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ON KY FAN'S MINIMAX PRINCIPLE

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Abstract

A generalized version of the Knaster–Kuratowski–Mazurkiewicz theorem is obtained and used to generalize Ky Fan's minimax principle. This result is applied to a variational inequality.

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1. Introduction

In Ky Fan (1972) Ky Fan has proved a minimax principle by using his own generalized version (Ky Fan (1961)) of Knaster – Kuratowski – Mazurkiewicz's theorem. In a joint paper Brezis, Nirenberg, and Stampacchia (1972) have given a further extension of Knaster–Kuratowski–Mazurkiewicz's theorem and applied this extended theorem to a number of problems including a generalized Ky Fan's minimax principle. In this note we have obtained a result which is analogous to the extended Knaster–Kuratowski–Mazurkiewicz theorem of Brezis–Nirenberg–Stampacchia. Using our result we have proved a Ky Fan's minimax principle which includes the corresponding theorem of Brezis–Nirenberg–Stampacchia. We have also shown that our result is also applicable to the types of problems considered in Brezis, Nirenberg, and Stampacchia (1972). Our approach is via a simple fixed point theorem of Browder (1968) and is different from that in Brezis, Nirenberg, and Stampacchia (1972) and Ky Fan (1972).

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In the sequel, E will denote a Hausdorff topological vector space. For any finite subset $\{x_1, x_2, \cdots, x_n\}$ of E, $\langle x_1, x_2, \cdots, x_n\rangle$ will denote the convex hull of $\{x_1, x_2, \cdots, x_n\}$. We first consider the following lemma (see Brezis, Nirenberg, and Stampacchia (1972), p. 2).

LEMMA 2.1. *Let X be a nonempty subset of E. To each $x \in X$, let a nonempty subset $F(x)$ of E be given such that*

(i) *$\overline{F(x_0)} = L$ is compact for some $x_0 \in X$;*

(ii) *the convex hull of every finite subset $\{x_1, x_2, \cdots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$;*

(iii) *for each $x \in X$, the intersection of $F(x)$ with any finite dimensional subspace is closed;*

(iv) *for every convex subset D of E the following equality holds*

$$\left\{\overline{\bigcap_{x \in X \cap D} F(x)}\right) \cap D = \left(\bigcap_{x \in X \cap D} F(x)\right) \cap D.$$

Then $\bigcap_{x \in X} F(x) \neq \phi$.

The above lemma is a slight generalization of Ky Fan's generalization (see Ky Fan (1961) Lemma 1, p. 305) of the well known classical finite dimensional result of Knaster–Kuratowski–Mazurkiewicz (1929).

To obtain our lemma we shall use the following fixed point theorem of Browder (1968), Theorem 1, p. 285.

THEOREM 2.1. (Browder). *Let K be a compact convex subset of E. Let T be a multi-valued mapping of K into 2^K such that*

(i) *for each $x \in K$, $T(x)$ is a nonempty convex subset of K;*

(ii) *for each $x \in K$, $T^{-1}(x) = \{y \in K : x \in T(y)\}$ is open in K.*

Then there is a point $x_0 \in K$ such that $x_0 \in T(x_0)$.

We now prove the following preliminary lemma.

LEMMA 2.2. *Let X be a nonempty subset of E. To each $x \in X$, let a nonempty set $F(x)$ of E be given such that*

(a) *$x \in F(x)$ for each $x \in F(x)$;*

(b) *$F(x_0)$ is compact for some $x_0 \in X$;*

(c) *for each finite subset $\{x_1, x_2, \cdots, x_n\}$ of X and each $x \in S_n = \langle x_1, x_2, \cdots, x_n \rangle =$ the convex hull of $\{x_1, x_2, \cdots, x_n\}$, the set $A(x) = \{y \in S_n \cap X : x \notin F(y)\}$ has the property that whenever $A(x)$ is nonempty, it contains a nonempty convex subset $H(x)$ such that the set $P(x) = \{y \in S_n : x \notin H(y)\}$ is closed;*

(d) *$F(x_0) \cap F(x)$ is closed for each $x \in C$.*

Then $\bigcap_{x \in X} F(x) \neq \phi$.

