Canad. J. Math. Vol. 75 (4), 2023 pp. 1369–1392
http://dx.doi.org/10.4153/S0008414X22000426

Two-moment characterization of spectral measures on the real line

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Abstract. In Kiukas, Lahti, and Ylinen (2006, *Journal of Mathematical Physics* 47, 072104), the authors asked the following general question. *When is a positive operator measure projection valued?* A version of this question formulated in terms of operator moments was posed in Pietrzycki and Stochel (2021, *Journal of Functional Analysis* 280, 109001). *Let T be a self-adjoint operator, and let F be a Borel semispectral measure on the real line with compact support. For which positive integers $p < q$ do the equalities $T^k = \int_{\mathbb{R}} x^k F(\mathrm{d}x)$, $k = p, q$, imply that F is a spectral measure?* In the present paper, we completely solve the second problem. The answer is affirmative if p is odd and q is even, and negative otherwise. The case $(p, q) = (1, 2)$ closely related to intrinsic noise operator was solved by several authors including Kruszyński and de Muynck, as well as Kiukas, Lahti, and Ylinen. The counterpart of the second problem concerning the multiplicativity of unital positive linear maps on C^*-algebras is also provided.

1 Introduction

One of the most important concepts in mathematics and physics is the notion of a normalized positive-operator-valued measure (POV measure) also known as a probability-operator-valued measure or a generalized observable, or else semispectral measure. This concept was introduced in the 1940s by Naimark (see [37–39]). POV measures play a significant role in operator theory [1, 7, 26, 43] and are a standard tool in quantum information theory and quantum optics [12, 13, 23, 53]. Recall that a map $F\colon \mathscr{A} \to \boldsymbol{B}(\mathcal{H})$ defined on a σ-algebra $\mathscr{A}$ of subsets of a set X is said to be:

- a *POV measure* if $\langle F(\cdot)h, h\rangle$ is a positive measure for every $h \in \mathcal{H}$,
- a *semispectral measure* if F is a POV measure such that $F(X) = I$,
- a *spectral measure* if F is a semispectral measure such that $F(\Delta)$ is an orthogonal projection for every $\Delta \in \mathscr{A}$,

where $\boldsymbol{B}(\mathcal{H})$ is the collection of all bounded linear operators on a Hilbert space $\mathcal{H}$ and I is the identity operator on $\mathcal{H}$. The celebrated Naimark's dilation theorem (see [39] and [34, Theorem 6.4]) states that a POV measure $F\colon \mathscr{A} \to \boldsymbol{B}(\mathcal{H})$ can always be represented as the R-compression $R^* E(\cdot) R$ of a spectral measure $E\colon \mathscr{A} \to \boldsymbol{B}(\mathcal{K})$, where $\mathcal{K}$ is a Hilbert space and R is a bounded linear operator from $\mathcal{H}$ to $\mathcal{K}$. By [34, p. 14], $\mathcal{K}$ can be made minimal in the sense that $\mathcal{K} = \bigvee\{E(\Delta)R(\mathcal{H})\colon \Delta \in \mathscr{A}\}$. If F is

Received by the editors May 2, 2022; revised August 28, 2022; accepted September 6, 2022.
Published online on Cambridge Core September 15, 2022.
The research of both authors was supported by the National Science Center (NCN) Grant OPUS No. DEC-2021/43/B/ST1/01651.
AMS subject classification: **47B15**, **44A60**, 47A63, 46G10, 46L05.
Keywords: Semispectral measure, spectral measure, operator moment, positive linear map, multiplicativity, operator monotone function.

semispectral, then $\mathcal{H}$ is a subspace of $\mathcal{K}$ and R is the (isometric) embedding of $\mathcal{H}$ into $\mathcal{K}$, and so the minimality condition takes the form $\mathcal{K} = \bigvee\{E(\Delta)\mathcal{H}: \Delta \in \mathscr{A}\}$.

It turns out that, from a mathematical and physical point of view, it is important to investigate the relationship between semispectral and spectral measures. In the classical von Neumann description of quantum mechanics, self-adjoint operators or, equivalently, Borel spectral measures on the real line represent observables. This approach is insufficient in describing many natural properties of measurements, such as measurement inaccuracy. Therefore, in standard modern quantum theory, the generalization to semispectral measures is widely used. In particular, this is the case in quantum information theory and in quantum optics (to represent measurement statistics). Among the papers undertaking this line of research, the following are noteworthy [3–6, 14, 22, 25, 30, 31, 37].

By a *Borel POV measure on* $\mathbb{R}$, we mean a POV measure $F: \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$, where $\mathfrak{B}(\mathbb{R})$ stands for the σ-algebra of all Borel subsets of the real line $\mathbb{R}$ (below, the algebra $\boldsymbol{B}(\mathcal{H})$ will not be explicitly mentioned unless necessary). For an integer $n \geqslant 1$ and a Borel POV measure F on $\mathbb{R}$ with compact support,[1] the integral

$$\int_{\mathbb{R}} x^n F(\mathrm{d}x)$$

is a (bounded) self-adjoint operator, which is called the *nth operator moment* of F. A straightforward application of the Weierstrass approximation theorem and the uniqueness part of the Riesz representation theorem shows that a normalized Borel POV measure on $\mathbb{R}$ with compact support is uniquely determined by its operator moments. One of the features of a Borel spectral measure on $\mathbb{R}$ is the multiplicativity of the corresponding Stone–von Neumann functional calculus. In particular, if E is a Borel spectral measure on $\mathbb{R}$ with compact support, then the following identities hold:[2]

$$\left(\int_{\mathbb{R}} x E(\mathrm{d}x)\right)^n = \int_{\mathbb{R}} x^n E(\mathrm{d}x), \quad n = 1, 2, \ldots. \tag{1.1}$$

Hence, all operator moments of E are determined by the first one, and according to the spectral theorem, there is a one-to-one correspondence between Borel spectral measures on $\mathbb{R}$ and their first operator moments. This is no longer true for general Borel semispectral measures on $\mathbb{R}$. It turns out, however, that the single equality in (1.1) with $n = 2$ guarantees spectrality.

Theorem 1.1 ([31, Proposition 1], [30, Theorem 5], and [43, Remark 5.3]) *A Borel semispectral measure F on $\mathbb{R}$ with compact support*[3] *is spectral if and only if*

$$\left(\int_{\mathbb{R}} x F(\mathrm{d}x)\right)^2 = \int_{\mathbb{R}} x^2 F(\mathrm{d}x).$$

[1] For more information on closed supports of spectral and semispectral measures, see [48, p. 69] and [26, p. 1799].

[2] The condition (1.1) holds even if the closed support of E is not compact. Since we only deal with bounded operators in this paper, the POV measures considered have compact supports (see Section 7 for more explanation).

[3] The first two references contain versions of this result for semispectral measures with noncompact supports.

It is worth mentioning that if F is a Borel semispectral measure on $\mathbb{R}$ with compact support, then the operator $\mathrm{Var}(F)$, called *(intrinsic) noise operator* (see [13, p. 177]), defined as

$$\mathrm{Var}(F) = \int_{\mathbb{R}} x^2 F(\mathrm{d}x) - \left(\int_{\mathbb{R}} x F(\mathrm{d}x)\right)^2, \tag{1.2}$$

is always positive (see Corollary 3.4; this can also be deduced from the Kadison inequality (2.4)). Thus, according to Theorem 1.1, equality holds in $\mathrm{Var}(F) \geqslant 0$ only for spectral measures.

In this connection, it is worth emphasizing that Theorem 1.1 was developed for the needs of quantum physics. Namely, the main purpose of the quantization proposed in [29, 30] was to construct observables that are not spectral measures, and this was done by applying the technique of operator moments. To achieve this goal, it was important to be able to use operator moments to determine whether a given observable is or is not a spectral measure. This led Kiukas, Lahti, and Ylinen to the following question (see [30, Section VI]; see also [28, Section 5]).

Question 1.2 When is a positive operator measure projection valued?

In a recent paper [43], we gave a solution to [16, Problem 1.1] concerning subnormal square roots of quasinormal operators. In fact, the paper [43] provides two solutions to this problem that use two different approaches. The first one appeals to the theory of operator monotone functions, in particular Hansen's inequality. The second one is based on the technique that utilizes operator moments of semispectral measures. A detailed analysis of both solutions led us to a new criterion for the spectrality of a Borel semispectral measure on $\mathbb{R}$ compactly supported in $[0, \infty)$, written in terms of its two operator moments. This criterion was used to solve a generalization of [16, Problem 1.1] (see [43, Theorem 4.1]).

Theorem 1.3 ([43, Theorem 4.2] and [44]) *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a positive operator, and let α, β be two distinct positive real numbers. Assume that $F: \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ is a semispectral measure compactly supported in $[0, \infty)$. Then the following conditions are equivalent*:

(i) *F is the spectral measure of T.*
(ii) *$T^n = \int_{[0,\infty)} x^n F(\mathrm{d}x)$ for all integers $n \geqslant 0$.*
(iii) *$T^r = \int_{[0,\infty)} x^r F(\mathrm{d}x)$ for all $r \in [0, \infty)$.*
(iv) *$T^r = \int_{[0,\infty)} x^r F(\mathrm{d}x)$ for $r = \alpha, \beta$.*

As shown in the proof of [43, Theorem 4.2], the implication (iv)⇒(i) is equivalent to the fact that a semispectral measure $F: \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ compactly supported in $[0, \infty)$ for which there exists $s \in (0, \infty) \backslash \{1\}$ such that

$$\left(\int_{[0,\infty)} x F(\mathrm{d}x)\right)^s = \int_{[0,\infty)} x^s F(\mathrm{d}x)$$

is spectral (see [43, Lemma 4.3]).

