Robin Hartshorne
Nagoya Math. J.
Vol. 43 (1971), 73–89

AMPLE VECTOR BUNDLES ON CURVES

ROBIN HARTSHORNE[1]

Introduction

In our earlier paper [4] we developed the basic sheaftheoretic and cohomological properties of ample vector bundles. These generalize the corresponding well-known results for ample line bundles. The numerical properties of ample vector bundles are still poorly understood. For line bundles, Nakai's criterion characterizes ampleness by the positivity of certain intersection numbers of the associated divisor with subvarieties of the ambient variety. For vector bundles, one would like to characterize ampleness by the numerical positivity of the Chern classes of the bundle (and perhaps of its restrictions to subvarieties and their quotients). Such a result, like the Riemann-Roch theorem, giving an equivalence between cohomological and numerical properties of a vector bundle, may be quite subtle. Some progress has been made by Gieseker [2], by Kleiman [8], and in the analytic case, by Griffiths [3].

Even on a complete non-singular curve over a field k, the problem is non-trivial. A line bundle is ample if and only if its degree is positive. The degree of an ample vector bundle on a curve is positive. Any quotient of an ample vector bundle is ample, and so its degree is also positive. This leads us to the following

QUESTION: Let X be a complete non-singular curve over a field k. Let E be a vector bundle, all of whose quotient bundles (including E itself) have degree positive. Then is E ample?

In our earlier paper, we found an answer to this question for bundles of rank 2 [4, 7.6 and 7.7]. In that case E is ample provided that either

Received August 20, 1970.
Revised December 11, 1970.
[1] The author gratefully acknowledges financial support from Nagoya University, Nagoya, the Tata Institute of Fundamental Research, Bombay, and the Institut des Hautes Etudes Scientifiques, Paris.

char. $k=0$, or char. $k=p$ and $\deg E > (2/p)(g-1)$. In this paper we show (for bundles of any rank) that E is ample, if X is an elliptic curve (§1), or if k is the complex numbers (§2), but not in general (§3). In a fourth section of the paper, independent of the others, we show that a curve generating an abelian variety has an ample normal bundle. This has applications to the theory of formal-rational functions and cohomological dimension.

To prove our theorem for elliptic curves in characteristic zero, we use Atiyah's classification of vector bundles and his explicit description of the multiplicative structure. For elliptic curves in characteristic p, we use a theorem of Oda which gives conditions for the frobenius map on cohomology to be injective. For curves of genus $g \geq 2$ over the complex numbers, we use a theorem of Narasimhan and Seshadri relating stable vector bundles to unitary representations of the fundamental group. In §3 we give a counter-example of Serre, which is a bundle of rank 2 on a curve of genus 3 over a field of characteristic 3. In this example the Hasse-Witt matrix of the curve is identically zero.

We have not been able to find a satisfactory criterion for ampleness on curves in characteristic p. Would the above result hold, for example, on a curve whose Hasse-Witt matrix is non-singular?*) The elucidation of this question calls for a more profound study of extensions of vector bundles, and their behavior under frobenius.

§1. Bundles on elliptic curves

In this section we show that an indecomposable bundle on an elliptic curve is ample if and only if its degree is positive. Atiyah [1] has classified vector bundles on an elliptic curve. In characteristic zero he has also described the multiplicative structure: for indecomposable bundles E_1 and E_2, he gives the indecomposable direct summands of $E_1 \otimes E_2$. The idea of our proof, in characteristic zero, is to calculate $E^{\otimes n}$ for large n, using Atiyah's explicit description. It is then easy to see when E is ample.

In characteristic p, we prove our theorem by descending induction on the degree of E. The crucial step is to replace E by its frobenius pullback f^*E, which has larger degree. We need to know that certain extensions

*) *Added in proof;* Recently Sumihiro and Tango, and independently Oda, have given examples which show that the Question of the introduction is false, even for curves with non-singular Hasse-Witt matrix.

of vector bundles do not split when pulled back by frobenius. For this we use a theorem of Oda, which gives conditions for the induced action of frobenius, $f^* : H^1(E) \longrightarrow H^1(f^*E)$, to be injective.

We begin by recalling the classification of vector bundles due to Atiyah. Fix an elliptic curve X over an algebraically closed field k. For any r, d, let $\mathscr{E}(r, d)$ be the set of indecomposable bundles on X of rank r and degree d. Then one can choose one bundle $E(r, d) \in \mathscr{E}(r, d)$, for each r, d, such that any other bundle $E \in \mathscr{E}(r, d)$ is of the form $E(r, d) \otimes L$ for some $L \in \mathscr{E}(1, 0)$. In particular, we choose $E(r, 0)$ to be the unique element of $\mathscr{E}(r, 0)$ with $H^0 \neq 0$, and we denote it by F_r. We will use this notation, and other results of Atiyah's paper as needed.

