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COVERING A POLYGON WITH TRIANGLES: A CARATHEODORY-TYPE THEOREM

M. J. C. BAKER

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Abstract

Generalizations are proved for theorems of Carathéodory (1907), Kirchberger (1903) and Watson (1973), the theme of these results being how thickly the convex hull of a family of points is covered by simplexes whose vertices are chosen from the points of the family.

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Summary

This paper pursues a train of thought suggested by the theorems of Carathéodory (1907) and Watson (1973). In two dimensions we ask how thickly covered with triangles is the convex hull of a family of points, the vertices of the triangles being points of the family. This leads to the following

THEOREM 1. *If F is a family of n points ($n > d$) in d-dimensional affine space R_d, then for any point $\alpha \in \operatorname{conv} F$ there are at least $\binom{n-d}{r-d}$ different selections of r of the points of F ($n \geqslant r > d$) that contain α in their convex hull.*

(conv F means the convex hull of F, which is the smallest convex set containing F.)

As a corollary a variant of the theorem of Kirchberger (1903) can be deduced.

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Introduction

Given a finite family of F of points in a plane, Carathéodory's theorem says that any point in $\operatorname{conv} F$ is in at least one triangle made from the points of F; or that $\operatorname{conv} F$ is completely covered with such triangles. Watson's theorem shows that we may restrict ourselves to triangles that have as one of their vertices an arbitrarily prescribed point of F; they will cover $\operatorname{conv} F$ by themselves.

It is clear that if F has more than three points then $\operatorname{conv} F$ must be covered with triangles at least two-deep everywhere; and a little experiment suggests that for n points the triangle-covering is at least $(n-2)$-deep everywhere. It certainly is exactly $(n-2)$-deep at a point on or just inside one of the sides of the polygonal boundary of $\operatorname{conv} F$; because a covering triangle for such a point must use the ends of that side as two of its vertices and can use as its third vertex any of the remaining $n-2$ points of F.

This argument, which generalizes to d dimensions, would leave little to do if we could be sure that the triangle-covering (in d dimensions the simplex-covering) always gets thicker as you go in from the edge of $\operatorname{conv} F$ towards the 'middle'. However, there is in Baker (1978) an example of a set of points with a central region less thickly covered with triangles than any of the regions immediately surrounding it.

As well as generalizing to d dimensions it is convenient also to give consideration to coverings not only by simplexes but also by polytopes defined as the convex hull of more than $d+1$ of the points of F.

Proof of Theorem 1

Let $k=r-d$. So k is an integer greater than zero. It will be kept constant throughout the proof (which works for any k). The proof is by induction on n and d.

(1) If $d=1$ the theorem holds for all n. For suppose that $x_1, x_2, \ldots, x_n$ are distinct points in that order on a line. Points within (x_1, x_2) are covered by each of the $\binom{n-1}{r-1}$ intervals which are the hulls of x_1 together with a selection of $r-1$ of the other $n-1$ points. And if $j<\frac{1}{2}n$ then (x_j, x_{j+1}) is not less thickly covered with such hulls than (x_{j-1}, x_j), because the only intervals that cover one but not the other of them—and this happens only if $j<r$ and $j>n-r+1$—are those having x_j as an end point, and in this case since $j<\frac{1}{2}n$ there are more of them to the right of x_j than to the left. Thus in the one-dimensional case the thickness of covering does increase monotonically until half-way, and then decreases down to $\binom{n-1}{r-1}$ again at the right-hand end.

Observe two things: first, that the boundary points of the intervals into which $\operatorname{conv} F$ is divided are covered at least as thickly as other points of $\operatorname{conv} F$ in their neighbourhood; and, secondly, that the result remains true if the points x_i are not necessarily all distinct (although distinctly labelled). To see the latter we need only separate any coincident points by a small amount, apply the above argument, and close them up again.