In view of Question 1.2 and Theorems 1.1 and 1.3, it seems natural to pose the following general problem in which Ξ is a fixed nonempty set of positive integers (if the set Ξ is finite, then we always order its elements in a nondecreasing manner).

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Problem 1.4 can be regarded as a generalization of [43, Problem 5.2], which deals with two-element sets Ξ.

Problem 1.4 *Let $F\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ be a semispectral measure with compact support, and let $T \in \boldsymbol{B}(\mathcal{H})$ be a self-adjoint operator. Does the system of equations*

$$T^k = \int_{\mathbb{R}} x^k F(\mathrm{d}x), \quad k \in \Xi, \tag{1.3}$$

imply that F is spectral?

This problem can be rephrased equivalently in terms of dilation theory as follows (use Naimark's dilation theorem and Lemma 3.2).

Problem 1.5 *Let $F\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ be a semispectral measure with compact support, let $E\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{K})$ be a minimal spectral dilation of F (i.e., E is a spectral measure satisfying (3.2) and (3.4)), let S be the first operator moment of E (i.e., $S = \int_{\mathbb{R}} xE(\mathrm{d}x)$), and let $T \in \boldsymbol{B}(\mathcal{H})$ be a self-adjoint operator. Does the system of equations*

$$T^k = PS^k|_{\mathcal{H}}, \quad k \in \Xi,$$

imply that P and S commutes?

It turns out that Problem 1.4 is closely related to the question of multiplicativity of unital positive linear maps on C^*-algebras (see Remark 4.3). In fact, the two problems are logically equivalent regardless of the cardinality of the set Ξ (see Remark 4.2). The C^*-algebra counterpart of Problem 1.4 takes the following form.

Problem 1.6 *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, let $\Phi\colon \mathcal{A} \to \mathcal{B}$ be a unital positive linear map, and let $a \in \mathcal{A}$ and $b \in \mathcal{B}$ be self-adjoint. Does the system of equations*

$$b^k = \Phi(a^k), \quad k \in \Xi,$$

imply that Φ restricted to the unital subalgebra generated by $\{a\}$ is multiplicative?

The correspondence between Problems 1.4 and 1.5 allows us to use the theories of operator monotone functions and positive linear maps on C^*-algebras, as well as related operator inequalities, to prove the main results of this paper, which provide complete solutions to Problems 1.4–1.6 for two-element sets Ξ. We begin with the affirmative solutions.

Theorem 1.7 *Let $F\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ be a semispectral measure with compact support, and let p, q be positive integers such that $p < q$, p is odd, and q is even. If $T \in \boldsymbol{B}(\mathcal{H})$ is self-adjoint, then the following conditions are equivalent:*

(i) *F is the spectral measure of T.*
(ii) *$T^k = \int_{\mathbb{R}} x^k F(\mathrm{d}x)$ for all integers $k \geqslant 0$.*
(iii) *$T^k = \int_{\mathbb{R}} x^k F(\mathrm{d}x)$ for $k = p, q$.*

The affirmative solution to Problem 1.6 takes the form.

Theorem 1.8 *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, let $\Phi\colon \mathcal{A} \to \mathcal{B}$ be a unital positive linear map, let $a \in \mathcal{A}$ be self-adjoint, and let p, q be positive integers such that $p < q$, p is odd, and q is even. Then the following conditions are equivalent:*

(i) *Φ restricted to the unital subalgebra generated by $\{a\}$ is multiplicative.*
(ii) *There exists a self-adjoint element $b \in \mathcal{B}$ such that $b^k = \Phi(a^k)$, for $k = p, q$.*

Moreover, if (ii) *holds, then $b = \Phi(a)$.*

In the complementary result, we show that the set

$$\Omega := \{(p,q) \in \mathbb{N}^2 : p < q,\ p \text{ odd and } q \text{ even}\}, \tag{1.4}$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$, is the largest possible subset of $\{(p,q) \in \mathbb{N}^2 : p \leqslant q\}$ for which Problem 1.4 has an affirmative solution for $\Xi = \{p, q\}$. Surprisingly, suitable counterexamples can be constructed even when the underlying Hilbert space $\mathcal{H}$ is one-dimensional (see Theorem 5.2 for more details).

Theorem 1.9 *Let $(p,q) \in \mathbb{N}^2 \setminus \Omega$ be such that $p \leqslant q$. Then there exist a Hilbert space $\mathcal{H}$, a self-adjoint operator $T \in \boldsymbol{B}(\mathcal{H})$, and a semispectral measure $F : \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ with compact support which is not spectral and such that*

$$T^k = \int_{\mathbb{R}} x^k F(\mathrm{d}x), \quad k = p, q.$$

The proofs of Theorems 1.7–1.9 will be given in Sections 3–5, respectively. In Section 2, we provide the basic facts on operator monotone functions, positive linear maps on C^*-algebras, and related operator inequalities needed in this paper. Section 6 contains additional counterexamples (including the case of infinite-dimensional spaces) related to the Fibonacci sequence. Finally, in Section 7, we discuss the possibility of adapting the two-moment characterizations of spectral measures given in Theorems 1.3 and 1.7 to the case of semispectral measures whose closed supports are not compact.

2 Prerequisites

In this paper, we use the following notation. The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The symbols $\mathbb{N}$, $\mathbb{Z}_+$, and $\mathbb{R}_+$ stand for the sets of positive integers, nonnegative integers, and nonnegative real numbers, respectively. We write $\mathfrak{B}(X)$ for the σ-algebra of all Borel subsets of a topological Hausdorff space X. The C^*-algebra of all continuous complex functions on a compact Hausdorff space K equipped with supremum norm is denoted by $C(K)$. For $\lambda \in \mathbb{R}$, δ_λ stands for the Borel probability measure on $\mathbb{R}$ concentrated on $\{\lambda\}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be (complex) Hilbert spaces. Denote by $\boldsymbol{B}(\mathcal{H}, \mathcal{K})$ the Banach space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. If $A \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$, then A^*, $\mathcal{N}(A)$, and $\mathcal{R}(A)$ stand for the adjoint, the kernel, and the range of A, respectively. It is well known that $\boldsymbol{B}(\mathcal{H}) := \boldsymbol{B}(\mathcal{H}, \mathcal{H})$ is a C^*-algebra with unit I, where $I = I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$. We say that $A \in \boldsymbol{B}(\mathcal{H})$ is *self-adjoint* if $A = A^*$, *positive* if $\langle Ah, h\rangle \geqslant 0$ for all $h \in \mathcal{H}$, and an *orthogonal projection* if $A = A^*$ and $A = A^2$.

Let $\mathscr{A}$ be a σ-algebra of subsets of a set X, and let $F : \mathscr{A} \to \boldsymbol{B}(\mathcal{H})$ be a semispectral measure. Denote by $L^1(F)$ the vector space of all $\mathscr{A}$-measurable functions $f : X \to \mathbb{C}$ such that $\int_X |f(x)| \langle F(\mathrm{d}x)h, h\rangle < \infty$ for all $h \in \mathcal{H}$. Then, for every $f \in L^1(F)$, there exists a unique operator $\int_X f \mathrm{d}F \in \boldsymbol{B}(\mathcal{H})$ such that (see, e.g., [51, Appendix])

$$\left\langle \int_X f \mathrm{d}F h, h \right\rangle = \int_X f(x) \langle F(\mathrm{d}x)h, h\rangle, \quad h \in \mathcal{H}. \tag{2.1}$$

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If F is a spectral measure, then $\int_X f \mathrm{d}F$ coincides with the usual spectral integral. In particular, if F is the spectral measure of a self-adjoint operator $A \in \boldsymbol{B}(\mathcal{H})$, then we write $f(A) = \int_{\mathbb{R}} f \mathrm{d}F$ for any F-essentially bounded Borel function $f: \mathbb{R} \to \mathbb{R}$; the map $f \mapsto f(A)$ is called the Stone–von Neumann functional calculus. For more information needed in this article on spectral integrals, including the spectral theorem for self-adjoint operators and the Stone–von Neumann functional calculus, we refer the reader to [46, 48, 54].

Let $J \subseteq \mathbb{R}$ be an interval (which may be open, half-open, or closed; bounded or unbounded). A continuous function $f: J \to \mathbb{R}$ is said to be *operator monotone* if $f(A) \leqslant f(B)$ for any two self-adjoint operators $A, B \in \boldsymbol{B}(\mathcal{H})$ such that $A \leqslant B$ and the spectra of A and B are contained in J. In [33], Löwner proved that a continuous function defined on an open interval is operator monotone if and only if it has an analytic continuation to the complex upper half-plane which is a Pick function (see also [18, 21]). Operator monotone functions have integral representations with respect to suitable positive Borel measures. In particular, a continuous function $f: (0, \infty) \to \mathbb{R}$ is operator monotone if and only if there exists a positive Borel measure ν on $[0, \infty)$ such that $\int_0^\infty \frac{1}{1+\lambda^2} \mathrm{d}\nu(\lambda) < \infty$ and

$$f(t) = \alpha + \beta t + \int_0^\infty \left(\frac{\lambda}{1+\lambda^2} - \frac{1}{t+\lambda} \right) \mathrm{d}\nu(\lambda), \quad t \in (0, \infty),$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$ (see [21, Theorem 5.2] or [9, p. 144]). The most important example of an operator monotone function is $f: [0, \infty) \ni t \to t^p \in \mathbb{R}$, where $p \in (0, 1)$. This function has the following integral representation (see [9, Exercise V.1.10(iii)] or [9, Exercise V.4.20]):

$$t^p = \frac{\sin p\pi}{\pi} \int_0^\infty \frac{t\lambda^{p-1}}{t+\lambda} \mathrm{d}\lambda, \quad t \in [0, \infty). \tag{2.2}$$

Operator monotone functions are related to the Hansen inequality [20]. In [52, Lemma 2.2], Uchiyama gave a necessary and sufficient condition for equality to hold in the Hansen inequality when the external factor is a nontrivial orthogonal projection (see the "moreover" part of Theorem 2.1; see also the paragraph before [43, Theorem 2.4] showing why the separability of $\mathcal{H}$ can be dropped).

