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SEMIGROUPS OF HIGH RANK II DOUBLY NOBLE SEMIGROUPS

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1. Introduction

This paper is a sequel to [**2**]. By a *semigroup of high rank* we mean a semigroup such that for $s_1 \neq s_2$, $\langle S\backslash\{s_1,s_2\}\rangle \subset S$ (properly). Semigroups of high rank such that $\langle S\backslash\{s\}\rangle \subset S$ (*royal semigroups*) were classified in [**2**], where it was also shown that for a *noble semigroup* (i.e. a semigroup of high rank such that there exists a superfluous element z in S for which $\langle S\backslash\{z\}\rangle = S$) there exists either exactly one superfluous element or exactly two superfluous elements [**2**, Theorem 3.7].

The main results of [**2**] give structure theorems for *singly noble semigroups* (for which there is a unique superfluous element). The purpose of this paper is to describe the structure of *doubly noble semigroups*, i.e. semigroups S in which there exist two distinct elements z_1, z_2 for which

$$\langle S\backslash\{z_1\}\rangle = \langle S\backslash\{z_2\}\rangle = S.$$

In one important respect the description is easier than in the singly noble case, for a doubly noble semigroup must be a band [**2**, Theorem 3.7]. Moreover, if we express such a band in the standard way as a semilattice of rectangular bands, then not only must all rectangular bands be in $\mathbf{RZ} \cup \mathbf{LZ}$ [**2**, Lemma 2.1], but also the underlying semilattice must be a chain Theorem 3.9. A full structure theorem for doubly noble semigroups is therefore not very hard to obtain, and this is given as Theorem 3.10.

2. Doubly noble semigroups

Let B be a doubly noble semigroup. Then B is a band [**2**, Theorem 3.7] containing two elements z_1, z_2 such that $\langle B\backslash\{z_1\}\rangle = \langle B\backslash\{z_2\}\rangle = B$, but there do not exist distinct s_1 and s_2 such that $B\backslash\{s_1, s_2\}$ generates B. From [**2**, Section 3] we have

$$(\forall s, t \in B) \qquad st \in \{s, t, z_1, z_2\} \tag{2.1}$$

$$(\forall s, t \in B\backslash\{z_i\}) \qquad st \in \{s, t, z_i\} \qquad i = 1, 2 \tag{2.2}$$

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We also have

$$(\forall s,t\in B\backslash\{z_1,z_2\}) \qquad st\in\{s,t\} \tag{2.3}$$

for if st were equal to z_1 or z_2 it would follow that $B\backslash\{z_1,z_2\}$ generates B.

Next we have

Theorem 2.4 *Let B be a doubly noble band, with $\langle B\backslash\{z_1\}\rangle=\langle B\backslash\{z_2\}\rangle=B$. Then either*

$$(\exists x,y\in B\backslash\{z_1,z_2\}) \qquad z_1=z_2x \quad \textit{and} \quad z_2=z_1y \tag{2.5}$$

or

$$(\exists x,y\in B\backslash\{z_1,z_2\}) \qquad z_1=xz_2 \quad \textit{and} \quad z_2=yz_1. \tag{2.6}$$

Proof. From (2.3) and from the fact that $\langle B\backslash\{z_1\}\rangle=B$ we must have either $z_1=z_2x$ or $z_1=xz_2$ for some x in $B\backslash\{z_1,z_2\}$. Equally there exist y in $B\backslash\{z_1,z_2\}$ such that $z_2=z_1y$ or $z_2=yz_1$. If

$$z_1=z_2x \quad \text{and} \quad z_2=yz_1$$

then

$$z_1=z_2x=yz_1x=yz_2x^2=yz_2x=yz_1=z_2$$

a contradiction. Similarly

$$z_1=xz_2 \quad \text{and} \quad z_2=z_1y$$

lead to a contradiction. The result follows.

