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HEREDITARY TORSION THEORIES OF A LOCALLY NOETHERIAN GROTHENDIECK CATEGORY

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Abstract

Let $\mathcal{A}$ be a locally noetherian Grothendieck category. We construct closure operators on the lattice of subcategories of $\mathcal{A}$ and the lattice of subsets of ASpec $\mathcal{A}$ in terms of associated atoms. This establishes a one-to-one correspondence between hereditary torsion theories of $\mathcal{A}$ and closed subsets of ASpec $\mathcal{A}$. If $\mathcal{A}$ is locally stable, then the hereditary torsion theories can be studied locally. In this case, we show that the topological space ASpec $\mathcal{A}$ is Alexandroff.

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1. Introduction

The classification of subcategories of an abelian category is an important area widely studied by numerous authors in recent years (see [1, 3, 10, 11]). The subject originates from a result of Gabriel [1] classifying localising and Serre subcategories of R-Mod in terms of specialisation closed subsets of Spec$R$, the prime spectrum of R, when R is a commutative noetherian ring.

In 1997, Herzog [2] and Krause [6] gave a classification of localising subcategories of finite type for locally coherent Grothendieck categories in terms of indecomposable injectives. Recently, for an abelian category $\mathcal{A}$, Kanda [4] defined and studied the atom spectrum ASpec$\mathcal{A}$ of $\mathcal{A}$, the class of atoms in $\mathcal{A}$, which is analogous to the prime spectrum of a commutative ring. As Kanda [4, 5] has shown, this notion is easier to use, but the study of indecomposable injectives has a longer history.

Given an object M of $\mathcal{A}$, we define *the associated atoms* of M, denoted by AAss(M), a subclass of ASupp(M), by

$$\text{AAss}\, M = \{\overline{H} \in \text{ASupp}(M) \mid \text{there exists } H' \in \overline{H} \text{ which is a subobject of } M\}.$$

More generally, for any subcategory $\mathcal{X}$ of A, we define

$$\text{AAss}(\mathcal{X}) = \bigcup_{M \in \mathcal{X}} \text{AAss}\, M.$$

Throughout this paper, $\mathcal{A}$ is assumed to be a locally noetherian Grothendieck category. We will show that associated atoms of subcategories can construct closure operators on the lattice of subcategories of $\mathcal{A}$ and the lattice of subsets of $\mathrm{ASpec}\mathcal{A}$. We first show that hereditary torsion theories of $\mathcal{A}$ correspond to closed subsets of the topological space $\mathrm{ASpec}\mathcal{A}$. More precisely, we prove that the map $\mathcal{F} \mapsto \mathrm{AAss}(\mathcal{F})$ establishes a one-to-one correspondence between torsion-free subcategories of $\mathcal{A}$ corresponding to some hereditary torsion theory and closed subsets of $\mathrm{ASpec}\mathcal{A}$. The inverse map is given by $V \mapsto \mathrm{AAss}^{-1}(V)$ (Theorem 2.7).

For any object M of $\mathcal{A}$ with injective envelope $E(M)$, the localising subcategory defined by M, denoted by $\mathcal{X}(M)$, consists of all objects N such that $\mathrm{Hom}(N, E(M)) = 0$. We prove that for any localising subcategory $\mathcal{X}$ of $\mathcal{A}$, there exists an object M such that $\mathcal{X} = \mathcal{X}(M)$ (Proposition 2.17). Moreover, we prove that, for every $\alpha = \overline{H} \in \mathrm{ASpec}\mathcal{A}$ with monoform object H of $\mathcal{A}$, $\mathcal{X}(H) = \mathcal{X}(\alpha)$, where $\mathcal{X}(\alpha) = \mathrm{ASupp}^{-1}(\mathrm{ASpec}\mathcal{A} \setminus \overline{\{\alpha\}})$ (Theorem 2.10).

For every subcategory Y of $\mathcal{A}$, let $(\mathcal{T}(Y), \mathcal{F}(Y))$ be the torsion theory cogenerated by Y. It is shown that $\mathcal{T}(Y)$ is localising if Y is closed under subobjects and injective envelopes (Theorem 2.12). We also show that Y is closed under subobjects, injective envelopes and direct unions if and only if $\mathrm{AAss}^{-1}(\mathrm{AAss}(Y)) = Y$. We note that $\mathcal{X}(M) = \mathcal{T}(\mathrm{AAss}^{-1}(\mathrm{AAss}(M)))$ for any object M of $\mathcal{A}$.

We prove that the map $Y \mapsto \mathcal{F}(\mathrm{AAss}^{-1}(\mathrm{AAss}(Y)))$ is a closure operator on the lattice of all subcategories Y of $\mathcal{A}$ and, symmetrically, $V \mapsto \mathrm{AAss}(\mathcal{F}(\mathrm{AAss}^{-1}(V)))$ is a closure operator on the set of all subsets of $\mathrm{ASpec}\mathcal{A}$. Finally, in Theorem 2.19, for an object M of $\mathcal{A}$, we determine $\mathrm{AAss}\,\mathcal{Y}(M)$ when M is finitely generated or $\mathcal{A}$ is locally stable. As a corollary, we prove that the topological space $\mathrm{ASpec}\mathcal{A}$ of a locally stable Grothendieck category is Alexandroff (Corollary 2.21).

2. The main results

We first recall some concepts and definitions of abelian categories. Monoform objects and the atom spectrum of an abelian category were defined by Kanda [4].