Theorem 2.1 [20, 52] *Let $A \in \boldsymbol{B}(\mathcal{H})$ be a positive operator, let $T \in \boldsymbol{B}(\mathcal{H})$ be a contraction, and let $f: [0, \infty) \to \mathbb{R}$ be a continuous operator monotone function such that $f(0) \geqslant 0$. Then*

$$T^* f(A) T \leqslant f(T^* A T). \tag{2.3}$$

Moreover, if f is not an affine function and T is an orthogonal projection such that $T \neq I$, then equality holds in (2.3) if and only if $TA = AT$ and $f(0) = 0$.

The reader is referred to [9, 18, 20, 21, 33, 49] for the fundamentals of the theory of operator monotone functions.

A linear map $\Phi: \mathcal{A} \to \mathcal{B}$ between unital C^*-algebras is said to be *positive* if $\Phi(a) \geqslant 0$ for every $a \in \mathcal{A}$ such that $a \geqslant 0$. The map Φ is called *unital* if it preserves the units. If Φ is positive and unital, then the following inequality, called Kadison's inequality (see

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[27, Theorem 1]), holds:

$$\Phi(a^2) \geqslant \Phi(a)^2 \text{ for all } a \in \mathcal{A} \text{ such that } a = a^*. \tag{2.4}$$

In this paper, we will need the Lieb–Ruskai inequality, which can be thought of as a Kadison-type inequality.

Theorem 2.2 [32, Theorem 2] *Let $R \in \boldsymbol{B}(\mathcal{H},\mathcal{K})$, and let $\Phi: \boldsymbol{B}(\mathcal{K}) \to \boldsymbol{B}(\mathcal{H})$ be the positive linear map defined by*

$$\Phi(X) = R^* X R, \quad X \in \boldsymbol{B}(\mathcal{K}).$$

Then, for all $A, B \in \boldsymbol{B}(\mathcal{K})$, the net $\{\Phi(A^*B)(\Phi(B^*B) + \varepsilon I)^{-1}\Phi(B^*A)\}_{\varepsilon>0}$ is convergent in the strong operator topology as $\varepsilon \downarrow 0$ and

$$\Phi(A^*A) \geqslant (\mathrm{sot})\lim_{\varepsilon\downarrow 0} \Phi(A^*B)(\Phi(B^*B) + \varepsilon I)^{-1}\Phi(B^*A).$$

3 Proof of Theorem 1.7

We begin with the following lemma, which gives a necessary and sufficient condition for equality to hold in a Kadison-type inequality (cf. (2.4)). Although this is a known fact even for unbounded operators (see [31, Lemmas 1 and 2]), we will provide a brief algebraic proof for the reader's convenience. Note also that part (iii) of Lemma 3.1 is [19, Lemma in Section 6].

Lemma 3.1 *Let $T \in \boldsymbol{B}(\mathcal{H})$ be a self-adjoint operator, and let $P \in \boldsymbol{B}(\mathcal{H})$ be an orthogonal projection. Then the following statements are valid*:

(i) $(PTP)^2 \leqslant PT^2P$.
(ii) *Equality holds in* (i) *if and only if $PT = TP$.*
(iii) *If T is an orthogonal projection, then PTP is an orthogonal projection if and only if $PT = TP$.*

Proof (i) This is a direct consequence of the following algebraic identities:

$$\begin{aligned} PT^2P - (PTP)^2 &= PT^2P - PTPTP \\ &= PT(I-P)TP \\ &= (TP)^*(I-P)TP \geqslant 0. \end{aligned} \tag{3.1}$$

(ii) It follows from (3.1) that equality holds in (i) if and only if

$$(TP)^*(I-P)TP = 0,$$

or equivalently if and only if

$$\mathcal{R}(TP) \subseteq \mathcal{N}((I-P)^{\frac{1}{2}}) = \mathcal{N}(I-P),$$

which in turn is equivalent to $(I-P)TP = 0$. The last equality holds if and only if $TP = PTP$, which by $(PTP)^* = PTP$ is equivalent to $PT = TP$.

(iii) This is a direct consequence of (ii) because PTP is an orthogonal projection if and only if $(PTP)^2 = PT^2P$. ∎

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For our further considerations, the following fact is fundamental. In particular, in view of Naimark's dilation theorem (see the Introduction), it shows that Problems 1.4 and 1.5 are logically equivalent.

Lemma 3.2 *Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces such that $\mathcal{H} \subseteq \mathcal{K}$ and $P \in \boldsymbol{B}(\mathcal{K})$ be the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. Suppose that $F: \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ is a semispectral measure and $E: \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{K})$ is a spectral measure such that*

$$F(\varDelta) = PE(\varDelta)|_{\mathcal{H}}, \quad \varDelta \in \mathfrak{B}(\mathbb{R}). \tag{3.2}$$

Set[4] *$S := \int_{\mathbb{R}} x E(\mathrm{d}x)$. Then the following statements are valid:*

(i) *F is spectral if and only if P commutes with E (equivalently, $\mathcal{H}$ reduces E).*

(ii) *If $S \in \boldsymbol{B}(\mathcal{K})$, then F has compact support and*

$$PS^k|_{\mathcal{H}} = \int_{\mathbb{R}} x^k F(\mathrm{d}x), \quad k \in \mathbb{Z}_+. \tag{3.3}$$

(iii) *If F has compact support and $\mathcal{K}$ is minimal, that is,*

$$\mathcal{K} = \bigvee \{E(\varDelta)\mathcal{H} : \varDelta \in \mathfrak{B}(\mathbb{R})\}, \tag{3.4}$$

then E has compact support, $S \in \boldsymbol{B}(\mathcal{K})$, and $S = S^*$.

Proof (i) Set $\hat{F}(\varDelta) = F(\varDelta) \oplus 0$ for $\varDelta \in \mathfrak{B}(\mathbb{R})$, where 0 stands for the zero operator on $\mathcal{K} \ominus \mathcal{H}$. Then, by (3.2), $\hat{F}(\varDelta) = PE(\varDelta)P$. Hence, observing that $F(\varDelta)$ is an orthogonal projection if and only if $\hat{F}(\varDelta)$ is an orthogonal projection and using Lemma 3.1(iii), we obtain (i).

(ii) It follows from (3.2) that the closed support of F is contained in the closed support of E. Since E has compact support (because $S \in \boldsymbol{B}(\mathcal{K})$; see [48, Theorem 5.9]), so does F. Applying the Stone–von Neumann functional calculus, we get

$$\begin{aligned}\langle PS^k|_{\mathcal{H}} h, h\rangle = \langle S^k h, h\rangle &= \int_{\mathbb{R}} x^k \langle E(\mathrm{d}x)h, h\rangle \\ &\overset{(3.2)}{=} \int_{\mathbb{R}} x^k \langle F(\mathrm{d}x)h, h\rangle \\ &\overset{(2.1)}{=} \Big\langle \int_{\mathbb{R}} x^k F(\mathrm{d}x)h, h\Big\rangle, \quad h \in \mathcal{H},\ k \in \mathbb{Z}_+,\end{aligned}$$

which implies (3.3).

(iii) By (3.2) and (3.4), the closed supports of the POV measures E and F coincide (see the proofs of [24, Theorem 4.4] and [26, Proposition 4(iii)]). Hence, the closed support of E is compact. As a consequence, the operator $\int_{\mathbb{R}} x E(\mathrm{d}x)$ is bounded and self-adjoint (see [48, Theorem 5.9]). This completes the proof. ■

We are now in a position to prove the main result of this paper, which provides a two-moment characterization of spectral measures.

Proof of Theorem 1.7 (i)⇒(ii) This is immediate from the Stone–von Neumann functional calculus.

(ii)⇒(iii) Obvious.

[4] Note that *a priori* the operator S may be unbounded (see [48, Theorem 5.9] for more details).

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(iii)⇒(i) It follows from Naimark's dilation theorem (see the Introduction) that there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a spectral measure $E\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{K})$ such that (3.2) and (3.4) hold, where $P \in \boldsymbol{B}(\mathcal{K})$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. By Lemma 3.2, E has compact support, the operator $S := \int_{\mathbb{R}} xE(\mathrm{d}x)$ is bounded and self-adjoint, and the following equalities are satisfied:

$$T^k = PS^k|_{\mathcal{H}}, \quad k = p, q. \tag{3.5}$$

First, we prove that F is a spectral measure. In view of Lemma 3.2(i), it suffices to show that P commutes with E. For this, we consider two cases.

Case 1. $p \leqslant \frac{q}{2}$.

