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A NOTE ON CYCLE TIMES IN TREE-LIKE QUEUEING NETWORKS

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Abstract

Cycle-time distribution is shown to take the form of a linear combination of M Erlang-N density functions in a cyclic queueing network of M servers and N customers. For paths of m servers in tree-like networks, the components in the more complex linear combination are convolutions of Erlang-N with at most $m-1$ negative exponentials.

1. Introduction

Some progress has been made recently in the derivation of cycle-time distributions in networks of queues, but so far only the Laplace–Stieltjes transform (LST) has been derived for cyclic networks [2], [7], tree-like networks [5], and overtake-free paths [3], [6]. Moreover, the expressions derived have not been simplified fully except in very special cases, e.g. 2-cycles [2]. It is shown here that the LST of cycle-time distributions in Markovian cyclic networks of M first-come–first-served (FCFS) exponential servers with population N, is a linear combination of M terms, and the result is then generalised to networks which are tree-like, essentially those in which all paths are overtake-free. The simplicity of the new formulae permits immediate inversion in the cyclic case, giving a weighted sum of M Erlang-N distributions, corresponding to each server in the network. The corresponding weights are more complex in the case of tree-like networks, and each component distribution is the convolution of the Erlang-N and at most $m-1$ negative exponentials for a path of length m.

2. Definitions and notation

We consider closed tree-like queueing networks of the Gordon–Newell type [4], in which all servers have negative exponential service-time distributions with constant rates, and FCFS queueing discipline. A formal definition of a tree-like network is given in [5]. Informally, it is one with a single *head* server at the top of a linear *root segment*, which has one or more servers, connected to a number of subtrees, such that there are no loops nor paths between different branches: i.e. subtrees are *disjoint*. The *leaf* centres of the tree are those which, after completing service, a customer next visits the head in a closed network, or departs from an open one. Thus cyclic networks are the special case of no branches, i.e. a root segment only. Cycle time is defined as the time elapsed between successive arrivals by some customer at the head.

Tree-like networks with FCFS queueing discipline therefore possess the non-overtaking property [8], and, if paths must all start at the same server, are the most general class for which it holds.

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Given a closed tree-like network, A, consisting of M servers with head server numbered 1, and having a population of N customers, define the following notation.

$$S=\left\{\boldsymbol{n}\;\middle|\;\sum_{i=1}^{M} n_i=N;\ n_i\geqq 0,\ 1\leqq i\leqq M\right\}\text{: state space of } A.$$

$S^I=\{\boldsymbol{n}\mid \boldsymbol{n}\in S;\ n_1>0\}$: subset of initial states in which some special customer has just arrived at the head server.

μ_i: constant service rate of server i $(1\leqq i, j\leqq M)$.

p_{ij}: routing probability between servers i, j $(1\leqq i, j\leqq M)$.

e_i: visitation rate of server i $(1\leqq i\leqq M)$.

$$G(N)=\sum_{\boldsymbol{n}\in S}\prod_{i=1}^{M}(e_i/\mu_i)^{n_i}\text{: normalising constant for } S.$$

3. Main results

Proposition 1. For distinct $\{x_i\mid 1\leqq i\leqq M\}$,

$$g_M(\boldsymbol{x})=\sum_{\boldsymbol{n}\in S}\prod_{i=1}^{M}x_i^{n_i}=\sum_{j=1}^{M}x_j^{N+M-1}\Big/\left\{\prod_{i\neq j}(x_j-x_i)\right\}.$$

The proof is by induction on M.

If $\{x_i\}$ is generate, say $x_{M-1}=x_M$, a similar result is easily derived, for example via l'Hôpital's rule.

It can be shown, e.g. [7], that for a cyclic network, A, in stochastic equilbrium, the cycle-time distribution has Laplace–Stieltjes transform

$$L(s)=\{G(N-1)\}^{-1}\sum_{\sum n_i=N-1}\prod_{i=1}^{M}\mu_i(s+\mu_i)^{-n_i-1}.$$

If the service rates μ_i $(1\leqq i\leqq M)$ are distinct, we have, by direct application of Proposition 1, the following result.

Lemma 1.

$$L(s)=\{G(N-1)\}^{-1}\left\{\prod_{i=1}^{M}\mu_i\right\}\sum_{j=1}^{M}\left\{\prod_{i\neq j}(\mu_i-\mu_j)\right\}^{-1}(s+\mu_j)^{-N}.$$

Degeneracy may be accommodated by coalescing degenerate servers and analysing the resulting smaller network with a few minor complications, cf. Proposition 1.