Definition 2.1.

(i) A nonzero object M in $\mathcal{A}$ is *monoform* if for any nonzero subobject N of M, there exists no common nonzero subobject of M and M/N: that is, there does not exist a nonzero subobject of M which is isomorphic to a subobject of M/N. We denote the class of all monoform objects of $\mathcal{A}$ by $\mathrm{ASpec}_0\mathcal{A}$.

(ii) Two monoform objects H and H' are said to be *atom-equivalent* if they have a common nonzero subobject.

(iii) By [4, Proposition 2.8], the atom equivalence establishes an equivalence relation on monoform objects; for a monoform object H, we denote the *equivalence class* of H by $\overline{H}$: that is $\overline{H} = \{G \in \mathrm{ASpec}_0\mathcal{A} \mid H \text{ and } G$ have a common nonzero subobject$\}$.

(iv) The *atom spectrum* $\mathrm{ASpec}\mathcal{A}$ of $\mathcal{A}$ is the quotient class of $\mathrm{ASpec}_0\mathcal{A}$ consisting of all equivalence classes induced by this equivalence relation.

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(v) A subclass Φ of $\operatorname{ASpec}\mathcal{A}$ is called *open* if, for any $\overline{H} \in \Phi$, there exists $H' \in \overline{H}$ such that $\operatorname{ASupp}(H') \subset \Phi$. The open subclasses are also called *closed under specialisation* as they correspond to the specialisation closed subsets of $\operatorname{Spec} A$, where A is a commutative ring (see [4]). A subclass Ψ of $\operatorname{ASpec}\mathcal{A}$ is called *closed* (or *closed under generalisation*) if $\operatorname{ASpec}\mathcal{A} \setminus \Psi$ is open.

(vi) For an object M of $\mathcal{A}$, we define a subclass $\operatorname{ASupp}(M)$ of $\operatorname{ASpec}\mathcal{A}$ by

$$\operatorname{ASupp} M = \{\overline{H} \in \operatorname{ASpec}\mathcal{A} \mid \text{there exists } H' \in \overline{H} \text{ which is a subquotient of } M\}.$$

We also define *the associated atoms* of M, denoted by $\operatorname{AAss}(M)$, a subclass of $\operatorname{ASupp}(M)$, by

$$\operatorname{AAss} M = \{\overline{H} \in \operatorname{ASupp}(M) \mid \text{there exists } H' \in \overline{H} \text{ which is a subobject of } M\}.$$

For any subcategory $\mathcal{X} \subset \mathcal{A}$, we define

$$\operatorname{ASupp}(\mathcal{X}) = \bigcup_{M \in \mathcal{X}} \operatorname{ASupp} M, \quad \operatorname{AAss}(\mathcal{X}) = \bigcup_{M \in \mathcal{X}} \operatorname{AAss} M.$$

Obviously, $\operatorname{ASupp}(M)$ is an open subclass of $\operatorname{ASpec}\mathcal{A}$ for any object M and it follows that $\operatorname{ASupp}(\mathcal{X})$ is open for any subcategory $\mathcal{X}$ of $\mathcal{A}$.

Definition 2.2. An abelian category $\mathcal{A}$ is called a *Grothendieck category* if it has exact direct limits and a generator. A Grothendieck category $\mathcal{A}$ is called *locally noetherian* if there exists a generating set of $\mathcal{A}$ consisting of noetherian objects.

Remark 2.3. If $\mathcal{A}$ is a locally noetherian Grothendieck category, by [4, Proposition 5.2], $\operatorname{noeth}\mathcal{A}$, the full subcategory of $\mathcal{A}$ consisting of all noetherian objects, is skeletally small so that it is a noetherian abelian category. Thus $\operatorname{ASpec}(\operatorname{noeth}\mathcal{A})$ forms a set. On the other hand, by [4, Proposition 5.3], $\operatorname{ASpec}\mathcal{A}$ coincides with $\operatorname{ASpec}(\operatorname{noeth}\mathcal{A})$ as topological spaces. This ensures that $\operatorname{ASpec}\mathcal{A}$ is a set. So we can replace the notion open (closed) subclasses of $\operatorname{ASpec}\mathcal{A}$ by open (closed) subsets.

Recall from [5] that $\operatorname{ASpec}\mathcal{A}$ can be regarded as a partially ordered set together with a specialisation order $\leq$ as follows. For any atoms α and β in $\operatorname{ASpec}\mathcal{A}$, we define $\alpha \leq \beta$ if and only if, for any open subclass Φ of $\operatorname{ASpec}\mathcal{A}$ satisfying $\alpha \in \Phi$, $\beta \in \Phi$. For more details, we refer the reader to [5].

Definition 2.4. A torsion theory for $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of subcategories of $\mathcal{A}$ satisfying:

(i) $\operatorname{Hom}(T, F) = 0$ for all $T \in \mathcal{T}, F \in \mathcal{F}$;
(ii) if $\operatorname{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$, then $F \in \mathcal{F}$; and
(iii) if $\operatorname{Hom}(T, F) = 0$ for all $F \in \mathcal{F}$, then $T \in \mathcal{T}$.

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$\mathcal{T}$ is called a *torsion subcategory* and its objects are *torsion objects*, while $\mathcal{F}$ is a *torsion-free subcategory* consisting of *torsion-free objects*. It is easy to see that $\mathcal{T}$ is closed under quotients, direct sums and extensions, while $\mathcal{F}$ is closed under subobjects, products and extensions.