Let $\hat{T} \in \boldsymbol{B}(\mathcal{K})$ be defined by $\hat{T} = T \oplus 0$, where 0 stands for the zero operator on $\mathcal{K} \ominus \mathcal{H}$. Set $q' = \frac{q}{2}$. Using Lemma 3.1(i) and then applying Theorem 2.1 to the positive operator $S^{2q'}$ and the operator monotone function $f(t) = t^{\frac{p}{q'}}$ (see (2.2)), we deduce that

$$\begin{aligned}\hat{T}^{2p} = (\hat{T}^p)^2 &\overset{(3.5)}{=} (PS^pP)^2 \\ &\leqslant PS^{2p}P \\ &\leqslant (PS^{2q'}P)^{\frac{p}{q'}} = (PS^qP)^{\frac{2p}{q}} \overset{(3.5)}{=} (\hat{T}^q)^{\frac{2p}{q}} \overset{(*)}{=} \hat{T}^{2p},\end{aligned}$$

where $(*)$ can be inferred from the hypothesis that q is even. This implies that

$$(PS^pP)^2 = PS^{2p}P.$$

It follows from Lemma 3.1(ii) that

$$PS^p = S^pP.$$

Hence, by [48, Theorem 5.1], P commutes with E_p, the spectral measure of S^p. By [10, Theorem 6.6.4], E_p is of the form

$$E_p(\Delta) = E(\varphi_p^{-1}(\Delta)), \quad \Delta \in \mathfrak{B}(\mathbb{R}), \tag{3.6}$$

where $\varphi_p\colon \mathbb{R} \to \mathbb{R}$ is a function given by

$$\varphi_p(x) = x^p, \quad x \in \mathbb{R}. \tag{3.7}$$

Since the map $\mathfrak{B}(\mathbb{R}) \ni \Delta \mapsto \varphi_p^{-1}(\Delta) \in \mathfrak{B}(\mathbb{R})$ is surjective (because p is odd), we deduce from (3.6) that P commutes with E.

Case 2. $p > \frac{q}{2}$.

Without loss of generality, we can assume that $P \neq I_{\mathcal{K}}$. Set $q' = \frac{q}{2}$ and $r = p - q'$. Since $p < q$ and q is even, we see that $r, q' \in \mathbb{N}$ and $0 < \frac{r}{q'} < 1$. By Theorem 2.1 applied to the positive operator $S^{2q'}$ and the operator monotone function $f(t) = t^{\frac{r}{q'}}$, we get

$$\hat{T}^{2r} = (\hat{T}^{2q'})^{\frac{r}{q'}} \overset{(3.5)}{=} (PS^{2q'}P)^{\frac{r}{q'}} \geqslant PS^{2r}P. \tag{3.8}$$

This implies that

$$T^{2r} = (PS^{2q'}|_{\mathcal{H}})^{\frac{r}{q'}} \geqslant PS^{2r}|_{\mathcal{H}}. \tag{3.9}$$

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Let $\Phi\colon \boldsymbol{B}(\mathcal{K}) \to \boldsymbol{B}(\mathcal{H})$ be the positive unital linear map defined by

$$\Phi(X) = PX|_{\mathcal{H}}, \quad X \in \boldsymbol{B}(\mathcal{K}).$$

Applying Theorem 2.2 to $A = S^r$, $B = S^{q'}$, and $R \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ defined by $Rh = h$ for $h \in \mathcal{H}$ leads to

$$PS^{2r}|_{\mathcal{H}} = \Phi(S^{2r}) \geqslant (\mathrm{sot})\lim_{\varepsilon\downarrow 0}\Phi(S^p)(\Phi(S^q) + \varepsilon I)^{-1}\Phi(S^p). \tag{3.10}$$

Let $G\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ be the spectral measure of T. Using the Stone–von Neumann functional calculus, we obtain

$$\begin{aligned}\Phi(S^p)(\Phi(S^q) + \varepsilon I)^{-1}\Phi(S^p) &\overset{(3.5)}{=} T^p(T^q + \varepsilon I)^{-1}T^p \\ &= \int_{\mathbb{R}} \frac{x^{2p}}{x^q + \varepsilon} G(\mathrm{d}x), \quad \varepsilon \in (0, \infty).\end{aligned} \tag{3.11}$$

Applying Lebesgue's monotone convergence theorem and the hypothesis that q is even and $2p - q \in \mathbb{N}$, we deduce that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\langle \Phi(S^p)(\Phi(S^q) + \varepsilon I)^{-1}\Phi(S^p)h, h\rangle &\overset{(3.11)}{=} \lim_{\varepsilon\downarrow 0}\Big\langle \int_{\mathbb{R}} \frac{x^{2p}}{x^q + \varepsilon} G(\mathrm{d}x)h, h\Big\rangle \\ &= \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}} \frac{x^{2p}}{x^q + \varepsilon}\langle G(\mathrm{d}x)h, h\rangle \\ &= \int_{\mathbb{R}} x^{2p-q}\langle G(\mathrm{d}x)h, h\rangle \\ &= \langle T^{2p-q}h, h\rangle \\ &= \langle T^{2r}h, h\rangle, \quad h \in \mathcal{H}.\end{aligned}$$

Combined with (3.10), this implies that

$$PS^{2r}|_{\mathcal{H}} \geqslant T^{2r}. \tag{3.12}$$

Using (3.9) and (3.12), we get

$$T^{2r} = (PS^{2q'}|_{\mathcal{H}})^{\frac{r}{q'}} \geqslant PS^{2r}|_{\mathcal{H}} \geqslant T^{2r}.$$

This yields

$$(PS^{2q'}|_{\mathcal{H}})^{\frac{r}{q'}} = PS^{2r}|_{\mathcal{H}},$$

or equivalently

$$(PS^{2q'}P)^{\frac{r}{q'}} = PS^{2r}P,$$

which implies that equality holds in the Hansen inequality (3.8). Thus, by the moreover part of Theorem 2.1, $PS^q = S^qP$ (recall that $q = 2q'$). Hence,

$$\hat{T}^{qn} \overset{(3.5)}{=} (PS^qP)^n = (PS^q)^n = PS^{qn}P, \quad n \in \mathbb{N}.$$

Therefore, we have

$$T^{qn} = PS^{qn}|_{\mathcal{H}}, \quad n \in \mathbb{N}. \tag{3.13}$$

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Take any $n_0 \in \mathbb{N}$ such that $p \leqslant \frac{qn_0}{2}$. Then, by (3.5) and (3.13), we obtain

$$T^k = PS^k|_{\mathcal{H}}, \qquad k = p, qn_0.$$

Since $p \leqslant \frac{qn_0}{2}$, we can apply Case 1 to the pair (p, qn_0) in place of (p,q). We then obtain that P commutes with E.

Summarizing, we have proved that in both cases F is a spectral measure. Therefore, to complete the proof, it remains to show that F is the spectral measure of T. Since $T^p = \int_{\mathbb{R}} x^p F(\mathrm{d}x)$ (by (iii)) and $T^p = \int_{\mathbb{R}} x^p G(\mathrm{d}x)$ (by Stone–von Neumann functional calculus), an application of [10, Theorem 6.6.4] shows that $F \circ \varphi_p^{-1}$ and $G \circ \varphi_p^{-1}$ are spectral measures of T^p, where G is the spectral measure of T, φ_p is as in (3.7) and

$$(F \circ \varphi_p^{-1})(\Delta) = F(\varphi_p^{-1}(\Delta)) \text{ and } (G \circ \varphi_p^{-1})(\Delta) = G(\varphi_p^{-1}(\Delta)) \text{ for } \Delta \in \mathfrak{B}(\mathbb{R}).$$

By the uniqueness part of [10, Theorem 6.1.1], $F \circ \varphi_p^{-1} = G \circ \varphi_p^{-1}$. Since the map $\mathfrak{B}(\mathbb{R}) \ni \Delta \mapsto \varphi_p^{-1}(\Delta) \in \mathfrak{B}(\mathbb{R})$ is surjective (because p is odd), we deduce that $F = G$, so F is the spectral measure of T. This completes the proof. ∎

We conclude this section by providing some inequalities for operator moments of a semispectral measure on the real line. Although it is a well-known fact (see [11] and the references therein), we outline its short proof for the reader's convenience.

Proposition 3.3 *Let $F\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ be a semispectral measure with compact support. Then*

$$\begin{bmatrix} I & \int_{\mathbb{R}} xF(\mathrm{d}x) & \cdots & \int_{\mathbb{R}} x^n F(\mathrm{d}x) \\ \int_{\mathbb{R}} xF(\mathrm{d}x) & \int_{\mathbb{R}} x^2 F(\mathrm{d}x) & \cdots & \int_{\mathbb{R}} x^{n+1} F(\mathrm{d}x) \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\mathbb{R}} x^n F(\mathrm{d}x) & \int_{\mathbb{R}} x^{n+1} F(\mathrm{d}x) & \cdots & \int_{\mathbb{R}} x^{2n} F(\mathrm{d}x) \end{bmatrix} \geqslant 0, \quad n \in \mathbb{Z}_+.$$

Proof By Naimark's dilation theorem (see the Introduction), there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a spectral measure $E\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{K})$ that satisfy (3.2) and (3.4). By Lemma 3.2(iii), E has compact support. Applying the Stone–von Neumann functional calculus, we obtain

$$\begin{aligned} \sum_{j,k=0}^{n} \Big\langle \int_{\mathbb{R}} x^{j+k} F(\mathrm{d}x) h_k, h_j \Big\rangle &= \sum_{j,k=0}^{n} \Big\langle \int_{\mathbb{R}} x^{j+k} E(\mathrm{d}x) h_k, h_j \Big\rangle \\ &= \Big\| \sum_{k=0}^{n} \int_{\mathbb{R}} x^k E(\mathrm{d}x) h_k \Big\|^2 \geqslant 0, \end{aligned}$$

for all finite sequences $\{h_k\}_{k=0}^n \subseteq \mathcal{H}$. ∎

Corollary 3.4 *Let $F\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ be a semispectral measure with compact support. Then $\mathrm{Var}(F) \geqslant 0$, where $\mathrm{Var}(F)$ is as in (1.2).*

Proof Apply Proposition 3.3 with $n = 1$ and use the following well-known fact (see [17, Lemma 1]; see also [35, Theorem 5.1]): if $A, B \in \boldsymbol{B}(\mathcal{H})$ are self-adjoint, A is invertible in $\boldsymbol{B}(\mathcal{H})$ and $X \in \boldsymbol{B}(\mathcal{H})$, then $\left[\begin{smallmatrix} A & X \\ X^* & B \end{smallmatrix}\right] \geqslant 0$ if and only if $B \geqslant X^* A^{-1} X$. ∎

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4 Proof of Theorem 1.8

We begin by stating a lemma needed in the proof of Theorem 1.8, which seems to be a folklore-type result. For the convenience of the reader, we provide its proof.

