Each share is $3x^2 - 2x - 5$, the quantum being $3x^2$; each recipient of a quantum must return $+ 2x + 5$ to the fund-box.

Out of the fund-box move $6x^4$ pounds to A, and there distribute in $3x^2$-quanta to $2x^2$ persons, who each return $2x + 5$ and therefore a total of $4x^3 + 10x^2$ to the box as shown. Next move $15x^2$ out of the box to position B and there give out in $3x^2$ quanta to $5x$ persons who each return $2x + 5$, that is, a total of $10x^2$ at C and $25x$ at D. Next move out $27x^2$ etc. until the fund is reduced to $45x + 49$ and further quantum distribution is impossible. The quotient is $2x^2 + 5x + 9$ people and the remainder $45x + 49$ pounds.

I found, long ago, a pupil calculating without division the remainder of a polynomial divided by $x^2 - 2x + 3$; he was substituting $2x - 3$ for x^2 wherever x^2 occurred in the dividend. The boy's argument was that he was applying the remainder theorem. He was right, but I might have seen that the boy was applying Horner's method. For obviously quantum-division consists in pushing every x^2-quantum out of the fund-box and receiving, for every quantum pushed out, an exchange of $2x - 3$. And have we not here a new way to the remainder theorem? Teach quantum-division, say in two steps as outlined above, emphasising perhaps the exchange for every quantum distributed till the degradation of the fund leaves a remainder with quantum output impossible. Then come to the remainder theorem; the remainder after division by $x - h$ will be the degraded fund after every x-quantum in it has been exchanged for h.

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Elementary methods in the theory of numbers

By S. A. Scott.

Introduction.

§ 1. The importance of proving inequalities of an essentially algebraic nature by "elementary" methods has been emphasised by Hardy

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(Prolegomena to a Chapter on Inequalities), and by Hardy, Littlewood and Pólya (Inequalities). The object of this Note is to show how some of the results in the early stages of Number Theory can be obtained by making a *minimum* appeal to irrational numbers and the notion of a limit. We use the elementary notion of a logarithm to a base " a " > 1, and make no appeal to the exponential function. The Binomial Theorem is only used for a positive integer index. Our minimum appeal rests in the assumption that a bounded monotone sequence tends to a limit. We adopt throughout the usual notation. Finally, it need scarcely be added that the methods employed are not claimed to be new.

§ 2. We write $\phi(r) = \prod_{r<p\leqq 2r} p$, where r is an integer, and p runs through primes only.

Lemma 1: $$\phi(r) < 2^{2r}.$$

We observe first that $\binom{2r}{r}$ is clearly an integer. Each prime p for which $r < p \leqq 2r$ divides $(2r)!$ but not $(r!)^2$. Hence $\phi(r)$ divides $\binom{2r}{r}$, and thus

$$\phi(r) \leqq \binom{2r}{r} < \sum_{m=0}^{2r} \binom{2r}{m} = (1+1)^{2r} = 2^{2r}.$$

Lemma 2: $$\prod_{p\leqq x} p < 16^x.$$

There exists an integer m such that $2^{m-1} \leqq x < 2^m$. Hence we have

$$\prod_{p\leqq x} p \leqq \prod_{p\leqq 2^m} p = \prod_{n=1}^{n=m} . \prod_{2^{n-1}<p\leqq 2^n} p$$

$$= \prod_{n=1}^{n=m} \phi(2^{n-1}) < \prod_{n=1}^{n=m} 2^{2^n} = 2^{2^{m+1}-2} < 2^{4.2^{m-1}} \leqq 2^{4x} = 16^x.$$

Lemma 3: $$\sum_{p\leqq x} \frac{\log p}{p(p-1)} \text{ is bounded.}$$

Here the logarithms are taken to a general base a not yet determined ($a > 1$).

If n is a positive integer, it easily follows by the Binomial Theorem that $\left(1+\frac{1}{n}\right)^n$ is monotone increasing and always less than 3. Hence it tends to a limit $b \leqq 3$. It follows that for $a \geqq b$, we have $\left(1+\frac{1}{n}\right)^n < a$.

Hence

$$\log\left(1+\frac{1}{n}\right) < \frac{1}{n} \text{ for } n \geqq 1. \qquad (1).$$

Now we have

$$0 < \sum_{p \leqq x} \frac{\log p}{p(p-1)} < \sum_{n=2}^{n=x} \frac{\log n}{n(n-1)} = \sum_{n=2}^{x} \log n\left(\frac{1}{n-1}-\frac{1}{n}\right)$$

$$= \log 2 + \sum_{n=2}^{n=x-1} \frac{1}{n} \log\left(1+\frac{1}{n}\right) - \frac{\log x}{x} \qquad \text{(by partial summation)}$$

$$< \log 2 + \sum_{n=2}^{n=x-1} \frac{1}{n^2} < \log 2 + \sum_{n=2}^{n=x-1} \frac{1}{n(n-1)} = \log 2 + 1 - \frac{1}{x-1} < \log 2 + 1.$$

Lemma 4: $x! > x^x d^{-x}$ for all positive integers x, where d is a certain constant.