A torsion theory $(\mathcal{T}, \mathcal{F})$ is called *hereditary* if $\mathcal{T}$ is closed under subobjects and so, in this case, $\mathcal{T}$ is called a *localising* subcategory. If $\mathcal{X}$ is a subcategory of $\mathcal{A}$ closed under subobjects, quotients, direct sums and extensions, then it is a localising subcategory of some hereditary torsion theory (see [9, Ch. VI, Proposition 2.1]).

A *radical functor* $r(-)$ of $\mathcal{A}$ is a subfunctor of the identity functor $I(-) : \mathcal{A} \to \mathcal{A}$ in the functor category $\mathrm{Func}(\mathcal{A}, \mathcal{A})$ such that, for any object M of $\mathcal{A}$:

(i) $r(r(M)) = r(M)$; and
(ii) $r(M/r(M)) = 0$.

In view of [9, Ch. VI, Proposition 2.3], every torsion theory $(\mathcal{T}, \mathcal{F})$ induces a unique radical functor on $\mathcal{A}$.

In the rest of this paper, $\mathcal{A}$ is a locally noetherian Grothendieck category.

Proposition 2.5. *Let V be a closed subset of* $\mathrm{ASpec}\mathcal{A}$. *Then* $\mathrm{AAss}^{-1}(V)$ *is a torsion-free subcategory of a hereditary torsion theory of* $\mathcal{A}$.

Proof. Since V is closed, $U = \mathrm{ASpec}\mathcal{A} \setminus V$ is an open subset of $\mathrm{ASpec}\mathcal{A}$. Thus, according to [4, Theorem 5.7], there exist a localising subcategory $\mathcal{T} = \mathrm{ASupp}^{-1}(U)$ and a radical functor $t_{\mathcal{T}}(-)$ such that, for every object M in $\mathcal{A}$, the object $t_{\mathcal{T}}(M)$ is the largest subobject of M contained in $\mathcal{T}$. Then $\mathcal{T}$ can be included in a hereditary torsion theory $(\mathcal{T}, \mathcal{F})$, where $\mathcal{F} = \{N \in \mathcal{A} \mid t_{\mathcal{T}}(N) = 0\}$. We now assert that $\mathrm{AAss}^{-1}(V) = \mathcal{F}$. Assume that $M \in \mathrm{AAss}^{-1}(V)$ from which we aim to show that $t_{\mathcal{T}}(M) = 0$. Suppose, on the contrary, that $t_{\mathcal{T}}(M) \neq 0$. Then $\mathrm{AAss}(t_{\mathcal{T}}(M)) \subseteq \mathrm{ASupp}(t_{\mathcal{T}}(M)) \subseteq \mathrm{ASupp}(\mathcal{T}) = U$. But $\mathrm{AAss}(t_{\mathcal{T}}(M)) \subseteq \mathrm{AAss}(M) \subseteq V$, which is a contradiction. Conversely, assume that $M \in \mathcal{F}$. If $\mathrm{AAss}(M) \nsubseteq V$, then there exists $\beta \in \mathrm{AAss}(M) \setminus V$ and so $\beta \in U$. Hence there exists a monoform object H such that $\overline{H} = \beta$ and $\mathrm{ASupp}(H) \subseteq U = \mathrm{ASupp}(\mathcal{T})$. This implies that $H \in \mathcal{T}$, by [4, Theorem 5.7]. On the other hand, there exists a monoform object H_1 with $\beta = \overline{H_1}$ such that H_1 is a subobject of M. Since H and H_1 have a common nonzero subobject, H_1 contains a subobject H_2 belonging to $\mathcal{T}$. But this implies that $t_{\mathcal{T}}(M)$ is nonzero, which contradicts $M \in \mathcal{F}$. □

Proposition 2.6. *Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory of $\mathcal{A}$. Then* $\mathrm{ASupp}\,\mathcal{T} = \mathrm{ASpec}\mathcal{A} \setminus \mathrm{AAss}(\mathcal{F})$ *and so* $\mathrm{AAss}(\mathcal{F})$ *is a closed subset of* $\mathrm{ASpec}\mathcal{A}$.

Proof. Assume that $\alpha \in \mathrm{ASupp}\,\mathcal{T}$. Since $\mathrm{ASupp}\,\mathcal{T}$ is open, there exists a monoform object H with $\alpha = \overline{H}$ and $\mathrm{ASupp}\,H \subseteq \mathrm{ASupp}\,\mathcal{T}$. Thus [4, Theorem 5.7] implies that $H \in \mathcal{T}$. On the other hand, if $\alpha \in \mathrm{AAss}\,\mathcal{F}$, there exists a monoform subobject $H_1 \in \mathcal{F}$ with $\alpha = \overline{H_1}$. But H and H_1 have a common nonzero subobject belonging to $\mathcal{F} \cap \mathcal{T} = 0$, which is a contradiction. Conversely, if α is not in $\mathrm{AAss}(\mathcal{F})$, then $t_{\mathcal{T}}(H) \neq 0$ for any monoform object H with $\alpha = \overline{H}$. Hence $\alpha \in \mathrm{ASupp}(t_{\mathcal{T}}(H)) \subseteq \mathrm{ASupp}(\mathcal{T})$ because $t_{\mathcal{T}}(H) \in \mathcal{T}$. □

Associated atoms establish a one-to-one correspondence between hereditary torsion theories of $\mathcal{A}$ and closed subsets of $\mathrm{ASpec}\mathcal{A}$.