Lemma 4.1 *Let $\mathcal{A}$ be a unital C^*-algebra, let $\Phi: \mathcal{A} \to \boldsymbol{B}(\mathcal{H})$ be a unital positive linear map, and let a be a self-adjoint element of $\mathcal{A}$. Then there exists a unique semispectral measure $F: \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ such that $x^n \in L^1(F)$ for all $n \in \mathbb{Z}_+$ and*

$$\Phi(a^n) = \int_{\mathbb{R}} x^n F(\mathrm{d}x), \quad n \in \mathbb{Z}_+.$$

Moreover, F possesses the following properties:

(i) *F has compact support.*
(ii) *The closed support of F is contained in $\mathbb{R}_+$ whenever $a \geqslant 0$.*

First proof of Lemma 4.1 Replacing $\mathcal{A}$ by the unital C^*-algebra generated by $\{a\}$, we may assume without loss of generality that $\mathcal{A}$ is commutative. According to [41, Corollary 2.9], Φ is contractive and therefore

$$\|\Phi(a^n)\| \leqslant \|a\|^n, \quad n \in \mathbb{Z}_+. \tag{4.1}$$

Since a is self-adjoint, we see that

$$\begin{bmatrix} e & a^1 & \cdots & a^n \\ a^1 & a^2 & \cdots & a^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a^n & a^{n+1} & \cdots & a^{2n} \end{bmatrix} = \begin{bmatrix} e & a^1 & \cdots & a^n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}^* \begin{bmatrix} e & a^1 & \cdots & a^n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \geqslant 0,$$

where e denotes the unit of $\mathcal{A}$. By the Stinespring theorem (see [50, Theorem 4]), Φ is completely positive, so $[\Phi(a^{j+k})]_{j,k=0}^n \geqslant 0$. In particular, we have

$$\sum_{j,k=0}^{n} \bar{\lambda}_j \lambda_k \Phi(a^{j+k}) \geqslant 0, \quad \{\lambda_j\}_{j=0}^n \subseteq \mathbb{C},\ n \in \mathbb{Z}_+. \tag{4.2}$$

Using (4.1) and (4.2), we deduce from [11, Theorem 2] that there exists a semispectral measure $F: \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ such that

$$\langle \Phi(a^n)h, h\rangle = \int_{\mathbb{R}} x^n \langle F(\mathrm{d}x)h, h\rangle, \quad n \in \mathbb{Z}_+,\ h \in \mathcal{H}. \tag{4.3}$$

($F(\mathbb{R}) = I$ because Φ is unital). Since $\lim_{r\to\infty} \|f\|_r = \|f\|_\infty$ whenever $\|f\|_r < \infty$ for some $r < \infty$ (see [47, Exercise 4, p. 71]) and

$$\lim_{n\to\infty} \Big(\int_{\mathbb{R}} x^{2n} \langle F(\mathrm{d}x)h, h\rangle\Big)^{\frac{1}{2n}} \overset{(4.3)}{=} \lim_{n\to\infty} \langle \Phi(a^{2n})h, h\rangle^{\frac{1}{2n}} \overset{(4.1)}{\leqslant} \|a\|, \quad h \in \mathcal{H},$$

we deduce that

$$\langle F(\{x \in \mathbb{R}: |x| > \|a\|\})h, h\rangle = 0, \quad h \in \mathcal{H}.$$

Thus, the closed support of F is contained in $[-\|a\|, \|a\|]$. Combined with (2.1) and (4.3), this implies that $x^n \in L^1(F)$ for all $n \in \mathbb{Z}_+$ and

$$\Phi(a^n) = \int_{\mathbb{R}} x^n F(\mathrm{d}x), \quad n \in \mathbb{Z}_+.$$

That F is unique follows from (2.1) and the fact that a Hamburger moment sequence having a representing measure with compact support is determinate (see [19]).

Assume $a \geqslant 0$. By the square root theorem (see [36, Theorem 2.2.1]), we have

$$\begin{bmatrix} a^1 & a^2 & \cdots & a^{n+1} \\ a^2 & a^3 & \cdots & a^{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ a^{n+1} & a^{n+2} & \cdots & a^{2n+1} \end{bmatrix} = \begin{bmatrix} a^{\frac{1}{2}} & a^{\frac{3}{2}} & \cdots & a^{\frac{2n+1}{2}} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}^* \begin{bmatrix} a^{\frac{1}{2}} & a^{\frac{3}{2}} & \cdots & a^{\frac{2n+1}{2}} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \geqslant 0.$$

Hence, by [50, Theorem 4], $[\Phi(a^{j+k+1})]_{j,k=0}^n \geqslant 0$, which implies that

$$\sum_{j,k=0}^{n} \bar{\lambda}_j \lambda_k \Phi(a^{j+k+1}) \geqslant 0, \quad \{\lambda_j\}_{j=0}^n \subseteq \mathbb{C},\ n \in \mathbb{Z}_+. \tag{4.4}$$

Combining (4.2), (4.4), and the Stieltjes theorem [8, Theorem 6.2.5] with the fact that a Hamburger moment sequence having a representing measure with compact support is determinate, we see that the closed support of F is contained in $\mathbb{R}_+$.

Second proof of Lemma 4.1 As in the first proof, there is no loss of generality in assuming that $\mathcal{A}$ is commutative. Applying the Stinespring dilation theorem (see [50, Theorems 1 and 4]), we deduce that there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a $*$-representation $\pi\colon \mathcal{A} \to \boldsymbol{B}(\mathcal{K})$ such that

$$\Phi(u) = P\pi(u)|_{\mathcal{H}}, \quad u \in \mathcal{A}, \tag{4.5}$$

where $P \in \boldsymbol{B}(\mathcal{K})$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. By [46, Theorem 12.22], there exists a spectral measure $E\colon \mathfrak{B}(\mathfrak{M}) \to \boldsymbol{B}(\mathcal{K})$ such that

$$\pi(u) = \int_{\mathfrak{M}} \widehat{\pi(u)}\, \mathrm{d}E, \quad u \in \mathcal{A}, \tag{4.6}$$

where $\mathfrak{M}$ is the maximal ideal space of the unital commutative C^*-algebra $\overline{\pi(\mathcal{A})}$, the (operator norm) closure of $\pi(\mathcal{A})$ in $\boldsymbol{B}(\mathcal{K})$, and $\widehat{\pi(u)}\colon \mathfrak{M} \to \mathbb{C}$ is the Gelfand transform of $\pi(u)$. Set $M = PE|_{\mathcal{H}}$. It follows from (4.5) and (4.6) that

$$\Phi(u) = \int_{\mathfrak{M}} \widehat{\pi(u)}\mathrm{d}M, \quad u \in \mathcal{A}. \tag{4.7}$$

By [46, Theorem 11.18] and the assumption that $a = a^*$, we see that $\widehat{\pi(a)}(\mathfrak{M}) \subseteq \mathbb{R}$. Define the semispectral measure $F\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ by

$$F(\Delta) = M\big(\widehat{\pi(a)}^{-1}(\Delta)\big), \quad \Delta \in \mathfrak{B}(\mathbb{R}).$$

Since $\widehat{\pi(a)}$ is continuous and $\mathfrak{M}$ is a compact Hausdorff space, we deduce that $\widehat{\pi(a)}(\mathfrak{M})$ is a compact subset of $\mathbb{R}$ such that $F\big(\mathbb{R}\setminus\widehat{\pi(a)}(\mathfrak{M})\big) = 0$, which implies that F has compact support. Applying (2.1) and the measure transport theorem (cf. [2, Theorem 1.6.12]), we conclude that

$$\Phi(a^n) \overset{(4.7)}{=} \int_{\mathfrak{M}} \widehat{\pi(a)}^n \mathrm{d}M = \int_{\mathbb{R}} x^n F(\mathrm{d}x), \quad n \in \mathbb{Z}_+.$$

The proof of the uniqueness of F proceeds as before.

If $a \geqslant 0$, then by the square root theorem and [46, Theorem 11.18], we get

$$\widehat{\pi(a)} = \widehat{\pi(a^{\frac{1}{2}})}^2 \geqslant 0,$$

which implies that the closed support of F is contained in $\mathbb{R}_+$.

Proof of Theorem 1.8 (i)⇒(ii) Since the map Φ preserves self-adjointness, $b := \Phi(a)$ does the job.

