Rheticus (1514–1574), who defined the six trigonometric functions based on triangles. Subsequent discussion concerns the evolution of trigonometric tables and familiar formulae derived by the algebraic methods of Viète (1540–1603).

Since logarithms became important for trigonometric calculations, chapter 2, 'Logarithms', describes the work of Napier and Briggs. Referring to Napier, van Brummelen surmises that Napier invented logarithms for the benefit of trigonometers such as himself. But although trigonometry thus became indebted to logarithms, it is suggested elsewhere that Napier's idea of calculating products by a process of addition came from trigonometry, via the formula

$$\sin A \times \sin B = \tfrac{1}{2}\big(\cos(A - B) - \cos(A + B)\big).$$

Chapter 3, 'Calculus', concerns the transition of trigonometry from a branch of geometry to an object of study in analysis (as embodied by the newly emerging calculus). This change was partly motivated by the discovery of the periodicity of the trigonometric functions and partly by the need to establish tangency and the quadrature of trigonometric curves such as the cycloid. Newton solved quadrature problems by representing curves by infinite series, while Leibniz used geometry based upon the differential triangle to determine derivatives and integrals. Van Brummelen concludes from extensive discussion of Euler's work on trigonometry that it was he who completed the project of bringing trigonometry into analysis. A surprising development was the use of complex numbers in the context of trigonometry, and it is shown how Euler, in 1748, derived his hallmark contribution in the form $e^{iv} = \cos v + i \sin v$. Thirty-four years previously the Englishman Roger Cotes used a totally different method to obtain a result that would now be expressed in the form $i\varphi = \ln(\cos\varphi + i \sin\varphi)$. Was this known to Euler?

Notable innovations in trigonometry after Euler (Chapter 5: 'Europe after Euler') included the appearance of Fourier series for problems such as heat transfer and

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vibrating strings. There is an intriguing (non-post Eulerian) account of the initial ideas on hyperbolic trigonometry due to Vincenzo Riccati (1757) and, more completely, by Johann Lambert (1768). The early nineteenth century witnessed the discovery of non-Euclidean geometries that invoked the application of trigonometry to surfaces other than the Euclidean plane.

In *The Mathematics of the heavens and the earth* Van Brummelen provided extensive treatment of trigonometry in India and the Islamic world. In this book, chapter 4 provides an account of the work of Chinese trigonometers. It also outlines the assimilation of Indian and Islamic ideas into Chinese mathematics, and it concludes with a description of the eventual importation of European trigonometry.

The depth and breadth of historical analysis and the extensive mathematical scope of this scholarly work are partially evidenced by its extensive bibliography (1000 references), yet Van Brummelen's entertaining narrative makes it compulsive reading. There is reference to about 280 different contributors to the field of trigonometry and its applications. Most of these are less well known, but emphasis is placed upon many of the 'greats', such as de Moivre, Newton, James Gregory, Leibniz, Euler and the Bernoullis. For many of these, biographical detail is supplied, often with portrayal of the surrounding cultural and political context.

This is yet another enjoyable mathematical gem from Glen Van Brummelen.

*References*

1. Glen Van Brummelen, *The Mathematics of the heavens and the earth*, Princeton University Press (2009), reviewed in *Math. Gaz.* **95** (March 2011), pp. 150-151.

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**Introduction to linear and matrix algebra** by Nathaniel Johnston, pp. 482, £49.99 (hard), ISBN 978-3-03052-810-2, Springer Verlag (2021)

This book starts with basic coordinate-system vectors as met in most of the UK at GCSE, and progresses to eigenvectors and eigenvalues of matrices. It is organised in three long chapters, each of which consists of theory followed by detailed explorations of some applications, called 'Extra Topics'. Chapter 1 introduces matrices and their use to describe linear transformations, largely in two dimensions, as well as the dot product, with extra topics on the cross product and paths in graphs. Chapter 2 looks at linear equations, elementary matrices and matrix inverses, vector spaces in a purely geometric context (and hence called just 'subspaces' here), bases and dimensions, with extra topics on finite fields, linear programming, rank decomposition and the LU decomposition. Only in Chapter 3 do we meet determinants formally, followed by eigenvectors and diagonalisation. The extra topics here give more on determinants, including permutations and an algebraic proof of $\det AB = \det A \det B$; power iteration to find the eigenvalue of largest magnitude, including the PageRank algorithm; complex eigenvalues; and linear recurrence relations. There are appendices of mathematical preliminaries, including complex numbers, polynomials and proof techniques, so although the blurb says that readers are assumed to have completed one or two university-level mathematics courses, the author seems to have aimed at students with much less background.