Since $\{\lambda/(s+\lambda)\}^N$ is the LST of the Erlang-N distribution with parameter λ, we have the following theorem.

Theorem 1. The probability density function of cycle time in the cyclic network defined above is

$$\{G(N-1)\}^{-1}\left\{\prod_{i=1}^{M}\mu_i\right\}\{t^{N-1}/(N-1)!\}\sum_{j=1}^{M}\left\{\prod_{i\neq j}(\mu_i-\mu_j)^{-1}e^{-\mu_j t}.\right.$$

Example (Chow [2]). For $M=2$, the result is

$$\{\mu_1\mu_2/G(N-1)\}\{t^{N-1}/(N-1)!\}\{e^{-\mu_1 t}-e^{-\mu_2 t}\}/(\mu_2-\mu_1)$$
$$=(\mu_1\mu_2)^N\{t^{N-1}/(N-1)!\}\{e^{-\mu_1 t}-e^{-\mu_2 t}\}/(\mu_2^N-\mu_1^N).$$

Now let Z denote the set of all paths, i.e. sequences of centres entered in passage through the network A. Then if $\mathbf{z}=(z_1, z_2, \cdots, z_k)\in Z$, $z_1=1$, z_k is a leaf centre, and the order of Z is the number of leaf centres. Let $p(\mathbf{z})$ be the probability of choosing path $\mathbf{z}$, equal to the product of the routing probabilities between successive component centres.

For the tree-like network A, the LST of cycle-time distribution, conditional on choice of path $z\in Z$, is easily seen, by generalising the argument for the cyclic case, to be

$$L(s \mid z)=\{G(N-1)\}^{-1}\sum_{\mathbf{n}\in S'}\prod_{i=1}^{M}(e_i/\mu_i)^{n_i}\prod_{j=1}^{|z|}\{\mu_{z_j}/(s+\mu_{z_j})\}^{n_{z_j}+1}$$

where $|z|$ is the number of centres in path z, and S' is the state space of A when its population is $N-1$.

Now let $G^z(n)$ be the normalising constant corresponding to all servers not in path z, and population n; a function only of $\{e_i, \mu_i \mid \exists j$ such that $z_j=i\}$ and n. G^z is simply computed, as a by-product of G, by the algorithm in [1].

Corollary 1. If the centres in path $z\in Z$ have distinct visitation: service-rate ratios, e_i/μ_i $(1\leqq i\leqq M)$

$$L(s \mid z)=\{G(N-1)\}^{-1}\sum_{h=1}^{N}G^z(N-h)\left\{\prod_{i=1}^{k}\mu_i\right\}\sum_{j=1}^{k}e_j^h\left\{\prod_{\substack{i=1\\ i\neq j}}^{k}C_{ij}\right\}^{-1}(s+\mu_j)^{-h}$$

where

$$C_{ij}=\begin{cases}\mu_i-\mu_j & \text{if}\quad e_i=e_j \qquad (\neq 0 \text{ by hypothesis})\\ (e_j-e_i)(s+\lambda_{ij}) & \text{if}\quad e_i\neq e_j\end{cases}$$

and

$$\lambda_{ij}=(\mu_i e_j-e_i\mu_j)/(e_j-e_i) \qquad (\neq 0 \text{ by hypothesis}).$$

Proof. We partition the sum over S' according to the total number of customers, $h+1$, at servers in the path z. Assume without loss of generality that $z=(1, 2, \cdots, k)$. Then

$$G(N-1)L(s \mid z)=\sum_{h=1}^{N}\sum_{\sum_{i>j}n_i=N-h}\left\{\prod_{i>k}(e_i/\mu_i)^{n_i}\right\}\sum_{\sum_{i\leq j}n_i=h-1}\prod_{i\leqq k}\{\mu_i/(s+\mu_i)\}\prod_{i\leqq k}\{e_i/(s+\mu_i)\}^{n_i}.$$

Let $x_j=e_j/(s+\mu_j)$ in Proposition 1, so that $x_j-x_i=C_{ij}/\{(s+\mu_i)(s+\mu_j)\}\neq 0$. The result then follows by definition of G^z.

Corollary 2. The unconditional LST is $L(s)=\sum_{z\in Z}p(z)L(s \mid z)$ where $p(z)=\prod_{i=1}^{k-1}p_{z_iz_{i+1}}$.

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