(ii)⇒(i) In view of the Gelfand–Naimark theorem (see [46, Theorem 12.41]), there is no loss of generality in assuming that $\mathcal{B} = \boldsymbol{B}(\mathcal{H})$. By Lemma 4.1, there exists a semispectral measure $F\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ with compact support such that

$$\Phi(a^n) = \int_{\mathbb{R}} x^n F(\mathrm{d}x), \quad n \in \mathbb{Z}_+. \tag{4.8}$$

Therefore, by (ii), we have

$$b^k = \Phi(a^k) \overset{(4.8)}{=} \int_{\mathbb{R}} x^k F(\mathrm{d}x), \quad k = p, q.$$

Applying Theorem 1.7 to $T = b$, we conclude that F is the spectral measure of b. Using the Stone–von Neumann functional calculus, we get

$$\Phi(a^n) \overset{(4.8)}{=} \Big(\int_{\mathbb{R}} xF(\mathrm{d}x)\Big)^n = b^n, \quad n \in \mathbb{Z}_+,$$

so $b = \Phi(a)$, which yields $\Phi(a^n) = \Phi(a)^n$ for all $n \in \mathbb{Z}_+$. This implies (i). ∎

Remark 4.2 We have deduced Theorem 1.8 from Theorem 1.7. It turns out that, conversely, Theorem 1.7 can be derived from Theorem 1.8. To do this, it suffices to show that under the assumptions and notation of Theorem 1.7, (iii) implies (i). So suppose that the condition (iii) of Theorem 1.7 holds. Define the unital positive linear map $\Phi\colon C(K) \to \boldsymbol{B}(\mathcal{H})$ by

$$\Phi(f) = \int_K f(x)F(\mathrm{d}x), \quad f \in C(K),$$

where K stands for the closed support of F. Let $a \in C(K)$ be the function defined by $a(x) = x$ for $x \in K$, and let $b = T$. Then $b^k = \Phi(a^k)$ for $k = p, q$. Hence, by Theorem 1.8, $T = \int_K xF(\mathrm{d}x)$ and

$$\int_K x^n F(\mathrm{d}x) = \Phi(a^n) = T^n = \int_K x^n G(\mathrm{d}x), \quad n \in \mathbb{Z}_+,$$

where G is the spectral measure of T. Using (2.1) and the fact that a Hamburger moment sequence having a representing measure with compact support is determinate, we conclude that $F = G$. This completes the proof.

A careful inspection of the proof of Theorem 1.8 in conjunction with the above discussion shows that, in fact, Problems 1.4 and 1.6 are logically equivalent regardless of the cardinality of the set Ξ.

Remark 4.3 At first glance, it seems that Theorem 1.8 is related to a result of Petz (see [42, Theorem]), which shows that equality holds in Jensen's inequality $f(\Phi(a)) \leqslant \Phi(f(a))$ if and only if Φ restricted to the unital subalgebra generated by $\{a\}$ is multiplicative, where Φ is a unital positive linear map between unital C^*-algebras,

f is a non-affine operator convex function on an open subinterval J of $\mathbb{R}$, and a is a self-adjoint element with spectrum in J. The main difference between Petz's result and Theorem 1.8 is that the monomial x^n with $n \in \mathbb{Z}_+$ is a non-affine operator convex function on $J = \mathbb{R}$ if and only if $n = 2$; this can be deduced from [9, Exercises V.1.3 and V.2.11] (cf. [45]). In the very special case of positive invertible elements, Petz's theorem implies Theorem 1.8 (and Theorem 4.4) as shown in Remark 4.5. However, the authors see no way to deduce Theorem 1.8 from Petz's theorem in the situation of general self-adjoint (or even positive noninvertible) elements. For this reason, the Lieb–Ruskai inequality turned out to be a key tool in the proof of Theorem 1.7, which in view of Remark 4.2 is equivalent to Theorem 1.8.

In the case where the elements a and b are positive, we get the following version of Theorem 1.8.

Theorem 4.4 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras, $\Phi{:}\,\mathcal{A} \to \mathcal{B}$ is a unital positive linear map, $a \in \mathcal{A}$ is positive, and p, q are distinct positive integers. Then the following conditions are equivalent*:

(i) *Φ restricted to the unital subalgebra generated by $\{a\}$ is multiplicative.*
(ii) *There exists a positive element $b \in \mathcal{B}$ such that $b^k = \Phi(a^k)$, for $k = p, q$.*

Moreover, if (ii) *holds, then $b = \Phi(a)$.*

Proof It suffices to show the implication (ii)⇒(i). Its proof is the same as that of Theorem 1.8, except that we use Theorem 1.3 instead of Theorem 1.7. We leave the details to the reader. ∎

Remark 4.5 If the element a is invertible, then in this case we can give another proof of the implication (ii)⇒(i) of Theorem 4.4, which relies on a result of Petz. Without loss of generality, we can assume that $0 < p < q$. Since a is positive and invertible and Φ is a unital positive linear map, we deduce that a^q and thus $\Phi(a^q)$ are positive and invertible. Let $f{:}\,(0,\infty) \to \mathbb{R}$ be the function given by $f(x) = -x^{\frac{p}{q}}$ for $x \in (0,\infty)$. It follows from [9, Theorems V.1.9 and V.2.5] that f is an operator convex function. Using (ii) and the Stone–von Neumann functional calculus, we get

$$f(\Phi(a^q)) = -\Phi(a^q)^{\frac{p}{q}} = -(b^q)^{\frac{p}{q}} = -b^p$$

and

$$\Phi(f(a^q)) = -\Phi((a^q)^{\frac{p}{q}}) = -\Phi(a^p) = -b^p.$$

Consequently,

$$f(\Phi(a^q)) = \Phi(f(a^q)).$$

Combined with [42, Theorem] and the fact that Φ is continuous (see [41, Corollary 2.9]), this implies that Φ restricted to the unital C^*-algebra generated by $\{a^q\}$ is multiplicative. Applying the Stone–von Neumann functional calculus and the Weierstrass approximation theorem, one can show that the unital C^*-algebras generated by $\{a\}$ and $\{a^q\}$ coincide (in fact, this is a very special case of the Müntz–Szász theorem; see [47, Theorem 15.26]). Hence, (i) holds.

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5 Proof of Theorem 1.9

We begin with a simple observation related to Problems 1.4 and 1.5. Namely, if T and F satisfy (1.3), then by (2.1) and the measure transport theorem for every $\tau \in \mathbb{R}\backslash\{0\}$, τT and F_τ satisfy (1.3), where $F_\tau\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ is the semispectral measure with compact support given by

$$F_\tau(\Delta) = F(\tau^{-1}\Delta), \quad \Delta \in \mathfrak{B}(\mathbb{R}).$$

Moreover, F is spectral if and only if F_τ is spectral. A similar observation applies to Problem 1.5. In other words, rescaling preserves the affirmative or negative solutions to Problems 1.4 and 1.5.

Next, we prove a lemma that is central to the proof of Theorem 5.2.

Lemma 5.1 *Suppose that $(p,q) \in \mathbb{N}^2\backslash\Omega$ and $p \leqslant q$, where Ω is as in (1.4). Then, for every $\tau \in \mathbb{R}\backslash\{0\}$, there exist $\alpha, \beta \in (0,1)$ and distinct $\lambda_1, \lambda_2 \in \mathbb{R}$ such that*

$$\begin{cases} \alpha + \beta = 1, \\ \alpha\lambda_1^p + \beta\lambda_2^p = \tau^p, \\ \alpha\lambda_1^q + \beta\lambda_2^q = \tau^q. \end{cases} \tag{5.1}$$

Proof We may assume, without loss of generality, that $\tau = 1$. Using the substitution

$$\alpha = \frac{a}{a+b} \quad \text{and} \quad \beta = \frac{b}{a+b}$$

with $a, b \in (0,\infty)$, we obtain an equivalent system of equations:

$$\begin{cases} \frac{a}{a+b}\lambda_1^p + \frac{b}{a+b}\lambda_2^p = 1, \\ \frac{a}{a+b}\lambda_1^q + \frac{b}{a+b}\lambda_2^q = 1. \end{cases}$$

Multiplying both sides of the above equalities by $a+b$ and rearranging gives

$$\begin{cases} a(\lambda_1^p - 1) + b(\lambda_2^p - 1) = 0, \\ a(\lambda_1^q - 1) + b(\lambda_2^q - 1) = 0. \end{cases} \tag{5.2}$$

The determinant of the above system of equations (with unknowns a, b) is

$$D(\lambda_1,\lambda_2) = \det\begin{bmatrix} \lambda_1^p - 1 & \lambda_2^p - 1 \\ \lambda_1^q - 1 & \lambda_2^q - 1 \end{bmatrix} = (\lambda_1^p - 1)(\lambda_2^q - 1) - (\lambda_2^p - 1)(\lambda_1^q - 1).$$

Observe that if $D(\lambda_1,\lambda_2) \neq 0$, then the system of equations (5.2) has only one solution $a = b = 0$. Thus, the only chance to find nonzero solutions a, b of the system (5.2) is when $D(\lambda_1,\lambda_2) = 0$. Note that

$$\text{if } \lambda_1, \lambda_2 \in \mathbb{R}\backslash\{-1,1\}, \text{ then } D(\lambda_1,\lambda_2) = 0 \text{ if and only if } \frac{\lambda_1^p - 1}{\lambda_1^q - 1} = \frac{\lambda_2^p - 1}{\lambda_2^q - 1}. \tag{5.3}$$

We will consider four cases.

Case 1. $p = q$.

It is easily seen that for any $\alpha, \beta \in (0,1)$ such that $\alpha + \beta = 1$, there are plenty of two-element subsets $\{\lambda_1, \lambda_2\}$ of $(0,\infty)$ solving (5.1).

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Case 2. Both p and q are even.

Set $\lambda_1 = -1$, $\lambda_2 = 1$, and $\beta = 1 - \alpha$, where $\alpha \in (0,1)$. Then it is easily seen that (5.1) is satisfied.