LEMMA 1.1 *Let E be indecomposable of degree d. If $d > 0$, we have*

$$\dim H^0(E) = d$$
$$H^1(E) = 0.$$

If $d < 0$, we have

$$H^0(E) = 0$$
$$\dim H^1(E) = d.$$

Proof. If $d > 0$ the result follows from [1, Lemmas 8, 15]. If $d < 0$, consider $\check{E}$, which has degree > 0, and use duality.

PROPOSITION 1.2 *Let E be any bundle on the elliptic curve X. Then the following conditions are equivalent:*

(i) *each indecomposable direct summand E_i of E has degree > 0.*

(ii) *every quotient line bundle L of E has degree > 0.*

(iii) *every quotient vector bundle E' of E has degree > 0.*

Proof. (i) $\Longrightarrow$ (ii) Let $E = \sum E_i$ with E_i indecomposable, and let L be a quotient line bundle of E. Then for some i, $\mathrm{Hom}(E_i, L) \neq 0$. In other words $H^0(\underline{Hom}(E_i, L)) \neq 0$. But $\underline{Hom}(E_i, L)$ is indecomposable, so by the lemma, its degree must be ≥ 0. This degree is $r_i \deg L - \deg E_i$, where $r_i = \mathrm{rank}\, E_i$, so we have $\deg L \geq \frac{1}{r_i} \deg E_i > 0$.

(ii) $\Longrightarrow$ (iii). We may assume E' is indecomposable, and thus we reduce to showing if E is indecomposable and satisfies (ii), then $\deg E > 0$.

Suppose not. Then $\deg E \leq 0$. If $\deg E < 0$, then $\check{E}$ has a non-zero

https://doi.org/10.1017/S0027763000014379 Published online by Cambridge University Press

section, by the lemma, and hence it has a sub-line bundle M of degree ≥ 0. But then $\check{M}$ is a quotient line bundle of E of degree ≤ 0, which is impossible.

If $\deg E = 0$, then $\check{E} \cong F_r \otimes L$ for some line bundle L of degree 0. Thus $\check{E}$ has L as a sub-line bundle [1, Thm. 5], so E has $\check{L}$ as a quotient line bundle, which is impossible.

(iii) $\Longrightarrow$ (i) is trivial.

NOTE: It is easy to show that these equivalences fail on a curve of genus $g > 1$.

THEOREM 1.3* *A vector bundle E on the elliptic curve X is ample if and only if it satisfies the equivalent conditions of the proposition.*

If E is ample, then every quotient invertible sheaf L of E is ample, so has degree >0, so E satisfies the conditions of the proposition. For the converse, we will separate cases according to the characteristic of k.

PROPOSITION 1.4 *Assume* char. $k = 0$, *and let E be indecomposable of rank r and degree d. Then for any $n = rs$, $s > 0$, we have*

$$E^{\otimes n} \cong \sum (E_i \otimes L_i)$$

where each $E_i \cong F_{r_i}$ for some r_i, and where the L_i are all line bundles of the same degree (necessarily ds).

Proof. We will use Atiyah's results on the multiplicative structure of bundles on X in characteristic zero [1, Part III]. First note that $E \cong E(r, d) \otimes L$ for some $L \in \mathscr{E}(1, 0)$. Thus it is sufficient to treat the case $E = E(r, d)$. We will use induction on r, the case $r = 1$ being trivial.

<u>*Step* 1.</u> Reduction to the case $(r, d) = 1$. Let $(r, d) = h$. If $h > 1$, we have

$$E(r, d) \cong F_h \otimes E(r', d')$$

where $r' = r/h$, and $d' = d/h$ [1, Lemma 24], and by induction we may assume the statement true for $E(r', d')$. Given $n = rs$, we have $n = r's'$ with $s' = sh$, so

$$E(r', d')^{\otimes n} \cong \sum (E'_i \otimes L'_i)$$

where each $E'_i \cong F_{r_i}$ for some r_i, and the L'_i are all line bundles of the same degree. Therefore

* This theorem has also been proved independently by Gieseker [2, Thm. 2.3].

https://doi.org/10.1017/S0027763000014379 Published online by Cambridge University Press

$$E(r, d)^{\otimes n} \cong \sum (F_h^{\otimes n} \otimes E_i' \otimes L_i').$$

Now by [1, Thm. 8], any tensor product of bundles of the form F_r is a direct sum of such bundles again, so we have the result for $E(r, d)$.