Notice now that (2.5) implies that $z_1\mathscr{R}z_2$, hence, since B is a band,

$$z_1z_2=z_2, \quad z_2z_1=z_1. \tag{2.7}$$

Similarly (2.6) implies $z_1\mathscr{L}z_2$ and so

$$z_1z_2=z_1, \quad z_2z_1=z_2.$$

In the former case we say that B is a *dexter* doubly noble band, in the later case we say that B is a *sinister* doubly noble band. In what follows we shall confine ourselves to the dexter case; results for the sinister case will always follow by duality.

Notice now that the elements x and y appearing in (2.5) must be distinct, if we had $z_1=z_2x, z_2=z_1x$ it would follow that

$$z_1=z_2x=z_2x^2=z_1x=z_2.$$

We thus deduce that if B is a doubly noble semigroup, then $|B|\geq 4$.

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Theorem 2.8. *Let B be a dexter doubly noble band. Then*

(i) *for all s in* $B\backslash\{z_1\}$

$$sz_1=s\Rightarrow z_1s=z_2s=sz_2=s,$$

$$sz_1=z_1\Rightarrow z_1s\in\{s,z_1,z_2\},$$

(ii) *for all s in* $B\backslash\{z_2\}$

$$sz_2=s\Rightarrow z_1s=z_2s=sz_1=s,$$

$$sz_2=z_2\Rightarrow z_2s\in\{s,z_1,z_2\}.$$

Proof. It will be sufficient to prove (i). So suppose first that $sz_1=s$. Then by (2.1)

$$sz_2\in\{s,z_1,z_2\}$$

now

$$s=sz_1=sz_2z_1. \qquad \text{by} \quad (2.7)$$

If $sz_2=z_1$, then $z_1^2=s$, a contradiction. If $sz_2=z_2$, then $z_2z_1=s$, again a contradiction. Hence $sz_2=s$. By (2.1) we have

$$z_1s,\, z_2s\in\{s,z_1,z_2\}.$$

If $z_1s=z_1$, then

$$z_1=z_1s=z_1sz_2=z_1z_2=z_2$$

a contradiction. If $z_1s=z_2$ then

$$z_2=z_1s=z_1sz_1=z_2z_1=z_1$$

again a contradiction. Thus $z_1s=s$, and similarly $z_2s=s$. The second part of the statement (i) follows directly from (2.1).

Corollary. (i) *For all s in* $B\backslash\{z_1\}$,

$$z_1s\neq s\Rightarrow sz_1=z_1,\, sz_2=z_2.$$

(ii) *For all s in* $B\backslash\{z_2\}$,

$$z_2s\neq s\Rightarrow sz_1=z_1,\, sz_2=z_2.$$

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Example. Let B_4 be the subsemigroup of $\mathscr{T}(\{1,2,3,4\})$

$$x=\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 2 & 3 & 3\end{pmatrix} \qquad y=\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 1 & 4 & 4\end{pmatrix}$$

$$z_1=\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 3 & 3 & 3\end{pmatrix} \qquad z_2=\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 4 & 4 & 4\end{pmatrix}$$

This has the Cayley table

	x	y	z_1	z_2
x	x	y	z_1	z_2
y	x	y	z_1	z_2
z_1	z_1	z_2	z_1	z_2
z_2	z_1	z_2	z_1	z_2

and is a dexter doubly noble band generated by $B\backslash\{z_1\}$ and $B\backslash\{z_2\}$. It illustrates the fact that the rather weak statement in the second parts of (i) and (ii) in Theorem 2.8 cannot be strengthened. If $s=x$ then

$$sz_1=z_1 \quad \text{and} \quad z_1s=z_1,$$

if $s=y$ then

$$sz_1=z_1 \quad \text{and} \quad z_1s=z_2,$$

if $s=z_2$ then

$$sz_1=z_1 \quad \text{and} \quad z_1s=s.$$

We now show that B_4 is a rather significant example.