If we were to assume the properties of the exponential function, this would follow at once, since $x^x/x! < e^x$.

Write $f(x) = \dfrac{x!\, d^x}{x^x}$.

Then $\dfrac{f(x+1)}{f(x)} = \dfrac{d}{\left(1+\frac{1}{x}\right)^x} > 1$ if $d \geqq b$.

$\therefore$ $f(x)$ increases with x, and $f(x) > f(1) = d$ for $x > 1$.

§ 3. *Theorem* 1: $\sum_{p \leqq x} \frac{\log p}{p} - \log x$ is bounded.

We start with the well-known identity

$$[x]! = \prod_{p \leqq x} p^{[x/p]+[x/p^2]+\cdots\cdots},$$

where $[y]$ means the greatest integer contained in y. Suppose $x > 3$.

Then $x^x > [x]! > \prod_{p \leqq x} p^{[x/p]} > \prod p^{x/p-1}$

$$= \prod_{p \leqq x} p^{-1} . a^{x \sum_{p \leqq x} \log p/p} > 16^{-x} . a^{x \sum_{p \leqq x} \log p/p}.$$

Hence $a^{x \log x} > a^{-x \log 16 + x \sum_{p \leqq x} \log p / p}$.

It follows that $\sum_{p \leqq x} \frac{\log p}{p} - \log x < \log 16$. To prove the inequality in the opposite direction, we have by Lemma 4 (assuming, as we may without loss, that x is an integer)

$x^x b^{-x} < x! \leqq \prod_{p \leqq x} p^{x/p + x/p^2 + \ldots + x/p^k}$ where $\frac{x}{p^k}$ is the last term whose integer part does not vanish.

$$\text{But } \frac{x}{p^2} + \frac{x}{p^3} + \ldots + \frac{x}{p^k} < \frac{x}{p} \left\{ \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{k-1}} + \frac{\frac{1}{p^k}}{1 - \frac{1}{p}} \right\}$$

$$= \frac{x}{p} \cdot \frac{\frac{1}{p}}{1 - \frac{1}{p}} = \frac{x}{p(p-1)}.$$

Hence $x^x b^{-x} < a^{x \sum_{p \leqq x} \log p / p} . a^{kx}$, where k is a constant, by Lemma 3 with $a \geqq b$.

It follows that $\sum_{p \leqq x} \frac{\log p}{p} - \log x > -\log b - k$.

Theorem 2: $\pi(x) > c_2 . \frac{x}{\log x}$ where $\pi(x)$ is the number of primes $\leqq x$, and c_2 a constant.

By Theorem 1, we have $\log x - c_1 < \sum_{p \leqq x} \frac{\log p}{p} < \log x + c_1$. Let h be any number $> 2c_1$.

$$\text{Then } \sum_{a^{-h} . x < p \leqq x} \frac{\log p}{p} \leqq \frac{\log x}{a^{-h} . x} \sum_{p \leqq x} 1 = a^h . \pi(x) \frac{\log x}{x}.$$

$$\text{Also } \sum_{a^{-h} . x < p \leqq x} \frac{\log p}{p} = \sum_{p \leqq x} \frac{\log p}{p} - \sum_{p \leqq a^{-h} . x} \frac{\log p}{p}$$

$$> \log x - c_1 - \{\log(a^{-h} . x) + c_1\} = h - 2c_1.$$

$$\text{Hence } \pi(x) \Big/ \frac{x}{\log x} > a^{-h}(h - 2c_1) = c_2.$$

Theorem 3: $\pi(x) < c_3 . \frac{x}{\log x}$.

With the notation of Lemma 1, we have $\phi(n) < 2^{2n}$.

Write $\theta(x) = \sum_{p \leqq x} \log p$.

Then $\quad 2n \log 2 > \log \phi(n) = \theta(2n) - \theta(n)$.

Take $\quad n = 2^{r-1}$. $\quad$ Hence $2^r \log 2 > \theta(2^r) - \theta(2^{r-1})$.

Sum the inequalities obtained from this by giving r the values $1, 2, 3 \ldots . m$. $\quad$ We obtain $\theta(2^m) < 2 \log 2 \sum_{r=1}^{m} 2^{r-1} < 2^{m+1} \log 2$.