(2) If $n = d+k$ the theorem holds for all d. This is trivial, for we have $n = r$; and α, which is in $\operatorname{conv} F$, is certainly in the hull of the $\binom{n-d}{r-d} = 1$ selection of r points that can be made from the points of F.

(3) The inductive step: if the result holds (i) for all families of n points in $d-1$ dimensions and (ii) for all families of $n-1$ points in d dimensions, then it holds for any family of n points in d dimensions.

Single out one of the points of F, x, say. Now α, which is in $\operatorname{conv} F$, is either in $\operatorname{conv}(F-x)$ or it is not.

(a) Suppose $\alpha \in \operatorname{conv}(F-x)$. By the second part of the inductive hypothesis α is in at least $\binom{n-1-d}{r-d}$ of the hulls of selections of r of the points of $F-x$. Further, the ray from x through α must, when produced beyond α, emerge from $\operatorname{conv}(F-x)$ at a point in one of the faces of $\operatorname{conv}(F-x)$ which, using Carathéodory's theorem in $d-1$ dimensions, is thus in the hull of some d of the points of $F-x$.

These d points together with x form a simplex containing α. We may enlarge this set of $d+1$ points to a set of r points by selecting $r-d-1$ of the remaining $n-d-1$ points of F. Thus α is in at least $\binom{n-d-1}{r-d-1}$ hulls of selections of $r-1$ of the points of $F-x$ together with x itself. Altogether α is in at least

$$\binom{n-1-d}{r-d}+\binom{n-d-1}{r-d-1}=\binom{n-d}{r-d}$$

hulls formed from selections of r of the points of F.

(b) Suppose $\alpha \notin \operatorname{conv}(F-x)$. It follows that $x \notin \operatorname{conv}(F-x)$. Consequently we may separate x from $\operatorname{conv}(F-x)$ by a hyperplane π. Let primes denote the projections of points and sets onto π with vertex x. If α is coincident with x the required result is trivial. Otherwise we have $\alpha' \in (F-x)'$; for if not, α would not be in $\operatorname{conv} F$.

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$(F-x)'$ consists of $n-1$ points in $d-1$ dimensions. Therefore, by the first part of the inductive hypothesis, α' is in at least $\binom{n-d}{r-d}$ of the hulls of selections of $r-1$ of the points of $(F-x)'$. It follows that α itself is in each of the hulls formed from x together with the corresponding selection of $r-1$ of the points of $F-x$, that is x is in at least $\binom{n-d}{r-d}$ hulls formed from selections of r of the points of F. (Note that in accordance with the observations made earlier it does not matter if any of the image-points of the points of $F-x$ and of α under the projection coincide, for at each step of the induction the result may be extended to cover the possibility of coincident points.)

This completes the inductive step, and so establishes the theorem.

The number $\binom{n-d}{r-d}$ cannot be increased. For consider a point α just inside one of the boundary hyperfaces of F defined by exactly d of its points, α not being in the neighbourhood of any lower dimensional face of F. If α is in the hull of some r points of F, then these r points must include the d points of the hyperface. There are only $\binom{n-d}{r-d}$ ways of selecting $r-d$ more points.

An extension of Kirchberger's theorem

This extension, analogous to that just provided for Carathéodory's theorem, may be deduced in a routine way.

THEOREM 2. *If a family F of n points ($n > d+1$) in R_d, composed of two mutually exclusive sets of points B ('Black') and W ('White'), is 'unsplittable' in the sense that no hyperplane exists that separates the black points from the white points, then there are at least* $\binom{n-d-1}{r-d-1}$ *subfamilies of r of the points of F ($n \geqslant r > d+1$) that are similarly unsplittable.* (The separation here is understood to be non-strict.)

PROOF. Embed the R_d in R_{d+1} as a hyperplane not through the origin O, and consider the family $F'' = B \cup -W$, where $-W$ consists of the points that are the reflections in O of the points of W. We must have $O \in \operatorname{conv} F''$; for if not, then there must be some hyperplane π through O with F'' entirely to one side of it. π would then separate B from W, and the intersection of π with the original R_d would split the family F into black and white, which, by hypothesis, cannot be done.