Theorem 2.9. *Let B be a dexter doubly noble band generated by $B\backslash\{z_1\}$ and $B\backslash\{z_2\}$ and let x, y be elements of $B\backslash\{z_1,z_2\}$ satisfying* (2.5). *Then $\{x,y,z_1,z_2\}$ is a subband of B isomorphic to B_4.*

Proof. We know that $z_1z_2=z_2, z_2z_1=z_1$ from (2.7). From (2.5) it follows that

$$z_1x=z_2x^2=z_2x=z_1$$

and similarly that $z_2y=z_2$. From the corollary to Theorem 2.8 it follows that

$$xz_1=yz_1=z_1 \qquad xz_2=yz_2=z_2.$$

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From (2.3) it follows that $xy \in \{x, y\}$. Now $xy = x$ implies that

$$z_1 = z_2 x = z_2 xy = z_1 y = z_2$$

a contradiction. Hence $xy = y$. Similarly $yx = x$. The result is now clear.

3. A structure theorem for doubly noble bands

Following the pattern established in [2] we begin by examining the structure of a dexter doubly noble band with exactly two $\mathscr{J}$-classes.

Let us examine the following construction. Let A and Z be disjoint right zero semigroups where $|A| \geqq 2$, $|Z| \geqq 2$. Let P be a proper subset of A such that $|P| \geqq 1$ and let z_1 and z_2 be fixed elements of Z. Define a multiplication on $A \cup Z$ by the rules

$$\begin{aligned} az &= z && (z \in Z, a \in A) \\ za &= z && (z \in Z \backslash \{z_1, z_2\}, a \in A) \\ z_1 a &= z_2 a = z_1 && (a \in P) \\ z_1 a &= z_2 a = z_2 && (a \in A \backslash P). \end{aligned} \tag{3.1}$$

It is possible to check directly that this is an associative multiplication. It is easy also to see that S is a doubly noble band, with superfluous elements z_1 and z_2 and with two $\mathscr{J}$-classes, namely A and Z. By analogy with the notational devices used in [2] we can write

$$B = DNB_R^R\,(A, Z;\ P;\ z_1, z_2) \tag{3.2}$$

Notice that the example B_4 described following Theorem 2.8 is

$$B_4 = DNB_R^R\,(\{x, y\},\ \{z_1, z_2\};\ \{x\};\ z_1, z_2).$$

A sinister doubly noble band dual to (3.2) is easily described and is denoted by

$$DNB_L^L\,(A, Z;\ P;\ z_1, z_2). \tag{3.3}$$

We now prove

Theorem 3.4. *Every doubly noble semigroup with two $\mathscr{J}$-classes is isomorphic to a semigroup of type* (3.2) *or* (3.3).

Proof. Let B be a doubly noble band with elements z_1, z_2 such that

$$\langle B \backslash \{z_1\} \rangle = \langle B \backslash \{z_2\} \rangle = B.$$

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Suppose first that B is dexter. Then $z_1 \mathscr{R} z_2$ and so certainly z_1, z_2 are in the same $\mathscr{J}$-class Z, then $|Z| \geqq 2$. We know by [**2**, Lemma 2.1] that Z is either left or right zero, by (2.7) it follows that Z is right zero.

There exist x, y in $B \backslash \{z_1, z_2\}$ such that $z_1 = z_2 x, z_2 = z_1 y$. If $x \in Z$ then

$$x = z_2 x = z_1$$

a contradiction. Hence $x \notin Z$, and similarly $y \notin Z$. Hence $x, y \in A$ ($|A| \geqq 2$), the other $\mathscr{J}$-class of B. By Theorem 2.9, $\{x, y, z_1, z_2\}$ is a subband of B isomorphic to B_4. Hence in particular $xy = y$, $yx = x$, and so A, which must be either left zero or right zero, is in fact right zero. B is thus a two element chain of right zero semigroups A and Z, with

$$AZ \subseteq Z,\ ZA \subseteq Z. \tag{3.5}$$

For $a \in A$ and $z \in Z$ we have $az = z' \in Z$. In fact

$$z' = az = az^2 = z'z = z$$

so we have

$$az = z \qquad (a \in A, z \in Z). \tag{3.6}$$

If $z \in Z \backslash \{z_1, z_2\}$ then by (2.3) and (3.5)