PROOF. In view of (b) and (d) it suffices to prove that $\bigcap_{i=1}^n F(x_i) \neq \phi$ for each finite subset $\{x_1, x_2, \cdots, x_n\}$ of X. On the contrary we suppose that for some finite subset $\{x_1, x_2, \cdots, x_k\}$ of X we have $\bigcap_{i=1}^k F(x_i) = \phi$. Then for each $x \in S_k = \langle x_1, x_2, \cdots, x_k \rangle$ the set $A(x) = \{y \in S_k \cap X : x \notin F(y)\}$ is nonempty. Indeed, at least one of the points x_i, $i = 1, 2, \cdots, k$ must be in $A(x)$, for otherwise $\bigcap_{i=1}^k F(x_i)$ would be nonempty. We now define a multi-valued mapping $T: S_k \to 2^{S_k}$ by $T(x) = H(x)$, $x \in S_k$; T is well defined by virtue of (c). Now for each $x \in S_k$, $T^{-1}(x) = \{y \in S_k : x \in T(y)\} = \{y \in S_k : x \in H(y)\} =$ complement of $P(x)$ in S_K which is an open set in S_k by condition (c) ($P(x)$ being closed in S_k). Hence by the fixed point theorem of Browder there is a point $x_0 \in S_k$ such that $x_0 \in T(x_0)$. But then by definition of $T(x_0)$ we have $x_0 \notin F(x_0)$ which contradicts (a). Thus $\bigcap_{i=1}^k F(x_i) \neq \phi$.

We are now in a position to prove our main lemma.

LEMMA 2.3. *Let X be a nonempty subset of E. To each $x \in X$, let a nonempty subset $F(x)$ of E be given such that*

(α) *$x \in F(x)$ for each $x \in X$;*

(β) *$\overline{F(x_0)} = L$ is compact for some $x_0 \in X$;*

(γ) *for each finite subset $\{x_1, x_2, \cdots, x_n\}$ of X and each $x \in S_n = \langle x_1, x_2, \cdots, x_n \rangle$ the set $A(x) = \{y \in S_n \cap X : x \notin F(y)\}$ has the same property as laid down in* (c) *of Lemma* 2.2.

(δ) *for each $x \in X$, the intersection of $F(x)$ with any finite dimensional subspace is closed;*

(ω) *the Brezis–Nirenberg–Stampacchia condition holds, that is, for every convex subset D of E we have $(\overline{\bigcap_{x \in X \cap D} F(x)}) \cap D = (\bigcap_{x \in X \cap D} F(x)) \cap D$.*

Then $\bigcap_{x \in X} F(x) \neq \phi$.

PROOF. We may assume $x_0 = 0$. Let $(E_i)_{i \in I}$ be the class of all finite dimensional subspaces of E ordered by inclusion i.e. $i \geqq j$ means $E_j \subset E_i$. Restricting to E_i the conditions of Lemma 2.2 apply to $X_i = X \cap E_i$ and $F_i(x) = F(x) \cap E_i$. Clearly (a) and (c) are satisfied and (b) and (d) follow from (β) and (δ). By Lemma 2.2 there is $u_i \in L \cap E_i$ satisfying

$$u_i \in F_i(x) \subset F(x) \quad \text{for every} \quad x \in X_i.$$

We now repeat the argument of Brezis, Nirenberg, and Stampacchia (1972). Let $\phi_i = \bigcup_{j \geqq i} \{u_j\}$ and so $u \in F(z)$ for $u \in \phi_i$ and $z \in x_i$ and hence $\phi_i \subset \bigcap_{z \in x_i} F(z)$.

Suppose $\tilde{x} \in \bigcap_{i \in I} \overline{\phi_i}$—which is non-empty since $\overline{\phi_i} \subset L$ is compact and let i_0 be such that $\tilde{x} \in E_{i_0}$. For any $x \in X$ we can find $i \geqq i_0$ such that $x \in E_i$. We have

$$\tilde{x} \in \overline{\phi_i} \cap E_i \subset \left(\overline{\bigcap_{z \in X_i} F(z)} \right) \cap E_i = \bigcap_{z \in X_i} F_i(z)$$

by (ω). Therefore $\tilde{x} \in F_i(x) \subset F(x)$ and consequently $\tilde{x} \in \bigcap_{x \in X} F(x)$.

3. Comparison between Lemma 2.1 and Lemma 2.3

(**A**). *If condition* (γ) *of Lemma* 2.3 *is strengthened to the condition*: $(\gamma)'$ *for each* $x \in S_n = \langle x_1, x_2, \cdots, x_n \rangle$ *the set* $A(x) = \{y \in S_n \cap X : x \not\in F(y)\}$ *is convex, then Lemma* 2.3 *follows from Lemma* 2.1.