Step 2. Assume $(r, d) = 1$, and let $r = \Pi r_i$ be the factorization of r into prime powers. Then there are integers d_i such that

$$E(r, d) \cong \otimes E(r_i, d_i),$$

where $d/r = \sum(d_i/r_i)$, and $(r_i, d_i) = 1$ for each i [1, Lemma 29]. By induction, we may assume the statement true for each $E(r_i, d_i)$, so again using [1, Thm. 8], we deduce the result for $E(r, d)$. Thus we reduce to the case where $r = p^e$ is a prime power.

Step 3. Assume $(r, d) = 1$, and $r = p^e$ with p prime. First we show by induction on t, for $0 < t < p$, that

$$E(p^e, d)^{\otimes t} \cong \sum E(p^e, td).$$

This is trivial for $t = 1$. For $t > 1$, we write $t = (t-1) + 1$, and use the induction hypothesis. Thus

$$E(p^e, d)^{\otimes t} \cong (\sum E(p^e, (t-1)d)) \otimes E(p^e, d) = \sum E(p^e, td)$$

by [1, Thm. 14].

Now by the same method, we find that

$$E(p^e, d)^{\otimes p} \cong (\sum E(p^e, (p-1)d)) \otimes E(p^e, d) \cong (\sum L_i) \otimes E(p^{e-1}, d),$$

where the L_i are suitable line bundles of degree 0. By induction we may assume the proposition true for $E(p^{e-1}, d)$, and so we conclude also for $E(p^e, d)$.

Proof of theorem in characteristic 0. It is sufficient to show that any E indecomposable of rank r and degree $d > 0$ is ample. Take $s = 1$ in the Proposition. Then

$$E^{\otimes r} \cong \sum (E_i \otimes L_i)$$

where each $E_i \cong F_{r_i}$ for some r_i, and each L_i is a line bundle of degree $d > 0$.

First we show, by induction on r, that $F_r \otimes L$ is ample, if L is a line bundle of degree > 0. For $r = 1$ it is trivial. For $r > 1$ we use the exact

https://doi.org/10.1017/S0027763000014379 Published online by Cambridge University Press

sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow 0$$

[1, Thm. 5]. Tensoring with L, and using the fact that an extension of ample bundles is ample [4, 3.4], we find $F_r \otimes L$ is ample.

Thus in the direct sum above, each $E_i \otimes L_i$ is ample. Therefore $E^{\otimes r}$ is ample, so its quotient $S^r(E)$ is ample, and so E is ample [4, 2.4].

In characteristic p, we will need the following theorem of Oda. For a curve X over a field k of characteristic p, we define the *frobenius morphism* $f: X \longrightarrow X$ as the identity on the underlying topological spaces, and the p^{th} power homomorphism on the structure sheaves.

THEOREM (Oda [10, Thm. 2.17]): *Let E be an indecomposable bundle of negative degree on the elliptic curve X. Then the induced action of frobenius on cohomology,*

$$f^*: H^1(X, E) \longrightarrow H^1(X, f^*E)$$

is injective.

Proof of theorem in characteristic $p > 0$. Using condition (ii) of Proposition 1.2, it is sufficient to show if E is a bundle such that every quotient line bundle L has degree > 0, then E is ample.

We use induction on the rank r of E. If $r = 1$, then E is a line bundle of positive degree, hence ample. For indecomposable bundles of fixed rank r, we will also use descending induction on $d = \deg E$. For $d \gg 0$, E is ample by [4, 7.2].

By the induction on r, we may assume E is indecomposable. Then by Proposition 1.2, $\deg E > 0$, and so by Lemma 1.1, $H^0(E) \neq 0$. Hence there is a non-zero map $\mathcal{O}_X \longrightarrow E$, which determines a sub-line bundle M of E such that either $\deg M > 0$ or $M \cong \mathcal{O}_X$. Let E' be the quotient, so we have an exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow E' \longrightarrow 0.$$

Since every quotient line bundle of E' is also a quotient line bundle of E, E' also satisfies our hypothesis, and so by induction on r, E' is ample.

If $\deg M > 0$, then M is ample, and so E is ample [4, 3.4]. So we have only to consider the case $M = \mathcal{O}_X$. The extension E of E' by $\mathcal{O}$ is classified by an element $c \in H^1(\check{E}')$. And since by hypothesis, $\mathcal{O}$ cannot be a quotient of E, the extension is non-trivial, so $c \neq 0$. Now E' being ample, is a direct sum of indecomposable bundles E'_i of degree > 0. So $\check{E}' = \sum \check{E}'_i$, with the $\check{E}'_i$ indecomposable of degree < 0. By the theorem of Oda, the frobenius map

$$f^* : H^1(\check{E}') \longrightarrow H^1(f^*\check{E}')$$

is injective. In particular, $f^*(c) \neq 0$, which implies that the sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow f^*E \longrightarrow f^*E' \longrightarrow 0$$

does not split.