THEOREM 2.7. *The map $\mathcal{F} \mapsto \mathrm{AAss}(\mathcal{F})$ establishes a one-to-one correspondence between torsion-free subcategories of $\mathcal{A}$ corresponding to some hereditary torsion theory and closed subsets of* $\mathrm{ASpec}\mathcal{A}$. *The inverse map is given by $V \mapsto \mathrm{AAss}^{-1}(V)$.*

PROOF. Suppose that $\mathbb{F}$ is the class of all torsion-free subcategories corresponding to some torsion theory and that $\mathbb{C}$ is the set of all closed subsets of $\mathrm{ASpec}\mathcal{A}$. In view of Propositions 2.5 and 2.6, we can define the map $\mathbb{F} \longrightarrow \mathbb{C}$ by $\mathcal{F} \longrightarrow \mathrm{AAss}(\mathcal{F})$ and the map $\mathbb{C} \longrightarrow \mathbb{F}$ by $V \mapsto \mathrm{AAss}^{-1}(V)$ for any $\mathcal{F} \in \mathbb{F}$ and $V \in \mathbb{C}$. It only remains to show that $\mathrm{AAss}^{-1}(\mathrm{AAss}(\mathcal{F})) = \mathcal{F}$ and $\mathrm{AAss}(\mathrm{AAss}^{-1}(V)) = V$ for any $\mathcal{F} \in \mathbb{F}$ and $V \in \mathbb{C}$. Assume that $(\mathcal{T}, \mathcal{F})$ is the hereditary torsion theory corresponding to $\mathcal{F}$. For every $M \in \mathrm{AAss}^{-1}(\mathrm{AAss}(\mathcal{F}))$, we prove that $t_{\mathcal{T}}(M) = 0$ and so $M \in \mathcal{F}$. If $\mathrm{AAss}(t_{\mathcal{T}}(M))$ is a nonempty set, then $\mathrm{AAss}(t_{\mathcal{T}}(M)) \subseteq \mathrm{AAss}(M) \subseteq \mathrm{AAss}(\mathcal{F}) = \mathrm{ASpec}\mathcal{A} \setminus \mathrm{ASupp}(\mathcal{T})$. On the other hand, $\mathrm{AAss}(t_{\mathcal{T}}(M)) \subseteq \mathrm{ASupp}(\mathcal{T})$, which is a contradiction. The inclusion $\mathcal{F} \subseteq \mathrm{AAss}^{-1}(\mathrm{AAss}(\mathcal{F}))$ is clear. In order to verify the second equality, assume that $\alpha \in V$ with $\alpha = \overline{H}$ for some monoform object H. Then $\mathrm{AAss}(H) = \{\alpha\} \subseteq V$ and hence $H \in \mathrm{AAss}^{-1}(V)$. Therefore $\alpha \in \mathrm{AAss}(H) \subseteq \mathrm{AAss}(\mathrm{AAss}^{-1}(V))$. For the other inclusion, assume that $\alpha \in \mathrm{AAss}(\mathrm{AAss}^{-1}(V))$. Then there exists $M \in \mathrm{AAss}^{-1}(V)$ such that $\alpha \in \mathrm{AAss}(M) \subseteq V$ so that $\alpha \in V$. □

COROLLARY 2.8. *Let $\mathcal{F}$ be a torsion-free subcategory of $\mathcal{A}$ corresponding to some hereditary torsion theory and let $\alpha \in \mathrm{AAss}(\mathcal{F})$. Then $\alpha \subseteq \mathcal{F}$.*

PROOF. For every monoform H with $\alpha = \overline{H}$, $\mathrm{AAss}(H) = \{\alpha\}$. Now Theorem 2.7 implies that $H \in \mathrm{AAss}^{-1}(\mathrm{AAss}(\mathcal{F})) = \mathcal{F}$. □

DEFINITION 2.9. Let M be an object of $\mathcal{A}$ with the injective envelope $E(M)$. We recall from [7] *the localising subcategory defined by* M, denoted by $\mathcal{X}(M)$, which is

$$\mathcal{X}(M) = \{N \in \mathcal{A} \mid \mathrm{Hom}(N, E(M)) = 0\}.$$

We denote by $\mathcal{Y}(M)$ the torsion-free subcategory corresponding to $\mathcal{X}(M)$, which is

$$\mathcal{Y}(M) = \{N \in \mathcal{A} \mid \mathrm{Hom}(X, N) = 0 \text{ for all } X \in \mathcal{X}(M)\}.$$

Observe that if N is an essential subobject of M, then $\mathcal{X}(N) = \mathcal{X}(M)$.

For every $\alpha \in \mathrm{ASpec}\mathcal{A}$, the topological closure of α, denoted by $\overline{\{\alpha\}}$, consists of all $\beta \in \mathrm{ASpec}\mathcal{A}$ such that $\beta \leq \alpha$. According to [4, Theorem 5.7], for each atom α, there is a localising subcategory $\mathcal{X}(\alpha)$ induced by α: that is $\mathcal{X}(\alpha) = \mathrm{ASupp}^{-1}(\mathrm{ASpec}\mathcal{A} \setminus \overline{\{\alpha\}})$. We denote by $\mathcal{Y}(\alpha)$ the torsion-free subcategory corresponding to $\mathcal{X}(\alpha)$. Obviously, $\mathrm{AAss}(\mathcal{Y}(\alpha)) = \overline{\{\alpha\}}$.