We apply the first theorem to F_1'' which is a family of points in $d+1$ dimensions, and deduce that O is in at least $\binom{n-d-1}{r-d-1}$ of the hulls formed from selections of r of the points of F''. Thus, by the argument of the preceding paragraph, the corresponding subfamilies of F are unsplittable. This completes the proof.

An extension of Watson's theorem

It will be recalled that the characteristic feature of Watson's theorem was to prescribe one of the points of F to be included as a vertex of the simplexes covering $\operatorname{conv} F$. Let us incorporate this idea in a version of our Theorem 1.

THEOREM 3. *If F is a family of n points ($n > d$) in R_d, and if T is a proper subfamily of F with t points ($n-d \geqslant t \geqslant 1$), and if $\alpha \in \operatorname{conv} F$, then there are at least $\binom{n-t-d}{r-t-d}$ selections of r of the points of F ($n \geqslant r \geqslant d+t$) including all of the points of T that contain α in their convex hull.*

The proof is like that of the first theorem; so it is only necessary to indicate a few modifications: in the one-dimensional case we observe that the thickness of covering increases monotonically for $j < \frac{1}{2}(n-t)$ and decreases for $j > \frac{1}{2}(n-t)$. (Of course there is no change in thickness at x_j unless T lies entirely to one side of x_j.) The thickness over the first interval is $\binom{n-t}{r-t}$ or $\binom{n-t-1}{r-t-1}$ according as x_1 belongs to T or not; the second quantity being the smaller.

In proving the inductive step we single out $x \in T$:

(a) If $\alpha \in \operatorname{conv}(F-x)$ then it is in $\binom{(n-1)-(t-1)-d}{(r-1)-(t-1)-d}$ hulls of $r-1$ points involving the $t-1$ points of $T-x$, giving $\binom{n-t-d}{r-t-d}$ hulls of r points involving all of T.

(b) If $\alpha \notin \operatorname{conv}(F-x)$ then α' is in $\binom{(n-1)-(t-1)-(d-1)}{(r-1)-(t-1)-(d-1)}$ hulls of $r-1$ points including the $t-1$ points of $(T-x)'$, so α is in $\binom{n-t-d+1}{r-t-d+1}$ hulls of r points including all of T. And this is a greater number than $\binom{n-t-d}{r-t-d}$.

The number given in this result is again the largest possible. There is also a similarly obtained further extension of Kirchberger's theorem.

No extension of Helly's theorem

Helly's theorem (1923) is closely related to those of Carathéodory and Kirchberger, and one might expect an analogous 'theorem': If there is no point common to each of a family of n convex sets in R_d then there must be at least $n-d$ subfamilies each containing three of the sets of F and having no common point. This is not true, however, which can be seen from a counter-example.

Construct n convex sets S_j in a plane as follows: let S_1 be the interior of the square whose vertices are $(0, \pm 1)$, $(2, \pm 1)$; and let S_j $(1 < j < n)$ be the image of S_1 under an anticlockwise rotation of $\pi(j-1)/2(n-2)$ about the origin; and let S_n be the interior of the square whose vertices are $(2,0)$, $(4,2)$, $(2,4)$, $(0,2)$. The subfamily S_1, S_{n-1}, S_n just fails to have a common point $(1,1)$; but all other three-membered subfamilies clearly do have common points.

Concluding remarks

We have been concerned with statements of the form α is in *at least* so many hulls. Instead we might look for an upper bound to the number of hulls in which an arbitrary point of $\operatorname{conv} F$ may lie. This seems harder, for the answer depends on the arrangement of the points; see Baker (1978).

Rather similar investigations to the present one are reported by Birch (1959) and Katchalski (1977).

Conversation with D. R. Watson has stimulated me in this research, and I thank him.

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Royal Australian Air Force Academy
Point Cook
Victoria 3029
Australia