Note that rank $f^*E = r$, and $\deg f^*E = p.\deg E > \deg E$, since $\deg E > 0$. Furthermore, every quotient line bundle L of f^*E has positive degree. Indeed, let $f^*E \longrightarrow L$ be a quotient line bundle. This induces a map $\varphi : \mathcal{O}_X \longrightarrow L$. This map cannot be an isomorphism, because then the sequence would split. If $\varphi \neq 0$, then $\deg L > 0$. If $\varphi = 0$, then L is a quotient line bundle of f^*E'. But f^*E' is ample since E' is [4, 4.3], so L is ample, and $\deg L > 0$.

Now if f^*E is decomposable, we conclude it is ample by induction on r. If it is indecomposable, we conclude it is ample by induction on d. Therefore E is also ample [4, 4.3]. q.e.d.

§ 2. Ample vector bundles on curves over the complex numbers.

Here we will prove that a bundle E on a complete non-singular curve X over $\boldsymbol{C}$ is ample, if and only if $\deg E' > 0$ for every quotient bundle E' of E (including E itself). First we reduce to the case of a stable bundle, in the sense of Mumford. Then we use a theorem of Narasimhan and Seshadri to deduce that all the symmetric powers of a stable bundle are semi-stable. At this point we must work over the complex numbers. Now to prove that E is ample, we look at the tautological line bundle L on $\boldsymbol{P}(E)$, and we show that L is ample, using a new criterion for an ample divisor, due to Seshadri.

Recall that a bundle E on a curve X is *stable* if for every proper subbundle $E' \subseteq E$ we have

$$\frac{\deg E'}{\text{rank } E'} < \frac{\deg E}{\text{rank } E}.$$

The bundle E is *semi-stable* if only the weak inequality $\leq$ holds.

PROPOSITION 2.1 *Let X be a complete curve over a field k. Then the following conditions are equivalent*:

(i) *Every stable bundle of positive degree is ample.*

(ii) *Every bundle, all of whose quotients have positive degree, is ample.*

Proof: (i) $\Longrightarrow$ (ii). Assume (i), and let E be a bundle, all of whose quotients have positive degree. We use induction on the rank r of E. If $r=1$, E is a line bundle of positive degree, and so is ample.

In general, let E have rank r. First suppose that E has an ample subbundle E', and let E'' be the quotient E/E'. Then every quotient bundle of E'' has positive degree, and rank $E'' < \operatorname{rank} E$, so by the induction hypothesis, E'' is ample. But then E is an extension of ample bundles. and so is ample itself.

Next we will show that if E has no ample subbundles, then E is stable. Hence by (i) we conclude that E is ample. To show that E is stable, we will show that every subbundle $E' \subseteq E$ has degree ≤ 0. We use induction on $s = \operatorname{rank} E'$. If $s=1$, then E' is a line bundle. We must have $\deg E' \leq 0$, for otherwise E' would be ample. In general, let E' be a subbundle of rank s. Suppose $\deg E' > 0$. Then for any quotient E'' of E', we have also $\deg E'' > 0$. Indeed, let $E_0 = \ker (E' \longrightarrow E'')$. Then rank $E_0 < s$, so by induction $\deg E_0 \leq 0$. Hence $\deg E'' \geq \deg E' > 0$. But now, since $\operatorname{rank} E' < \operatorname{rank} E$, we can conclude by our first induction hypothesis that E' is ample. This is a contradiction, because we are assuming that E has no ample subbundles. Therefore $\deg E' \leq 0$, as required.

(ii) $\Longrightarrow$ (i) If E is a stable bundle of positive degree, then every quotient bundle has positive degree also. Indeed, let

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be an exact sequence. Then

$$\deg E'' = \deg E - \deg E'.$$

But the stability of E implies that $\deg E' < \deg E$. So $\deg E'' > 0$. We conclude by (ii) that E is ample.

PROPOSITION 2.2 *Let X be a complete non-singular curve of genus $g \geq 2$ over the complex numbers $\boldsymbol{C}$. Let E be a stable vector bundle on X. Then the symmetric powers $S^n(E)$ are semi-stable for all $n \geq 1$.*

https://doi.org/10.1017/S0027763000014379 Published online by Cambridge University Press

Proof. We will use some results of Narasimhan and Seshadri [9] relating stable vector bundles to certain group representations. Given integers $g, N \geq 1$, we define a group π by generators $a_1, b_1, \cdots, a_g, b_g, c$ and relations

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} c = 1$$
$$c^N = 1.$$

If X is a compact Riemann surface of genus g, then π can be realized as the group of cover transformations of a branched covering $f: Y \longrightarrow X$, where Y is a simply connected Riemann surface, and where f is ramified over a single point $x_0 \in X$, and has ramification order N over x_0. For any $n \geq 1$, if

$$\rho : \pi \longrightarrow GL(n, \boldsymbol{C})$$

is a representation of π, then we define the *associated vector bundle* E on X as follows: let π act on $\mathcal{O}_Y^n$ via ρ, and let E be the subsheaf of $f_*(\mathcal{O}_Y^n)$ consisting of sections stable under π. Then E is a vector bundle of rank n on X.