Case 3. $p < q$, p is even, and q is odd.

Consider the function $\phi\colon [0,1] \to [0,1]$ defined by

$$\phi(x) = \frac{1 - x^p}{1 + x^q}, \quad x \in [0,1].$$

Then ϕ is continuous, $\phi(0) = 1$, $\phi(1) = 0$, and $\phi((0,1)) \subseteq (0,1)$. By the Darboux property of continuous functions, $\phi((0,1)) = (0,1)$. Take $\lambda_1 > 1$. Then

$$\frac{1 - \lambda_1^p}{1 - \lambda_1^q} \in (0,1).$$

Therefore, there exists $x \in (0,1)$ such that

$$\frac{1 - \lambda_1^p}{1 - \lambda_1^q} = \phi(x). \tag{5.4}$$

Set $\lambda_2 = -x$. Then $\lambda_2 < 0$, $|\lambda_2| = x < 1$ and

$$\frac{1 - \lambda_1^p}{1 - \lambda_1^q} \overset{(5.4)}{=} \frac{1 - |\lambda_2|^p}{1 + |\lambda_2|^q} = \frac{1 - \lambda_2^p}{1 - \lambda_2^q},$$

which by (5.3) means that $D(\lambda_1, \lambda_2) = 0$, so the system of equations (5.2) is linearly dependent. Take any $a \in (0, \infty)$ and set

$$b = a\frac{\lambda_1^p - 1}{1 - \lambda_2^p} > 0.$$

Then the pair (a, b) is a solution of the system of equations (5.2).

Case 4. $p < q$ and both p and q are odd.

Consider the function $\psi\colon [1, \infty) \to (0,1]$ defined by

$$\psi(x) = \frac{1 + x^p}{1 + x^q}, \quad x \in [1, \infty).$$

Then ψ is continuous, $\psi(1) = 1$, $\lim_{x\to\infty} \psi(x) = 0$, and $\psi((1,\infty)) \subseteq (0,1)$. As a consequence of the Darboux property of continuous functions, $\psi((1,\infty)) = (0,1)$. Take $\lambda_1 > 1$. Observe that

$$\frac{1 - \lambda_1^p}{1 - \lambda_1^q} \in (0,1).$$

Hence, there exists $x \in (1, \infty)$ such that

$$\frac{1 - \lambda_1^p}{1 - \lambda_1^q} = \psi(x). \tag{5.5}$$

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Set $\lambda_2 = -x$. Then $\lambda_2 < 0$, $|\lambda_2| = x > 1$, and

$$\frac{1-\lambda_1^p}{1-\lambda_1^q} \overset{(5.5)}{=} \psi(|\lambda_2|) = \frac{1+|\lambda_2|^p}{1+|\lambda_2|^q} = \frac{1-\lambda_2^p}{1-\lambda_2^q}.$$

As in Case 3, taking any $a \in (0,\infty)$ and setting

$$b = a\frac{\lambda_1^p - 1}{1-\lambda_2^p} = a\frac{\lambda_1^p - 1}{1+|\lambda_2|^p} > 0,$$

we see that the pair (a,b) is a solution of the system of equations (5.2). This completes the proof. ∎

We are now ready to prove the main result of this section, which provides the counterexamples mentioned earlier in the Introduction. In fact, this is a stronger version of Theorem 1.9.

Theorem 5.2 *Suppose that $(p,q) \in \mathbb{N}^2\backslash\Omega$ and $p \leqslant q$, where Ω is as in (1.4). Let $\tau \in \mathbb{R}\backslash\{0\}$. Set $\mathcal{H} = \mathbb{C}$ and $T = \tau I$. Then there exist $\alpha,\beta \in (0,1)$ and $\lambda_1,\lambda_2 \in \mathbb{R}$ such that $\alpha+\beta = 1$, $\lambda_1 \neq \lambda_2$ and*

(i) *the semispectral measure $F\colon \mathfrak{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ defined by*

$$F(\Delta) = \alpha\delta_{\lambda_1}(\Delta)I + \beta\delta_{\lambda_2}(\Delta)I, \quad \Delta \in \mathfrak{B}(\mathbb{R}), \tag{5.6}$$

is not spectral and

$$T^k = \int_{\mathbb{R}} x^k F(\mathrm{d}x), \quad k = p,q, \tag{5.7}$$

(ii) *the self-adjoint operator $S \in \boldsymbol{B}(\mathcal{H}\oplus\mathcal{H})$ defined by*

$$S = \begin{bmatrix} \alpha\lambda_1+\beta\lambda_2 & \sqrt{\alpha\beta}(\lambda_1-\lambda_2) \\ \sqrt{\alpha\beta}(\lambda_1-\lambda_2) & \beta\lambda_1+\alpha\lambda_2 \end{bmatrix}$$

does not commute with $P := \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and

$$T^k = PS^k|_{\mathcal{H}}, \quad k = p,q, \tag{5.8}$$

(iii) *the unital positive linear map $\Phi\colon C(K) \to \boldsymbol{B}(\mathcal{H})$ defined by*

$$\Phi(f) = \int_K f\,\mathrm{d}F, \quad f \in C(K), \tag{5.9}$$

is not multiplicative, $C(K)$ is the unital algebra generated by $\{a\}$, and

$$b^k = \Phi(a^k), \quad k = p,q, \tag{5.10}$$

where F is as in (5.6), $K = \{\lambda_1,\lambda_2\}$, $a(x) = x$ for $x \in K$, and $b = T$.

Proof Let $\alpha,\beta,\lambda_1,\lambda_2$ be as in Lemma 5.1.

(i) By (5.1) and (5.6), F is a semispectral measure satisfying (5.7). However, F is not a spectral measure because $F(\{\lambda_1\}) = \alpha I$ and $\alpha \in (0,1)$.

(ii) Clearly, the operator S is self-adjoint. It follows from the first equality in (5.1) that the matrix

$$\begin{bmatrix} \sqrt{\alpha} & \sqrt{\beta} \\ -\sqrt{\beta} & \sqrt{\alpha} \end{bmatrix}$$

is unitary and consequently

$$\begin{bmatrix} \sqrt{\alpha} & \sqrt{\beta} \\ -\sqrt{\beta} & \sqrt{\alpha} \end{bmatrix}^{-1} = \begin{bmatrix} \sqrt{\alpha} & -\sqrt{\beta} \\ \sqrt{\beta} & \sqrt{\alpha} \end{bmatrix}. \tag{5.11}$$

Now, it is easily seen that the Jordan decomposition of S takes the form

$$S = \begin{bmatrix} \sqrt{\alpha} & -\sqrt{\beta} \\ \sqrt{\beta} & \sqrt{\alpha} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \sqrt{\alpha} & \sqrt{\beta} \\ -\sqrt{\beta} & \sqrt{\alpha} \end{bmatrix}.$$

Combined with (5.11), this implies that

$$S^n = \begin{bmatrix} \alpha\lambda_1^n + \beta\lambda_2^n & \sqrt{\alpha\beta}(\lambda_1^n - \lambda_2^n) \\ \sqrt{\alpha\beta}(\lambda_1^n - \lambda_2^n) & \beta\lambda_1^n + \alpha\lambda_2^n \end{bmatrix}, \quad n \in \mathbb{Z}_+. \tag{5.12}$$

By (5.1) and (5.12), the condition (5.8) is satisfied. Since $\lambda_1 \neq \lambda_2$, the operator S does not commute with P.

(iii) It is immediate from (5.9) and (i) that Φ is the unital positive linear map which satisfies (5.10). Clearly, $C(K)$ is the unital algebra generated by $\{a\}$. To show that Φ is not multiplicative, consider two polynomials $u(x) = x - \lambda_1$ and $v(x) = x - \lambda_2$ and note that $\Phi(u(a)v(a)) = 0$ while $\Phi(u(a))\Phi(v(a)) = -\alpha\beta(\lambda_1 - \lambda_2)^2 \neq 0$. This completes the proof. ∎

Remark 5.3 A careful inspection of the proofs of Lemma 5.1 and Theorem 5.2 shows that there is great freedom in the choice of the parameters λ_1 and λ_2; however, if $p < q$ and $\tau > 0$, then in view of [43, Theorem 4.2], at least one of these parameters must be negative. Note also that if $p < q$, p is odd, and q is even, then according to Theorems 1.7 and 1.8, there are no $\alpha, \beta, \lambda_1, \lambda_2$ satisfying the conclusion of Theorem 5.2. In this particular case, we can justify it in an elementary way. Namely, by the Hölder inequality, we infer from (5.1) that

$$\begin{aligned} |\tau|^p = |\alpha\lambda_1^p + \beta\lambda_2^p| &\leqslant \alpha|\lambda_1|^p + \beta|\lambda_2|^p \\ &\leqslant \sqrt[r]{\alpha + \beta}\ \sqrt[q/p]{\alpha|\lambda_1|^q + \beta|\lambda_2|^q} \\ &= \sqrt[r]{\alpha + \beta}\ \sqrt[q/p]{\alpha\lambda_1^q + \beta\lambda_2^q} = |\tau|^p, \end{aligned} \tag{5.13}$$

where $r \in (1, \infty)$ is such that $\frac{1}{r} + \frac{p}{q} = 1$. This means that equality in the Hölder inequality holds. As a consequence, we deduce that $|\lambda_1| = |\lambda_2|$. Combined with the first inequality in (5.13) and the assumption that p is odd, this implies that $\lambda_1 = \lambda_2$, which is a contradiction.