We point out that any two monoform objects H and H_1 with $\alpha = \overline{H} = \overline{H_1}$ have a common nonzero subobject X that is essential in H and H_1. This implies that $E(H) = E(H_1)$ and so we denote $E(H)$ by $E(\alpha)$ as it is independent of the choice of the monoform object H.

THEOREM 2.10. *Let $\alpha = \overline{H}$ be an atom for some monoform object H of $\mathcal{A}$. Then $\mathcal{X}(H) = \mathcal{X}(\alpha)$.*

PROOF. Assume that $M \in \mathcal{X}(\alpha)$ and so $\mathrm{ASupp}(M) \cap \overline{\{\alpha\}} = \emptyset$. We claim that $\mathrm{Hom}(M, E(\alpha)) = 0$. Suppose, on the contrary, that there exists a nonzero element $f \in \mathrm{Hom}(M, E(\alpha))$. Then $\mathrm{Im} f$ is a nonzero subobject of $E(\alpha)$ and $\mathrm{AAss}(\mathrm{Im} f) = \{\alpha\}$. This forces $\alpha \in \mathrm{ASupp}(M)$, which is a contradiction. Conversely, assume that $M \in \mathcal{X}(H)$ and so $\mathrm{Hom}(M, E(\alpha)) = 0$. Without loss of generality, we may assume that M is finitely generated. Then it follows, from [4, Theorem 5.9], that $\alpha \notin \mathrm{ASupp}(M)$. If $M \notin \mathcal{X}(\alpha)$, then $\mathrm{ASupp}(M) \nsubseteq \mathrm{ASupp}(\mathcal{X}(\alpha))$ and so there exists $\beta \in \mathrm{ASupp}(M)$ such that $\beta \notin \mathrm{ASupp}(\mathcal{X}(\alpha))$. Hence $\beta \in \overline{\{\alpha\}}$ and so $\beta \le \alpha$. Now, since $\beta \in \mathrm{ASupp}(M)$, $\alpha \in \mathrm{ASupp}(M)$, which is a contradiction. □

The following proposition shows that the localising subcategory defined by an object can be specified by the localising subcategory defined by its monoform subobjects.

PROPOSITION 2.11. *Let M be an object of $\mathcal{A}$. Then $\mathcal{X}(M) = \bigcap_{\alpha \in \mathrm{AAss}(M)} \mathcal{X}(\alpha)$.*

PROOF. By Matlis theorem and [4, Theorem 5.11], $E(M) = \bigoplus_{\alpha \in \mathrm{AAss}(M)} E(\alpha)^{\mu_\alpha}$. Now assume that $N \in \mathcal{X}(M)$. Since $\mathcal{X}(M)$ is closed under direct limits, without loss of generality, we may assume that N is finitely generated. Then

$$0 = \mathrm{Hom}(N, E(M)) \cong \bigoplus_{\alpha \in \mathrm{AAss}(M)} \mathrm{Hom}(N, E(\alpha))^{\mu_\alpha},$$

which implies that $\mathrm{Hom}(N, E(\alpha)) = 0$ for all $\alpha \in \mathrm{AAss}(M)$. Thus it follows, from Theorem 2.10, that $N \in \mathcal{X}(\alpha)$ for all $\alpha \in \mathrm{AAss}(M)$. The converse is obtained by a similar argument. □

For any subcategory Y of $\mathcal{A}$, we denote by $(\mathcal{T}(Y), \mathcal{F}(Y))$ *the torsion theory cogenerated by* Y: that is,

$$\mathcal{T}(Y) = \{T \in \mathcal{A} \mid \mathrm{Hom}(T, N) = 0 \text{ for all } N \in Y\},$$
$$\mathcal{F}(Y) = \{F \in \mathcal{A} \mid \mathrm{Hom}(T, F) = 0 \text{ for all } T \in \mathcal{T}(Y)\}.$$

THEOREM 2.12. *Let Y be a subcategory of $\mathcal{A}$. Then $\bigcap_{\alpha \in \mathrm{AAss}(Y)} \mathcal{X}(\alpha) \subseteq \mathcal{T}(Y)$. Moreover, if Y is closed under subobjects and injective envelopes, then $\mathcal{T}(Y) = \bigcap_{\alpha \in \mathrm{AAss}(Y)} \mathcal{X}(\alpha)$ and so $\mathcal{T}(Y)$ is a localising subcategory of $\mathcal{A}$.*

PROOF. Assume that $M \in \bigcap_{\alpha \in \mathrm{AAss}(Y)} \mathcal{X}(\alpha)$. It follows, from Proposition 2.11, that $M \in \mathcal{X}(N)$ for every object N in Y; and hence $\mathrm{Hom}(M, N) = 0$ for every object N in Y. For the second claim, suppose that $M \in \mathcal{T}(Y)$ and $\alpha \in \mathrm{AAss}(Y)$. By the assumption, Y contains a monoform object H such that $E(\alpha) = E(H) \in Y$. Then, using Theorem 2.10, $M \in \mathcal{X}(\alpha)$ and so $\mathcal{T}(Y) \subseteq \bigcap_{\alpha \in \mathrm{AAss}(Y)} \mathcal{X}(\alpha)$. □

REMARK 2.13. It follows immediately from Theorem 2.12 that if Y is a subcategory of A closed under subobjects and injective envelopes, then $\mathcal{T}(Y) = \bigcap_{M \in Y} \mathcal{X}(M)$. In particular, for any object M of $\mathcal{A}$, one can easily check that $(\mathcal{X}(M), \mathcal{Y}(M))$ is the hereditary torsion theory cogenerated by $\mathrm{AAss}^{-1}(\mathrm{AAss}(M))$.