$$za \in \{z, a\} \cap Z.$$

Thus

$$za = z \qquad (a \in A, z \in Z \backslash \{z_1, z_2\}). \tag{3.7}$$

Finally notice that by (2.1) and (3.5)

$$z_1 a \in \{a, z_1, z_2\} \cap Z = \{z_1, z_2\}.$$

If $z_1 a = z_1$ then by (2.5)

$$z_2 a = z_1 y a = z_1 a = z_1$$

(since A is a right zero semigroup). Similarly if $z_1 a = z_2$ then $z_2 a = z_2$. Thus A divides into complementary sets given by

$$P = \{a \in A \colon z_1 a = z_2 a = z_1\}$$

$$A \backslash P = \{a \in A \colon z_1 a = z_2 a = z_2\}.$$

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Both sets are non-empty, since $x \in P$ and $y \in A \backslash P$. Thus

$$\begin{aligned} z_1 a = z_2 a = z_1 \qquad & a \in P \\ z_1 a = z_2 a = z_2 \qquad & a \in A \backslash P. \end{aligned} \tag{3.8}$$

Comparing (3.6), (3.7) and (3.8) with (3.1) we see that

$$B \simeq DNB_R^R \quad (A, Z; P; z_1, z_2).$$

In obtaining a more general theory for doubly noble bands the following result is crucial.

Theorem 3.9. *Let $B = \mathscr{B}[Y:\{E_a:\alpha \in Y\}]$ be a doubly noble band, expressed as a semilattice Y of rectangular bands. Then Y is a chain.*

Proof. Suppose not, and let α be a branch point of Y. Thus there exist $\beta, \gamma > \alpha$ such that $\beta\gamma = \alpha$. Since $E_\beta E_\gamma \subset E_\alpha$ it follows that one element z of E_α can be expressed as a product xy with $x \in E_\beta$, $y \in E_\gamma$ and so $x \neq z$, $y \neq z$. Thus $\langle B \backslash \{z\} \rangle = B$. Hence $z \in \{z_1, z_2\}$ and so $\{z_1, z_2\} \in E_\alpha$. On the other hand $x, y \in B \backslash \{z_1, z_2\}$ and so $xy \in \{x, y\}$. Then $xy \in E_\alpha \cap \{x, y\} = \emptyset$ and we have a contradiction.

Now let B an arbitrary doubly noble semigroup and E_0 the $\mathscr{J}$-class of B containing z_1 and z_2. Then $E_0 \in \mathbf{RZ}$ and B is a chain

$$B = \mathscr{B}[Y:\{E_\alpha: \alpha \in Y\}]$$

where there is at least one element α in Y, such that $\alpha > 0$. There exist x, y in $B \backslash E_0$ such that

$$z_2 x = z_1, \; z_1 y = z_2$$

and we have seen for any x, y satisfying these equations we have $xy = y$, $yx = x$, giving $x \mathscr{R} y$. It follows that

$$\{u \in B: z_2 u = z_1\} \cup \{v \in B: z_1 v = z_2\}$$

is contained in a single $\mathscr{J}$-class, say E_α, $\alpha > 0$, for if $z_1 u = z_1$, then $u \mathscr{R} y$, and if $z_1 v = z_2$ then $v \mathscr{R} x$. Then $E_\alpha \in \mathbf{RZ}$ and the structure of the subband $E_\alpha \cup E_0$ is that of

$$DNB_R^R(E_0; E_\alpha; P; z_1, z_2).$$

Consider now the subband $E_\gamma \cup E_0$ where $\gamma > 0$ $\gamma \neq \alpha$. If $u \in E_\gamma$ then $z_1 u \neq z_2$ and so by (2.1)

$$z_1 u \in \{z_1, u\} \cap E_0 = \{z_1\}.$$

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Thus $z_1u=z_1$, and similarly $z_2u=z_2$. If $z\in E_0\backslash\{z_1,z_2\}$ then our previous argument using (2.3) gives

$$zu\in\{z,u\}\cap E_0$$

and so $zu=z$ for all $u\in E_\gamma$ and $z\in E_0$. Also for all $u\in E_\gamma$ and all $z\in E_0$, $uz\in E_0$ and so

$$uz=uz^2=(uz)z=z$$

by the right zero property of E_0. Thus $E_\gamma\cup E_0$ has the structure of royal semigroup, as described in [2].