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6 More examples

In this section, we illustrate Theorem 1.9 (cf. Theorem 5.2) with two interesting examples for the case $(p,q)=(2,3)$. Now, we use the dilation approach as stated in Lemma 3.2 (cf. Problems 1.4 and 1.5).

Example 6.1 Let $\{f_n\}_{n=0}^{\infty}$ be the Fibonacci sequence, that is, $f_0=0$, $f_1=1$, and $f_{n+1}=f_n+f_{n-1}$ for $n\in\mathbb{N}$. It is well known and easy to prove that

$$\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}^n=\begin{bmatrix}f_{n-1} & f_n\\ f_n & f_{n+1}\end{bmatrix},\quad n\in\mathbb{N}. \tag{6.1}$$

Set $\mathcal{H}=\mathbb{C}$, $T=I$, and

$$S=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}.$$

Then S is a self-adjoint operator and the spectral measure E of S is given by

$$E(\varDelta)=\frac{1}{1+\phi^2}\delta_{1-\phi}(\varDelta)\begin{bmatrix}\phi^2 & -\phi\\ -\phi & 1\end{bmatrix}+\frac{1}{1+\phi^2}\delta_{\phi}(\varDelta)\begin{bmatrix}1 & \phi\\ \phi & \phi^2\end{bmatrix},\quad \varDelta\in\mathfrak{B}(\mathbb{R}),$$

where $\phi=\frac{1+\sqrt{5}}{2}$ is the golden ratio. It follows from (6.1) that

$$T^k=PS^k|_{\mathcal{H}} \text{ for } k=2,3 \text{ and } T^k\neq PS^k|_{\mathcal{H}} \text{ for } k\in\mathbb{N}\backslash\{2,3\},$$

where $P=\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$ is the orthogonal projection of $\mathcal{H}\oplus\mathcal{H}$ onto $\mathcal{H}$. The semispectral measure $F:=PE|_{\mathcal{H}}$ (see (3.2) for the definition of $PE|_{\mathcal{H}}$) takes the form

$$F(\varDelta)=\frac{\phi^2}{1+\phi^2}\delta_{1-\phi}(\varDelta)I+\frac{1}{1+\phi^2}\delta_{\phi}(\varDelta)I,\quad \varDelta\in\mathfrak{B}(\mathbb{R}). \tag{6.2}$$

By Lemma 3.2(ii), we have

$$T^k=\int_{\mathbb{R}}x^kF(\mathrm{d}x) \text{ for } k=2,3 \text{ and } T^k\neq\int_{\mathbb{R}}x^kF(\mathrm{d}x) \text{ for } k\in\mathbb{N}\backslash\{2,3\}.$$

Clearly, by (6.2), F is not a spectral measure (also because P does not commute with S; see Lemma 3.2).

The next example is a modification of the previous one.

Example 6.2 Let $T\in\boldsymbol{B}(\mathcal{H})$ be a nonzero self-adjoint operator, and let $S\in\boldsymbol{B}(\mathcal{H}\oplus\mathcal{H})$ be the self-adjoint operator given by the 2×2 block matrix

$$S=\begin{bmatrix}0 & T\\ T & T\end{bmatrix}. \tag{6.3}$$

One can verify that

$$S^2=\begin{bmatrix}T^2 & T^2\\ T^2 & 2T^2\end{bmatrix} \text{ and } S^3=\begin{bmatrix}T^3 & 2T^3\\ 2T^3 & 3T^3\end{bmatrix}.$$

Thus, the operators T and S satisfy the following two identities:

$$T^k=PS^k|_{\mathcal{H}},\quad k=2,3, \tag{6.4}$$

where $P = \begin{bmatrix} I_{\mathcal{H}} & 0 \\ 0 & 0 \end{bmatrix}$ is the orthogonal projection of $\mathcal{H} \oplus \mathcal{H}$ onto $\mathcal{H}$. Let E be the spectral measure of S, and let $F := PE|_{\mathcal{H}}$ be the corresponding semispectral measure. Applying Lemma 3.2(ii) and using (6.4), we get

$$T^k = \int_{\mathbb{R}} x^k F(\mathrm{d}x), \quad k = 2, 3.$$

Since $T \neq 0$, the operator P does not commute with S, so by Lemma 3.2, F is not a spectral measure. In contrast to Example 6.1, here it is much easier to use Lemma 3.2 to see that the semispectral measure F is not spectral.

Remark 6.3 Regarding Example 6.2, note that the operator S given by (6.3) is unitarily equivalent to the tensor product

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \otimes T.$$

Combined with (6.1), this implies that

$$S^n = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \otimes T^n = \begin{bmatrix} f_{n-1}T^n & f_n T^n \\ f_n T^n & f_{n+1}T^n \end{bmatrix}, \quad n \in \mathbb{N}.$$

This means that tensoring and orthogonal summation enrich the class of counterexamples by allowing semispectral measures to have operator values on Hilbert spaces of arbitrary dimension.

7 Semispectral measures with noncompact supports

In this section, we extend the two-moment characterizations of spectral measures given in Theorems 1.3 and 1.7 to the case of semispectral measures with noncompact supports. When considering a Borel semispectral measure F on the real line with noncompact support, it may happen that the coordinate function $\mathbb{R} \ni x \mapsto x \in \mathbb{R}$ is not in $L^1(F)$ (cf. (2.1)), which means that the expression $\int_{\mathbb{R}} x^k F(\mathrm{d}x)$, where $k \in \mathbb{N}$, may not yield a bounded operator. On the other hand, if Problem 1.4 has an affirmative solution under the formally weaker assumption that the functions $\mathbb{R} \ni x \mapsto x^k \in \mathbb{R}$, $k \in \Xi$, are in $L^1(F)$, then F being *a posteriori* a spectral measure must have a compact support. Indeed, by the measure transport theorem $T^k = \int_{\mathbb{R}} x (F \circ \varphi_k^{-1})(\mathrm{d}x)$, where φ_k is as in (3.7) and k is any element of Ξ, and thus $F \circ \varphi_k^{-1}$ is the spectral measure of the (bounded) self-adjoint operator T^k. Hence, $F \circ \varphi_k^{-1}$ must have compact support (see [48, Theorem 5.9]). Consequently, F itself must have compact support. A similar argument applied in the case where a semispectral measure $F: \mathscr{A} \to \boldsymbol{B}(\mathcal{H})$ is considered on an abstract measurable space $(X, \mathscr{A})$, and the coordinate function is replaced by a measurable real-valued function ω on X, leads to the conclusion that ω is F-essentially bounded, that is, $F(\{x \in X : |\omega(x)| > r\}) = 0$ for some $r \in \mathbb{R}_+$ (equivalently, $\omega \in L^\infty(F)$). However, to get the spectrality of F, it is not enough to assume that ω is F-essentially bounded. It turns out that the "missing" property of ω is σ-surjectivity. We say that a measurable map $f: X \to Y$ between measurable spaces $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ (i.e., a map such that $f^{-1}(\Delta) \in \mathscr{A}$ for all $\Delta \in \mathscr{B}$) is σ-surjective if the corresponding map $\mathscr{B} \ni \Delta \longmapsto f^{-1}(\Delta) \in \mathscr{A}$ is surjective. If Y is a topological Hausdorff space, σ-surjectivity refers to $\mathscr{B} = \mathfrak{B}(Y)$. In case $\mathscr{A} = \mathfrak{B}(X)$

https://doi.org/10.4153/S0008414X22000426 Published online by Cambridge University Press

and $\mathscr{B} = \mathfrak{B}(Y)$, where X and Y are topological Hausdorff spaces, σ-surjectivity is called in [15] *Borel injectivity*. It is worth emphasizing here that the property of being σ-surjective was used in the proof of Theorem 1.7.

Applying the measure transport theorem together with Theorems 1.3 and 1.7, we get the following.

Theorem 7.1 *Assume that $(X, \mathscr{A})$ is a measurable space, $F\colon \mathscr{A} \to \boldsymbol{B}(\mathcal{H})$ is a semispectral measure, $T \in \boldsymbol{B}(\mathcal{H})$ is a self-adjoint operator, and p, q are positive integers such that $p < q$. Let $\omega\colon X \to \mathbb{R}$ be an F-essentially bounded σ-surjective function such that*

$$T^k = \int_X \omega(x)^k F(\mathrm{d}x), \quad k = p, q.$$

If p is odd and q is even, or if $T \geqslant 0$ and $\omega(X) \subseteq \mathbb{R}_+$, then F is a spectral measure.

It is worth pointing out that there is a wide class of measurable spaces admitting functions ω with the properties mentioned in Theorem 7.1. Namely, if X is a Borel subset of a complete separable metric space and K is a bounded Borel subset of $\mathbb{R}$ such that X and K have the same cardinality, then by [40, Theorem 2.12] there exists a bijection $\omega_0\colon X \to K$ such that ω_0 and ω_0^{-1} are Borel measurable. This implies that the function $\omega\colon X \to \mathbb{R}$ defined by $\omega(x) = \omega_0(x)$ for $x \in X$ is bounded and σ-surjective. It turns out that the notions of injectivity and σ-surjectivity coincide for continuous maps $f\colon X \to Y$ between topological Hausdorff spaces whenever X is σ-compact (see [15, Proposition 16]). Coming back to the case of $X = Y = \mathbb{R}$, let us recall the well-known example of a bounded continuous and injective (consequently, σ-surjective) function $\omega\colon \mathbb{R} \to \mathbb{R}$ given by $\omega(x) = \frac{x}{1+|x|}$ for $x \in \mathbb{R}$.

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https://doi.org/10.4153/S0008414X22000426 Published online by Cambridge University Press