Determine m now by $2^{m-1} \leqq x < 2^m$.

Hence $\theta(x) \leqq \theta(2^m) \leqq 4x \log 2$.

Again, $\theta(x) \geqq \sum_{x^{\frac{1}{2}} < p \leqq x} \log p \geqq \{\pi(x) - \pi(x^{\frac{1}{2}})\} \log x^{\frac{1}{2}}$.

Since $\pi(x^{\frac{1}{2}}) < x^{\frac{1}{2}}$, it follows that

$$\pi(x) \Big/ \frac{x}{\log x} < 2\,\frac{\theta(x)}{x} + \frac{\log x}{x^{\frac{1}{2}}} < 8 \log 2 + \frac{\log x}{x^{\frac{1}{2}}}.$$

Suppose now that $[x^{\frac{1}{2}}] = n$.

Then $\dfrac{\log x^{\frac{1}{2}}}{x^{\frac{1}{2}}} < \dfrac{\log(n+1)}{n} < 1$, the last inequality being proved by a simple induction with the aid of (1).

$$\text{Hence} \quad \pi(x) \Big/ \frac{x}{\log x} < 8 \log 2 + 2 = c_3.$$

§ 4. *Lemma* 5: $\sum_{n \leqq x} \dfrac{1}{n} - \log x$ is bounded, where the logarithm is taken to the *natural* base.

It can be proved by "elementary" methods (Hardy, Littlewood and Pólya, Inequalities: Theorem 35) that

$$\left(1 + \frac{1}{x}\right)^x \text{ increases steadily for all } x > 0, \text{ and}$$

$$\left(1 - \frac{1}{x}\right)^{-x} \text{decreases steadily for all } x > 1.$$

Hence $\left(1 - \dfrac{1}{x}\right)^{-x}$ tends to a finite limit c which is clearly the same as when $x \to \infty$ through integer values.

$$\text{Thus } \left(1 - \frac{1}{x}\right)^{-x} > c \text{ for all } x > 1. \tag{2}$$

Similarly $\left(1 + \dfrac{1}{x}\right)^x < a$, from which it follows, putting $\dfrac{1}{x} = y$, that $\log(1 + y) < y$, for $0 < y < 1$. $\qquad$ (3)

Now $\left(1 + \dfrac{1}{n}\right)^n = \left(1 + \dfrac{1}{m-1}\right)^{m-1}$ (if $n = m-1$) $= \left(1 - \dfrac{1}{m}\right)^{-m} . \left(1 - \dfrac{1}{m}\right)$.

It follows that $b = c$.

Taking $a \leqq c\,(=b)$ in (2) above, we have

$$\left(1-\frac{1}{x}\right)^{-x} > a. \quad \text{Hence } \log\left(1-\frac{1}{x}\right) < -\frac{1}{x} \text{ for } x > 1. \tag{4}$$

Since, in the previous work, we have required $a \geqq b$, we assume from now on that $a = b$.

Write $\sigma(x) = \sum_{n \leqq x} \frac{1}{n} - \log x \leqq 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n_0} - \log n_0$ if $n_0 = [x]$

$$= 1 + \sum_{n=2}^{n_0} \left\{\frac{1}{n} + \log\left(1 - \frac{1}{n}\right)\right\}$$

$$< 1 + \sum_{n=2}^{n_0} \left\{\frac{1}{n} - \frac{1}{n}\right\} = 1.$$

Also $\sigma(x) > \sigma(n_0+1) - \dfrac{1}{n_0+1} = 1 + \sum_{n=2}^{n_0+1}\left\{\dfrac{1}{n} - \log\left(1 + \dfrac{1}{n-1}\right)\right\} - \dfrac{1}{n_0+1}$

$$> 1 + \sum_{n=2}^{n_0+1}\left\{\frac{1}{n} - \frac{1}{n-1}\right\} - \frac{1}{n_0+1} = 1 + \left\{\frac{1}{n_0+1} - 1\right\} - \frac{1}{n_0+1} = 0.$$

Hence $0 < \sigma(x) < 1$.

It should be noted that the base of logarithms must here be the Napierian one.

Lemma 6: $\sum_{2 \leqq n \leqq x} \dfrac{1}{n \log n} - \log\log x$ is bounded, where (for simplicity) x is an integer.