A representation ρ is called *special* if $\rho(c) = \lambda I$, where $\lambda \in \boldsymbol{C}$, $\lambda^n = 1$, and I is the identity matrix. The results we need are

a) [9, Prop 10.4]. If ρ is a special, unitary representation of π, then the associated bundle E is semi-stable.

b) [9, Thm. 2]. For the case $n = N$, the operation "associated vector bundle" gives a one-to-one correspondence between the equivalence classes of special, irreducible, unitary representations of π and the isomorphism classes of stable vector bundles E on X of rank n, with $-n < \deg E \leq 0$.

Now for the proof of the proposition, let E be a stable vector bundle on X, of rank n. Let L be a line bundle on X, such that $-n < \deg E \otimes L \leq 0$. Then $E \otimes L$ is also stable. Furthermore, $S^n(E \otimes L) = S^n(E) \otimes L^n$, so to prove $S^n(E)$ is semistable, it is sufficient to prove $S^n(E \otimes L)$ semistable [9, Prop. 4.2]. Thus we reduce to the case

$$-n < \deg E \leq 0.$$

By the result (b) above, E is associated to a special, irreducible, unitary representation ρ of π, where $N = n = \operatorname{rank} E$. Then $S^n(E)$ is associated to the symmetric power $S^n(\rho)$ of the representation ρ. $S^n(\rho)$ is a special, unitary representation (not necessarily irreducible). By the result (a) above, $S^n(E)$ is semi-stable.

https://doi.org/10.1017/S0027763000014379 Published online by Cambridge University Press

Now we will need a criterion for an ample divisor, due to Seshadri. For any integral curve C, let $m(C)=\max_{P\in C} \mathrm{mult}_P(C)$. Note that $m(C)\geq 1$.

THEOREM. (Seshadri; for proof see [6, Thm. I.7.1]): *Let X be a complete scheme over a field k, and let D be a Cartier divisor on X. Then D is ample if and only if there is an $\varepsilon>0$ such that for every integral curve C in X, $(D.C)>\varepsilon m(C)$.*

PROPOSITION 2.3 *Let X be a complete non-singular curve over a field k. Let E be a vector bundle of positive degree, all of whose symmetric powers $S^n(E)$ are semi-stable. Then E is ample.*

Proof. Let E have rank r, and degree $d>0$. We consider the associated projective bundle $\boldsymbol{P}(E)$, and the tautological line bundle $L=\mathcal{O}_{P(E)}(1)$. To show that E is ample on X is equivalent to showing that L is ample on $\boldsymbol{P}(E)$ [4, 3.2]. We will use the criterion of Seshadri to show that L is ample on $\boldsymbol{P}(E)$. Thus we must find an $\varepsilon>0$ such that $(L.C)>\varepsilon m(C)$ for every integral curve $C\subseteq\boldsymbol{P}(E)$.

Let $\pi:\boldsymbol{P}(E)\longrightarrow X$ be the projection. If $\pi(C)$ is a point, then C is contained in a fibre of π, which is a projective $(r-1)$-space. In this case $(L.C)$ is the degree of C as a curve in $\boldsymbol{P}^{r-1}$. Clearly $\deg C\geq m(C)$, so we have $(L.C)\geq m(C)$.

Now suppose that $\pi(C)=X$. Let m be the degree of the finite morphism $\pi:C\longrightarrow X$. Considering C as a closed subscheme of $\boldsymbol{P}(E)$, we have an exact sequence of sheaves

$$0\longrightarrow I_C\longrightarrow\mathcal{O}_{\boldsymbol{P}(E)}\longrightarrow\mathcal{O}_C\longrightarrow 0,$$

where I_C is the sheaf of ideals of C. Tensoring with L^n, we have an exact sequence

$$0\longrightarrow I_C\otimes L^n\longrightarrow L^n\longrightarrow\mathcal{O}_C\otimes L^n\longrightarrow 0.$$

For n sufficiently large, the functor π_* will be exact, since L is relatively ample for π. So we have an exact sequence of sheaves on X

$$0\longrightarrow\pi_*(I_C\otimes L^n)\longrightarrow\pi_*(L^n)\longrightarrow\pi_*(\mathcal{O}_C\otimes L^n)\longrightarrow 0.$$

These are all locally free sheaves; the middle one, $\pi_*(L^n)$, is just $S^n(E)$, because of our construction. So we can apply the hypothesis that $S^n(E)$ is semi-stable. Note, by the way, that a vector bundle E is semi-stable if and only if for every quotient bundle E'' of E, one has

https://doi.org/10.1017/S0027763000014379 Published online by Cambridge University Press

$$\frac{\deg E}{\text{rank } E} \leq \frac{\deg E''}{\text{rank } E''}.$$

This follows immediately from the definition.