Proposition 2.14. *Let Y be a subcategory of $\mathcal{A}$. Then $\mathcal{F}(\mathrm{AAss}^{-1}(\mathrm{AAss}(Y)))$ is the smallest torsion-free subcategory of $\mathcal{A}$ containing Y corresponding to some hereditary torsion theory. In particular, if $\mathcal{S}$ is the lattice of all subcategories of $\mathcal{A}$, then $\mathcal{F} : \mathcal{S} \to \mathcal{S}$, given by $Y \mapsto \mathcal{F}(\mathrm{AAss}^{-1}(\mathrm{AAss}(Y)))$, is a closure operator.*

Proof. As $\mathrm{AAss}^{-1}(\mathrm{AAss}(Y))$ is closed under subobjects and injective envelopes, according to Theorem 2.12, $\mathcal{F}(\mathrm{AAss}^{-1}(\mathrm{AAss}(Y)))$ is a torsion-free subcategory corresponding to a hereditary torsion theory. Assume that $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory such that $Y \subseteq \mathcal{F}$. It follows, from Theorem 2.7, that $\mathrm{AAss}^{-1}(\mathrm{AAss}(Y)) \subseteq \mathcal{F}$. In order to prove $\mathcal{F}(\mathrm{AAss}^{-1}(\mathrm{AAss}(Y))) \subseteq \mathcal{F}$, it suffices to show that $\mathcal{T} \subseteq \mathcal{T}(\mathrm{AAss}^{-1}(\mathrm{AAss}(Y)))$. Assume that M is an arbitrary object belonging to $\mathcal{T}$. Then $\mathrm{Hom}(M, F) = 0$ for all $\mathcal{F} \in \mathcal{F}$ and so $\mathrm{Hom}(M, F) = 0$ for all $F \in \mathrm{AAss}^{-1}(\mathrm{AAss}(Y))$. This implies that $M \in \mathcal{T}(\mathrm{AAss}^{-1}(\mathrm{AAss}(Y)))$. The second claim is straightforward. □

Proposition 2.15. *Assume V is a subset of $\mathrm{ASpec}(\mathcal{A})$. Then $\overline{V} = \mathrm{Ass}(\mathcal{F}(\mathrm{Ass}^{-1}(V)))$, where $\overline{V}$ is the closure of V in the topological space $\mathrm{ASpec}\mathcal{A}$. In particular, if $\mathcal{L}$ is the lattice of all subsets of $\mathrm{ASpec}\mathcal{A}$, then $\Phi : \mathcal{L} \to \mathcal{L}$, given by $V \mapsto \mathrm{Ass}(\mathcal{F}(\mathrm{Ass}^{-1}(V)))$, is a closure operator.*

Proof. Since $\mathrm{AAss}^{-1}(V)$ is closed under subobjects and injective envelopes, according to Theorem 2.12, $\mathcal{F}(\mathrm{Ass}^{-1}(V))$ is a torsion-free subcategory corresponding to a hereditary torsion theory. It then follows, from Proposition 2.6, that $\mathrm{Ass}(\mathcal{F}(\mathrm{Ass}^{-1}(V)))$ is a closed subset of $\mathrm{ASpec}\mathcal{A}$ containing V. Now assume that D is any closed subset of $\mathrm{ASpec}\mathcal{A}$ containing V. Then $\mathrm{AAss}^{-1}(V) \subseteq \mathrm{AAss}^{-1}(D)$. But, in view of Proposition 2.5, $\mathrm{AAss}^{-1}(D)$ is a torsion-free subcategory corresponding to a hereditary torsion theory and hence $\mathcal{F}(\mathrm{AAss}^{-1}(D)) = \mathrm{AAss}^{-1}(D)$ so that $\mathcal{F}(\mathrm{AAss}^{-1}(V)) \subseteq \mathrm{AAss}^{-1}(D)$. Consequently, it follows, from Theorem 2.7, that $\mathrm{AAss}(\mathcal{F}(\mathrm{AAss}^{-1}(V))) \subseteq D$. The second claim is straightforward. □

We recall from [8] that a proper subobject N of an object M is *atomical* if $\mathrm{AAss}(M/N)$ has just one element. If M is a noetherian object, then an *atomical decomposition* of a subobject L of M is obtained by writing L as a finite intersection $L = L_1 \cap \cdots \cap L_n$ of atomical subobjects L_i of M, so that:

(i) the decomposition is irredundant; and
(ii) $\mathrm{AAss}(M/L_i) \neq \mathrm{AAss}(M/L_j)$ for $i \neq j$.