The expression in the lemma is $\sum_{2 \leqq n \leqq x} a_n$ where $a_2 = \dfrac{1}{2 \log 2} - \log\log 2$, and, for $n \geqq 3$,

$$a_n = \frac{1}{n \log n} + \log\left\{\frac{\log(n-1)}{\log n}\right\} = \frac{1}{n \log n} + \log\left\{1 + \frac{\log\left(1 - \frac{1}{n}\right)}{\log n}\right\}$$

$$< \frac{1}{n \log n} + \log\left\{1 + \frac{\left(-\frac{1}{n}\right)}{\log n}\right\} < \frac{1}{n \log n} - \frac{1}{n \log n} = 0 \text{ (using (4)).}$$

Hence $\sum_{n=2}^{x} a_n < a_2 = \dfrac{1}{2 \log 2} - \log\log 2$.

Again, for $n \geqq 3$, $a_n = \dfrac{1}{n \log n} - \log \left\{ 1 + \dfrac{\log\left(1 + \dfrac{1}{n-1}\right)}{\log (n-1)} \right\}$

$> \dfrac{1}{n \log n} - \log \left\{ 1 + \dfrac{1}{(n-1) \log (n-1)} \right\}$ by (1)

$> \dfrac{1}{n \log n} - \dfrac{1}{(n-1) \log (n-1)}$ (since $(n-1) \log (n-1) \geqq 2 \log 2$

$= \log 4 > 1$, as $a < 3$).

Hence $\sum_{2 \leqq n \leqq x} a_n > a_2 + \dfrac{1}{x \log x} - \dfrac{1}{2 \log 2} > a_2 - \dfrac{1}{2 \log 2} = -\operatorname{loglog} 2$.

Theorem 4: $\sum_{p \leqq x} \dfrac{1}{p} - \operatorname{loglog} x$ is bounded.

Let $a_n = \dfrac{\log n}{n} - \dfrac{1}{n}$ when n is a prime p; $a_n = -\dfrac{1}{n}$ otherwise.

$\therefore \sum_{n \leqq x} a_n = \sum_{p \leqq x} \dfrac{\log p}{p} - \sum_{n \leqq x} \dfrac{1}{n} = \sum_{p \leqq x} \dfrac{\log p}{p} - \log x + \left\{ \log x - \sum_{n \leqq x} \dfrac{1}{n} \right\}$.

By Theorem 1 and Lemma 5, each part is bounded, from which it follows easily that $\sum_{n \leqq x} \dfrac{a_n}{\log n}$ is bounded.

Thus $\sum_{p \leqq x} \dfrac{1}{p} - \sum_{2 \leqq n \leqq x} \dfrac{1}{n \log n}$ is bounded, and the result now follows from Lemma 6. In this theorem the base of logarithms is the natural base, since we have used Lemma 5.

Theorem 5: $\prod_{p \leqq x} \left(1 - \dfrac{1}{p}\right) < c_4 (\log x)^{-1}$.

We have seen that $\left(1 + \dfrac{1}{m}\right)^m$ is monotone-increasing. Putting $m = n - 1$, it follows that $\left(1 - \dfrac{1}{n}\right)^{-n+1}$ is monotone-increasing, and hence tends to a limit which is clearly $c\ (= a)$. Thus $\left(1 - \dfrac{1}{n}\right)^{-n+1} < a$ for $n > 1$.

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Hence $\log\left(1-\frac{1}{n}\right) > -\frac{1}{n-1}$.

We have, therefore,

$$\left|\log \prod_{p \leqq x}\left(1-\frac{1}{p}\right)+\sum_{p \leqq x} \frac{1}{p}\right| < \sum_{p \leqq x} \frac{1}{p(p-1)} < \sum_{2 \leqq n \leqq x}\left(\frac{1}{n-1}-\frac{1}{n}\right) < 1.$$

Also $\left|\sum_{p \leqq x} \frac{1}{p} - \log\log x\right| < c_5$.

Hence $\left|\log \prod_{p \leqq x}\left(1-\frac{1}{p}\right) + \log\log x\right| < c_6$, which proves the theorem.

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A note on some networks of polygons

By W. BARRETT.

Given an infinity of polygons which form the boundary of a finite number of polyhedra, we shall consider the complex K consisting of the polyhedra, and of the faces, edges and vertices of the polygons. We consider only those cases in which the Eulerian Characteristic N of K is finite. *Then if the mean number of sides meeting at a vertex is p, and the mean number of sides of a polygon is q, then*

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{2}.$$

The complex K is considered as the limit of a complex K' having a finite number v_0 of points, v_1 of edges, v_2 of polygons, and v_3 of polyhedra, when v_2 tends to infinity in a definite manner. Since $v_2 \leqq \sum_{r=1}^{v_1} \binom{v_1}{r}$, which is finite if v_1 is finite, it follows that v_1 is infinite if v_2 is infinite.

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