In our case, we have rank $E = r$, and $\deg E = d > 0$. An easy calculation shows that

$$\text{rank } S^n(E) = \binom{n+r-1}{r-1}$$

and

$$\deg S^n(E) = \frac{dn}{r} \cdot \binom{n+r-1}{r-1}.$$

Since $\pi : C \longrightarrow X$ is a finite morphism of degree m, we have

$$\text{rank } \pi_*(\mathcal{O}_C \otimes L^n) = m.$$

If M is any invertible sheaf on C, then one sees easily that

$$\deg \pi_* M = \deg M + \deg \pi_* \mathcal{O}_C.$$

In our case, $\deg \mathcal{O}_C \otimes L^n = n(L.C)$, so we have

$$\deg \pi_*(\mathcal{O}_C \otimes L^n) = n(L.C) + \deg \pi_* \mathcal{O}_C.$$

Now since $S^n(E)$ is semi-stable, and $\pi_*(\mathcal{O}_C \otimes L^n)$ is its quotient, we have

$$\frac{dn}{r} \leq \frac{n(L.C) + \deg \pi_* \mathcal{O}_C}{m}.$$

Dividing by n, and multiplying by m, we have

$$\frac{d}{r} m \leq (L.C) + \frac{1}{n} \deg \pi_* \mathcal{O}_C.$$

This is true for all n sufficiently large, so we conclude that

$$(L.C) \geq \frac{d}{r} m.$$

On the other hand, since X is non-singular, $m \geq m(C)$, so we have

$$(L.C) \geq \frac{d}{r} m(C).$$

Now if we take $0 < \varepsilon < \min\left(1, \frac{d}{r}\right)$ we will have $(L.C) > \varepsilon m(C)$ for all curves $C \subseteq \boldsymbol{P}(E)$, which proves that L is ample. Hence E is ample on X.

THEOREM 2.4. *Let X be a complete non-singular curve over the complex numbers $\boldsymbol{C}$. Let E be a vector bundle on X. Then E is ample if and only if every quotient bundle of E has positive degree.*

Proof. If E is ample, every quotient bundle E' of E is also ample, and so has positive degree. Conversely, to show that every bundle, all of whose quotients have positive degree, is ample, it is sufficient by Proposition 2.1 to consider only stable bundles. But if E is stable, and X is of genus at least 2 over $\boldsymbol{C}$, then all the symmetric powers $S^n(E)$ are semi-stable, by Proposition 2.2. It follows from Proposition 2.3 that E is ample. For curves of genus 1, the result was proved in the previous section, and for curves of genus 0 it is trivial.

§3. An Example of Serre

In this section we give an example, due to Serre, of a non-singular curve X of genus 3 over a field of characteristic 3, and a bundle E of rank 2 on X, of degree 1, such that every quotient bundle of E has positive degree, but E is not ample. In this example, the Hasse-Witt matrix of X is identically zero.

Let k be an algebraically closed field of characteristic 3. Let $X \subseteq \boldsymbol{P}_k^2$ be the curve given by the homogeneous equation

$$x^3y + y^3z + z^3x = 0.$$

One verifies easily that X is non-singular. Being a plane curve of degreee 4, it has genus 3.

LEMMA 3.1 *The Hasse-Witt matrix of X is identically zero, i.e., the action of frobenius on $H^1(X, \mathcal{O}_X)$ is 0.*

Proof. Let $g = x^3y + y^3z + z^3x$. Then we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}}(-4) \xrightarrow{g} \mathcal{O}_{\boldsymbol{P}} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

which gives rise to an isomorphism

$$H^1(X, \mathcal{O}_X) \xrightarrow{\cong} H^2(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(-4)). \tag{1}$$

This latter vector space, according to the explicit calculations of cohomology on projective space, has a basis consisting of the "negative monomials" of degree 4, namely

https://doi.org/10.1017/S0027763000014379 Published online by Cambridge University Press

$$\frac{1}{x^2yz}, \frac{1}{xy^2z}, \frac{1}{xyz^2}.$$

Under the isomorphism (1), the action of frobenius on $H^1(X, \mathcal{O}_X)$ becomes the composition

$$H^2(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(-4)) \xrightarrow{f^*} H^2(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(-12)) \xrightarrow{g^2} H^2(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(-4))$$

of frobenius on $\boldsymbol{P}^2$ with multiplication by g^2. Now frobenius takes our three monomials into their cubes,

$$\frac{1}{x^6y^3z^3}, \frac{1}{x^3y^6z^3}, \frac{1}{x^3y^3z^6}.$$

Every monomial of g^2 contains either x^6 or y^3 or z^3. Hence $g^2/x^6y^3z^3 = 0$ in $H^2(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(-4))$. By symmetry, g^2 also kills the other two monomials, so we find that frobenius on $H^1(X, \mathcal{O}_X)$ is 0.