To show this, it is enough to show that $(\gamma)'$ implies condition (ii) of Lemma 2.1. Let (α) hold and $\{x_1, x_2, \cdots, x_n\}$ be any finite subset of X. Suppose (ii) fails and $S_n = \langle x_1, x_2, \cdots, x_n \rangle \not\subseteq \bigcup_{i=1}^n F(x_i)$. Then there is $x \in S_n$ with $x \not\in \bigcup_{i=1}^n F(x_i)$, $x = \sum_{i=1}^n \lambda_i x_i$, $\lambda_i \geqq 0$, and $\sum_{i=1}^n \lambda_i = 1$. Since $x \not\in F(x_i)$, $x_i \in A(x)$ for all $i = 1, 2, \cdots, n$, and hence $x = \sum_{i=1}^n \lambda_i x_i \in A(x)$ by $(\gamma)'$. This means that $x \not\in F(x)$ contradicting (α). Thus (ii) of Lemma 2.1 and Lemma 2.3 follows from Lemma 2.1.

Remark. It is interesting to note that in this case we can take $H(x) = A(x)$ for each $x \in X$ since $P(x) = \{y \in S_n : x \not\in H(y) = A(y)\} = \{y \in S_n : x \not\in F(y)\}$ is automatically closed by (δ).

(**B**). *Lemma* 2.1 *applies to the following example although Lemma* 2.3 *does not apply.*

Let E be the plane R^2, $S = \{(u,v) \in R^2 : -1 \leqq u, v \leqq 1\}$, and $X = \{(u,v) \in S : |u| = |v| = 1\}$. For $x = (i,j) \in X$ set $F(x) = \{(u,v) \in R^2 : 0 \leqq iu, jv \leqq 1\}$. Clearly Lemma 2.1 applies and by inspection $\bigcap_{x \in X} F(x) = \{0,0)\}$. That Lemma 2.3 does not apply can be seen as follows. For x in S let $A(x) = \{y \in S \cap X : x \not\in F(y)\}$ so that for $x \neq (0,0)$, $A(x)$ is a non-empty subset of X. Let $H(x)$ be a non-empty convex subset of $A(x)$ for $x \neq (0,0)$. Suppose $H^{-1}(x) = \{y \in S : x \in H(y)\}$ is open in S for all x in S. Now $H^{-1}(x)$ is empty for x not in X and since $H(x)$ is a single element for $x \neq (0,0)$ non-empty $H^{-1}(x)$ are disjoint. Now $\bigcup_{x \in X} H^{-1}(x) = S - \{(0,0)\}$ is connected which is a contradiction.

(**C**). *Lemma* 2.3 *applies to the following example although Lemma* 2.1 *does not apply.*

Let E be the reals, $F(-3) = \{x \in R : -3 \leqq x \leqq -2$ or $|x| \leqq 1\}$ and $F(3) = \{x \in R : 2 \leqq x \leqq 3$ or $|x| \leqq 1\}$. Clearly Lemma 2.1 does not apply since $[-3,3]$ is not a subset of $F(3) \cup F(-3)$. Now Lemma 2.3 applies since for x in $[-3,3]$, $A(x) = \{y \in [-3,3] \cap \{-3,3\} : x \not\in F(y)\}$ and we may choose

$$H(x) = \begin{cases} -3, & \text{for } x > 1 \\ 3, & \text{for } x < -1. \end{cases}$$

Then $H(x)$ is a convex subset of $A(x)$ and $H^{-1}(x)$ is open in $[-3,3]$. The other conditions of Lemma 2.3 are clearly satisfied.

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4. Applications

THEOREM 4.1. (Minimax priciple). *Let K be a non-empty convex subset of E and $f(x,y)$ be a real valued function defined on $K \times K$ such that*

(i) $f(x,x) \leqq 0$ *for each* $x \in K$;

(ii) *for each finite subset* $\{x_1, x_2, \cdots, x_n\}$ *of* K *and* $x \in S_n = \langle x_1, x_2, \cdots, x_n \rangle$ *the set* $A(x) = \{y \in S_n : f(x,y) > 0\}$ *if non-empty contains a non-empty convex subset* $H(x)$ *such that the set*

$$P(x) = \{y \in S_n : x \notin H(y)\}$$

is closed;

(iii) *for each* $y \in K, f(x,y)$ *is a lower semicontinuous function of* x *on the intersection of* K *with any finite dimensional subspace of* E;

(iv) *there is a compact subset* L *of* E *and* $y_0 \in L \cap K$ *such that* $f(x,y_0) > 0$ *for* $x \in K, x \notin L$;

(v) *whenever* $x, y \in K$ *and* x_α *is a net on* K *converging to* x, *then* $f(x_\alpha, (1-t)x + ty) \leqq 0$ *for every* $t \in [0,1]$ *implies* $f(x,y) \leqq 0$.
Then there is a point $x_0 \in L \cap K$ *such that*

$$f(x_0, y) \leqq 0 \quad \textit{for all} \quad y \in K.$$

In particular, $\operatorname{Inf}_{x \in K} \sup_{y \in K} f(x,y) \leqq 0$.