Proposition 2.16. *Let Y be a subcategory of $\mathcal{A}$. Then $Y = \mathrm{AAss}^{-1}(\mathrm{AAss}(Y))$ if and only if Y is closed under subobjects, injective envelopes and direct unions.*

Proof. The 'only if' part is clear and so we only prove the 'if' part. Clearly, $Y \subseteq \mathrm{AAss}^{-1}(\mathrm{AAss}(Y))$ and so we have to prove the reverse of this inclusion. Assuming that $M \in \mathrm{AAss}^{-1}(\mathrm{AAss}(Y))$, we have $\mathrm{AAss}(M) \subseteq \mathrm{AAss}(Y)$. Since Y is closed under taking direct unions, we may assume that M is finitely generated. This implies that $\mathrm{AAss}(M) = \{\alpha_1, \ldots, \alpha_n\}$ is a finite set. Since Y is closed under subobjects, there exist monoform objects $H_i \in Y$ such that $\alpha_i = \overline{H_i}$. On the other hand, using [8, Propositions 2.5 and 2.7], the zero subobject of M has an atomical decomposition $0 = \bigcap_{i=1}^{n} Q_i$ such

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that $\mathrm{AAss}(M/Q_i) = \{\alpha_i\}$ and M is embedded in $\bigoplus_{i=1}^{n} M/Q_i$. But $E(M/Q_i) = E(H_i)$ for each i, and since Y is closed under injective envelopes, we deduce that $M/Q_i \in Y$. Therefore $\bigoplus_{i=1}^{n} M/Q_i \in Y$ and, consequently, $M \in Y$. □

Proposition 2.17. *For every hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$, there exists an object M of $\mathcal{F}$ such that $\mathcal{T} = \mathcal{X}(M)$ and $\mathcal{F} = \mathcal{Y}(M)$.*

Proof. (i). It suffices to show that $\mathcal{T} = \mathcal{X}(M)$ for some object M of $\mathcal{F}$. For every $\alpha \in \mathrm{ASpec}\mathcal{A} \setminus \mathrm{ASupp}(\mathcal{T})$, let $H(\alpha)$ be a monoform object such that $\alpha = \overline{H(\alpha)}$. Put $M = \bigoplus_{\alpha \notin \mathrm{ASupp}(\mathcal{T})} H(\alpha)$. Then

$$\mathcal{X}(M) = \bigcap_{\alpha \in \mathrm{AAss}(M)} \mathcal{X}(\alpha) = \bigcap_{\alpha \notin \mathrm{ASupp}(\mathcal{X})} \mathcal{X}(\alpha) = \mathcal{T},$$

where the first equality follows from Proposition 2.11 and the third follows from [5, Corollary 6.9]. On the other hand, the construction of M shows that $\mathrm{AAss}(M) = \mathrm{AAss}(\mathcal{F})$. Hence it follows, from Theorem 2.7, that M belongs to $\mathcal{F}$. □

Lemma 2.18. *Let M be an object of $\mathcal{A}$. Then $\mathcal{Y}(\alpha) \subseteq \mathcal{Y}(M)$ for every $\alpha \in \mathrm{AAss}(M)$.*

Proof. Since $\alpha \in \mathrm{AAss}(M)$, there exists a monoform object H with $\overline{H} = \alpha$ such that H is a subobject of M. Then $\mathcal{X}(M) \subseteq \mathcal{X}(H) = \mathcal{X}(\alpha)$ so that $\mathcal{Y}(\alpha) \subseteq \mathcal{Y}(M)$. □

From [1], a localising subcategory $\mathcal{T}$ of the Grothendieck category $\mathcal{A}$ is called *stable* if the injective envelope in $\mathcal{A}$ of any object of $\mathcal{T}$ is also an object of $\mathcal{T}$. Furthermore, a Grothendieck category is said to be *locally stable* if any localising subcategory is stable. We notice that if A is a commutative noetherian ring, then Mod-A is a locally stable category.

Theorem 2.19. *Assume that M is an object of $\mathcal{A}$. Then*

$$\bigcup_{\alpha \in \mathrm{AAss}(M)} \overline{\{\alpha\}} \subseteq \mathrm{AAss}(\mathcal{Y}(M)).$$

Furthermore, if M is finitely generated or $\mathcal{A}$ is locally stable, then

$$\mathrm{AAss}(\mathcal{Y}(M)) = \bigcup_{\alpha \in \mathrm{AAss}(M)} \overline{\{\alpha\}}.$$

Proof. For every $\alpha \in \mathrm{AAss}(M)$, it follows, from Lemma 2.18, that $\mathcal{Y}(\alpha) \subseteq \mathcal{Y}(M)$ and so $\overline{\{\alpha\}} = \mathrm{AAss}(\mathcal{Y}(\alpha)) \subseteq \mathrm{AAss}(\mathcal{Y}(M))$. Thus $\bigcup_{\alpha \in \mathrm{AAss}(M)} \overline{\{\alpha\}} \subseteq \mathrm{AAss}(\mathcal{Y}(M))$. In order to verify the the second claim, we first assume that M is a finitely generated object; and hence it suffices to show that $\bigcap_{\alpha \in \mathrm{AAss}(M)} \mathrm{ASupp}(\mathcal{X}(\alpha)) \subseteq \mathrm{ASupp}(\mathcal{X}(M))$. To do this, assume that $\beta \in \bigcap_{\alpha \in \mathrm{AAss}(M)} \mathrm{ASupp}(\mathcal{X}(\alpha))$. Then, for every $\alpha \in \mathrm{AAss}(M)$, there exists a monoform object $H(\alpha) \in \mathcal{X}(\alpha)$ such that $\beta = \overline{H(\alpha)}$. Since M is finitely generated, it follows, from [4, Theorem 2.9], that $\mathrm{AAss}(M)$ is a finite set. Consider $\mathrm{AAss}(M) = \{\alpha_1, \ldots, \alpha_n\}$. Then $H(\alpha_1), \ldots, H(\alpha_n)$ have a common nonzero subobject $X \in \bigcap_{\alpha \in \mathrm{AAss}(M)} \mathcal{X}(\alpha) = \mathcal{X}(M)$ so that $\beta \in \mathrm{ASupp}(\mathcal{X}(M))$. Now assume that