EXAMPLE 3.2 A bundle E of rank 2 on X, such that every quotient bundle of E has positive degree, but E is not ample.

Let $p \in X$ be a point, and consider an extension of sheaves on X,

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow \mathcal{O}(p) \longrightarrow 0,$$

classified by an element $\xi \in H^1(X, \mathcal{O}(-p))$. By Riemann-Roch, $H^1(X, \mathcal{O}(-P))$ has dimension 3. The degree of E is 1. If $\xi \neq 0$, then every quotient line bundle L of E has positive degree. Indeed, consider the composed map $\mathcal{O}_X \longrightarrow E \longrightarrow L$. If it is zero, then $\mathcal{O}_X(P) \subseteq L$, so $\deg L \geq 1$. If it is non-zero, but not an isomorphism, then $\deg L > 0$. If it is an isomorphism, then $\mathcal{O}_X$ is a direct summand of E, so $\xi = 0$, a contradiction.

Now consider the action of the frobenius morphism f on the extension above. We have a new extension

$$0 \longrightarrow \mathcal{O} \longrightarrow f^*E \longrightarrow \mathcal{O}(3P) \longrightarrow 0$$

classified by the element $f^*(\xi) \in H^1(X, \mathcal{O}(-3P))$. We have a commutative diagram

$$\begin{array}{ccc} H^1(X, \mathcal{O}(-P)) & \xrightarrow{\cong} & H^1(X, \mathcal{O}) \\ \downarrow f^* & & \downarrow f^* \\ H^1(X, \mathcal{O}(-3P)) & \longrightarrow & H^1(X, \mathcal{O}) \longrightarrow 0. \end{array}$$

By Riemann-Roch, $\dim H^1(X, \mathcal{O}_X(-3P)) = 5$. Now frobenius is 0 on $H^1(X, \mathcal{O}_X)$.

https://doi.org/10.1017/S0027763000014379 Published online by Cambridge University Press

So by counting dimensions, there is a $\xi \neq 0$ in $H^1(X, \mathcal{O}_X(-P))$ with $f^*(\xi) = 0$.

Let E be the corresponding bundle. Then every quotient bundle of E has positive degree (as we have seen), but E is not ample, because f^*E has $\mathcal{O}_X$ as a quotient, and so cannot be ample. On the other hand, E is ample if and only if f^*E is ample, by [4, 4.3].

Remarks. 1. The same construction would work on any curve X of genus g, over a field of characteristic $p > 0$, such that the rank σ of the Hasse-Witt matrix of X satisfies $\sigma < g - p + 1$.

2. The bundle E constructed above is stable, but its third symmetric product $S^3(E)$ is not semi-stable. The stability of E is obvious. Since $f^*E \subseteq S^3(E)$, we have also $\mathcal{O}_X(3P) \subseteq S^3(E)$. Now $S^3(E)$ has rank 4 and degree 6, so it is not semi-stable.

3. It would be interesting to know, for which curves in characteristic p is it true that the symmetric powers of a stable bundle are semi-stable. In the same vein, one can ask, when is it true that E stable implies f^*E semi-stable?

4. In the example above, we have an ample line bundle L, namely $\mathcal{O}_X(P)$, such that the frobenius map $f^* : H^1(\check{L}) \longrightarrow H^1(f^*\check{L})$ is not injective. However, the theorem of Oda (see §1) says that if E is an ample bundle on an elliptic curve, then $f^* : H^1(\check{E}) \longrightarrow H^1(f^*\check{E})$ is injective. Thus we may ask, for which curves X in characteristic p is it true that $f^* : H^1(\check{E}) \longrightarrow H^1(f^*\check{E})$ is injective for all ample bundles E? Is it sufficient that the Hasse-Witt matrix of the curve be non-singular?

§4. Curves in Abelian Varieties

Let X be a subvariety of an abelian variety A. We say that X *generates* A if the set of differences $\{x_1 - x_2 \mid x_1, x_2 \in X\}$ generate A as a group.

In this section we show that if X is a non-singular curve in an abelian variety A, which generates A, then the normal bundle to X in A is ample. In fact, our result is more general: it applies to any non-singular variety X in an abelian variety A, such that every curve in X generates A. We also give two applications.