PROOF. For each $z \in K$ we set

$$F(z) = \{x \in K : f(x,z) \leqq 0\}.$$

For each finite subset $\{x_1, x_2, \cdots, x_n\}$ of K and $x \in S_n = \langle x_1, x_2, \cdots, x_n \rangle$ the set $A(x) = \{y \in S_n : x \notin F(y)\} = \{y \in S_n : f(x,y) > 0\}$ has the property (γ) of Lemma 2.3 by (ii). While (α), (δ) and (ω) of Lemma 2.3 follow from (i), (iii) and (v) respectively (to see that (v) implies (ω) we refer to proof of application 2, Brezis, Nirenberg, and Stampacchia (1972), p. 4. Finally by (iv), $F(y_0)$ is compact and hence we have (β) of Lemma 2.3. Thus by Lemma 2.3 there is a point $x_0 \in L \cap K$ such that

$$x_0 \in \bigcap_{x \in K} F(x), \quad \text{that is,} \quad f(x_0, y) \leqq 0 \quad \text{for all} \quad y \in K.$$

We note that $x_0 \in L$ by virtue of (iv).

COROLLARY 4.1. (Brezis Nirenberg and Stampacchia (1972)). *Let K be a non-empty convex subset of E and $f(x,y)$ be a real valued function defined on $K \times K$ such that*

(i)′ $f(x,x) \leqq 0$ *for each* $x \in K$;

(ii)′ *for every* $x \in K$, *the set* $\{y \in K : f(x,y) > 0\}$ *is convex*;

(iii)′ *the condition* (iii) *of Theorem* 3.1 *holds*;
(iv)′ *the condition* (iv) *of Theorem* 3.1 *holds*;
(v)′ *the condition* (v) *of Theorem* 3.1 *holds.*
Then there exists a point $x_0 \in L \cap K$ *such that*

$$f(x_0, y) \leqq 0 \quad \textit{for all} \quad y \in K.$$

PROOF. As before we set

$$F(z) = \{x \in K : f(x, z) \leqq 0\} \quad \text{for each} \quad z \in K.$$

The set $A'(x) = \{y \in K : f(x, y) > 0\}$ is convex by (ii)′. Hence for any finite subset $\{x_1, x_2, \cdots, x_n\}$ of K and $x \in S_n = \langle x_1, x_2, \cdots, x_n \rangle$ the set $A(x) = \{y \in S_n : f(x, y) > 0\}$ is convex. Now we choose $H(x) = A(x)$ for each $x \in K$. The set $P(x) = \{y \in S_n : x \in H(y)\}$ is closed by (iii)′ because of the reason given in the remark following (**A**). Thus the conclusion of the corollary follows from the Theorem 4.1.

COROLLARY 4.2. (Ky Fan (1972)). *Let* K *be a non-empty compact convex subset of* E *and* $f(x, y)$ *be a real valued function defined on* $K \times K$ *such that*
(0) $f(x, x) \leqq 0$ *for each* $x \in K$;
(00) *for each* $x \in K$, *the set* $\{y : f(x, y) > 0\}$ *is convex*;
(000) *for each* $y \in K$, $f(x, y)$ *is a lower semicontinuous function of* x *on* K.
Then there is a point $x_0 \in K$ *such that* $f(x_0, y) \leqq 0$ *for all* $y \in K$.

PROOF. This follows from Corollary 4.1.

Let E be Hausdorff topological vectors space over the reals and K be a subset of E. Then a mapping A of K into E^* is called *pseudomonotone* if, whenever x_α is a net in K converging to x with $\limsup (Ax_\alpha, x_\alpha - x) \leqq 0$ then $\liminf (Ax_\alpha, x_\alpha - y) \geqq (Ax, x - y)$. Here $(.,.)$ denotes the pairing between E^* and E.

COROLLARY 4.3. (Brezis (1968), Corollary 29). *Let* K *be convex subset of* E (*over reals*) *and let* $f(x, y) = (Ax, x - y) + \phi(x) - \phi(y)$ *where* A *is a pseudo-monotone mapping from* K *into* E^* *and* ϕ *is a lower semicontinuous convex function. In addition we assume that* A *is continuous from any finite dimensional subspace of* E *to the weak topology of* E^* *and condition* (iv)′ *of Corollary* 4.1 *holds. Then there exists* $x_0 \in L \cap K$ *such that* $(Ax_0, x_0 - y) + \phi(x_0) - \phi(y) \leqq 0$ *for all* $y \in K$.

PROOF. The conditions (i)′, (ii)′, and (iii)′ of Corollary 4.1 follow immediately. To verify that (v)′ holds, see the proof of application 3, Brezis, Nirenberg, and Stampacchia (1972), p. 5.

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