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$\mathcal{A}$ is locally stable and that $N \in \mathcal{Y}(M)$. We show that $\mathrm{AAss}(N) \subseteq \bigcup_{\alpha \in \mathrm{AAss}(M)} \overline{\{\alpha\}}$. Consider $\beta \in \mathrm{AAss}(N)$ and a monoform subobject H of N with $\beta = \overline{H}$. Then $\overline{H} \in \mathrm{AAss}\,\mathcal{Y}(M) = \mathrm{ASpec}\mathcal{A} \setminus \mathrm{ASupp}\,\mathcal{X}(M)$. This implies that $H \notin \mathcal{X}(M)$. Then, in view of Proposition 2.11, there exists $\alpha \in \mathrm{AAss}(M)$ such that $H \notin \mathcal{X}(\alpha)$. Thus $\mathrm{ASupp}(H) \nsubseteq \mathrm{ASupp}\,\mathcal{X}(\alpha)$ and $\mathrm{ASupp}(H) \cap \overline{\{\alpha\}} \neq \emptyset$. Now assume that $\gamma \in \mathrm{ASupp}(H) \cap \overline{\{\alpha\}}$. By [8, Proposition 4.11], $\mathrm{Min}\,\mathrm{ASupp}(H) = \mathrm{AAss}(H) = \{\beta\}$ and so $\beta \leq \gamma$. This fact, together with $\gamma \leq \alpha$, forces $\beta \leq \alpha$, which implies that $\beta \in \overline{\{\alpha\}}$. □

Corollary 2.20. *Assume that M is an object of $\mathcal{A}$. Then*

$$\mathrm{ASupp}(\mathcal{X}(M)) \subseteq \bigcap_{\alpha \in \mathrm{AAss}(M)} \mathrm{ASupp}(\mathcal{X}(\alpha)).$$

Furthermore, if M is finitely generated or $\mathcal{A}$ is locally stable, then

$$\mathrm{ASupp}(\mathcal{X}(M)) = \bigcap_{\alpha \in \mathrm{AAss}(M)} \mathrm{ASupp}(\mathcal{X}(\alpha)).$$

Proof. From Proposition 2.11, $\mathcal{X}(M) \subseteq \mathcal{X}(\alpha)$ for every $\alpha \in \mathrm{AAss}(M)$. This implies that $\mathrm{ASupp}(\mathcal{X}(M)) \subseteq \mathrm{ASupp}(\mathcal{X}(\alpha))$ and the first inclusion follows. To prove the equality, we use Theorem 2.19: that is,

$$\mathrm{ASupp}(\mathcal{X}(M)) = \mathrm{ASpec}\mathcal{A} \setminus \mathrm{AAss}(\mathcal{Y}(M)) = \mathrm{ASpec}\mathcal{A} \Big\backslash \bigcup_{\alpha \in \mathrm{AAss}(M)} \overline{\{\alpha\}}$$
$$= \bigcap_{\alpha \in \mathrm{AAss}(M)} (\mathrm{ASpec}\mathcal{A} \setminus \overline{\{\alpha\}}) = \bigcap_{\alpha \in \mathrm{AAss}(M)} \mathrm{ASupp}(\mathcal{X}(\alpha)).$$ □

A topological space X is called *Alexandroff* if the intersection of any family of open subsets of X is also open.

Corollary 2.21. *If $\mathcal{A}$ is locally stable, then the topological space* $\mathrm{ASpec}\mathcal{A}$ *is Alexandroff.*

Proof. Let $\{U_r\}_{r \in \Lambda}$ be a family of open subsets of $\mathrm{ASpec}\mathcal{A}$ and, for every $r \in \Lambda$, assume that $\mathcal{T}_r = \mathrm{ASupp}^{-1}(U_r)$ is the corresponding localising subcategory of $\mathcal{A}$. We show that $\bigcap_\Lambda U_r$ is an open subset of $\mathrm{ASpec}\mathcal{A}$. For every $r \in \Lambda$, according to Proposition 2.17, there exists an object M_r such that $\mathcal{T}_r = \mathcal{X}(M_r)$. Hence, using Corollary 2.20,

$$\bigcap_\Lambda U_r = \bigcap_\Lambda \mathrm{ASupp}(\mathcal{T}_r) = \bigcap_\Lambda \mathrm{ASupp}(\mathcal{X}(M_r)) = \bigcap_\Lambda \bigcap_{\alpha \in \mathrm{AAss}(M_r)} \mathrm{ASupp}(\mathcal{X}(\alpha))$$
$$= \bigcap_{\alpha \in \mathrm{AAss}(\oplus_\Lambda M_r)} \mathrm{ASupp}(\mathcal{X}(\alpha)) = \mathrm{ASupp}(\oplus_\Lambda M_r).$$ □

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