Every subband $E_\nu\cup E_\mu$, such that $\nu,\mu\neq 0$ is a royal band.

We now show that α covers 0. For suppose that α does not cover 0 and let $b\in E_\delta$ where $0<\delta<\alpha$. Then both $E_\alpha\cup E_0$ and $E_\alpha\cup E_\delta$ are royal. Choose x in E_α such that $z_1=z_2x$: then

$$z_1=z_2x=(z_2b)x=z_2(bx)=z_2b=z_2,$$

a contradiction.

We have therefore proved the converse half of the following theorem; where Ω denotes the class of all non-zero cardinal numbers.

Theorem 3.10. *Let $(Y,\leqq)$ be a chain and let 0 be a fixed non maximal element of Y and suppose that Y contains an element 1 covering 0. Let $M\colon Y\to\Omega$ and $H\colon Y\to\{R,L\}$ be maps and suppose that $M(0)\geqq 2$, $M(1)\geqq 2$ and $H(0)=H(1)=R$. Let E_β, $\beta\in Y$, be a set containing $M(\beta)$ elements and having right or left-zero semigroup structure according as $H(\beta)$ is R or L. Let B the disjoint union $B=\cup\{E_\beta\colon\beta\in Y\}$ and $z_1,z_2\in E_0$. Let $\emptyset\neq P\subseteq E_1$.*

Extend the binary operation on E_β, $\beta\in Y$ to B giving $E_1\cup E_0$ the structure

$$DNB_R^R\,(E_1,E_0;P;z_1,z_2)$$

and all other unions $E_\beta\cup E_\gamma$ the structure

$$\mathrm{Roy}\,(\{\beta,\gamma\},M|\{\beta,\gamma\},H|\{\beta,\gamma\}) \tag{3.12}$$

(*see* [2, Theorem 2.2]). *Then B is a dexter doubly noble band, denoted by*

$$DNB(Y;M,H;P).$$

Conversely, every dexter doubly noble band is isomorphic to one constructed in this way.

Proof. All that remains is to show that the multiplication described in B is associative. Let $a\in E_\alpha$, $b\in E_\beta$, $c\in E_\gamma$, where α, β, γ are distinct elements of Y. By (3.12) the associativity is obvious from the properties of royal semigroups unless $\{0,1\}\subset\{\alpha,\beta,\gamma\}$.

If $\{\alpha,\beta,\gamma\}$ includes one element of Y that is lower than 0 then the associativity is again clear. Since $E_1\cup E_0\backslash\{z_1,z_2\}$ is royal we need only consider the six cases.

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(i) $a \in \{z_1, z_2\}$, $\beta = 1, \gamma > 1$,

(ii) $a \in \{z_1, z_2\}$, $\beta > 1, \gamma = 1$,

(iii) $b \in \{z_1, z_2\}$, $\alpha = 1, \gamma > 1$,

(iv) $b \in \{z_1, z_2\}$, $\alpha > 1, \gamma = 1$,

(v) $c \in \{z_1, z_2\}$, $\alpha = 1, \beta > 1$,

(vi) $c \in \{z_1, z_2\}$, $\alpha > 1, \beta = 1$.

All the verifications are routine and a sample verification will suffice. In case (i) $z_1 b \in \{z_1, z_2\} \subset E_0$ and so $bc = b$; hence

$$(z_1 b)c = z_1 b = z_1(bc).$$

The other cases are similar.

If $|\{\alpha, \beta, \gamma\}| < 3$ the associativity is obvious.

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