We need the following result of Gieseker:

THEOREM (Gieseker [2, Prop. 2.1]): *Let X be a complete scheme over a field k. Let E be a vector bundle on X which is generated by its global sections.*

https://doi.org/10.1017/S0027763000014379 Published online by Cambridge University Press

Then E is ample if and only if for every curve $C \subseteq X$, and for every quotient line bundle L of $E|_C$, $\deg L > 0$.

Proposition 4.1 *Let X be a non-singular variety, contained in an abelian variety A. Assume that every curve in X generates A. Then the normal bundle N to X in A is ample.*

Proof. Since A is an abelian variety, its tangent bundle T_A is trivial. Thus the normal bundle to X, being a quotient of $T_A|_X$, is generated by global sections. To apply the criterion of Gieseker, we must show that for every curve C in X, and for every quotient line bundle L of $N|_C$, $\deg L > 0$. Suppose to the contrary there is a C and an L with $\deg L \leq 0$. Now L is also generated by global sections, so we would have $\deg L = 0$ and in fact $L \cong \mathcal{O}_C$. Therefore the dual $\check{N}|_C$ has a subbundle isomorphic to θ_C, and in particular, $H^0(\check{N}|_C) \neq 0$. Now we have an exact sequence

$$0 \longrightarrow \check{N}|_C \longrightarrow \Omega_A^1|_C \longrightarrow \Omega_X^1|_C \longrightarrow 0$$

of sheaves on C, where Ω^1 denotes the sheaf of differential forms. On the other hand, there is a natural map of sheaves $\Omega_A^1|_C \longrightarrow \Omega_C^1$, which factors through $\Omega_X^1|_C$. Since C generates A, the map of global sections

$$H^0(\Omega_A^1) = H^0(\Omega_A^1|_C) \longrightarrow H^0(\Omega_C^1)$$

is injective. So the map

$$H^0(\Omega_A^1|_C) \longrightarrow H^0(\Omega_X^1|_C)$$

is also injective, and hence $H^0(\check{N}|_C) = 0$. This is a contradiction. So we conclude by the criterion of Gieseker that N is ample on X.

Remark. This result applies in particular whenever X is a non-singular curve which generates A, or whenever X is a non-singular variety of any dimension in a *simple* abelian variety A.

Example. It is not sufficient in the Proposition to assume merely that X generates A. For example, let $X_1 \subseteq A_1$ and $X_2 \subseteq A_2$ be non-singular curves which generate the abelian varieties A_1 and A_2. Then their normal bundles N_1 and N_2 are ample. Let $X = X_1 \times X_2 \subseteq A = A_1 \times A_2$. Then X generates A. However, the normal bundle to X is $N \cong p_1^* N_1 \oplus p_2^* N_2$, which is not ample.

COROLLARY 4.2. *Let X and A be as in the proposition. Then the field $K(\hat{A})$ of formal-rational functions on the formal completion $\hat{A}$ of A along X is a finite extension field of $K(A)$.* (We say that X is $G2$ in A. This is a special case of a theorem of Hironaka and Matsumura [7, Thm. 4.2], which says that whenever X generates A, it is $G2$ in A.)

Proof. This follows immediately from [5, Cor. 6.8].

COROLLARY 4.3. *Let X and A be as in the proposition. Assume furthermore that the characteristic of the ground field k is 0. Then $H^i(A-X, F)$ is finite-dimensional over k, for all coherent sheaves F, and for all $i \geq codim(X, A)$.*

Proof. This follows immediately from [5, Cor. 5.5].

EXAMPLE. It is not sufficient in this Corollary to assume merely that X generates A. We take the same example as above. Let H_1 be a hyperplane section of A_1 which does not contain X_1. Then $H_1 \cap X_1$ is a finite set of points.

Let $H \subseteq A$ be $p_1^{-1}(H_1)$. Let $\hat{H}$ be the formal completion of H along $H \cap X$, and let $\hat{H}_1$ be the formal completion of H_1 along $H_1 \cap X_1$. Then there is a natural map of formal schemes $\hat{H} \longrightarrow \hat{H}_1$. But $\hat{H}_1$ consists of a finite number of points, so $H^0(\mathcal{O}_{\hat{H}_1})$ is infinite-dimensional. This is included in $H^0(\mathcal{O}_{\hat{H}})$, which is also infinite-dimensional. Therefore, by [5, Prop. 4.3] there is a coherent sheaf F on H such that $H^{n-2}(H - H \cap X, F)$ is infinite-dimensional. Here we let $\dim A = n$, so $\dim H = n-1$. But F is likewise a coherent sheaf on $A - X$, so $H^{n-2}(A - X, F)$ is infinite-dimensional.

PROBLEM. Find necessary and sufficient conditions on a subvariety X of an abelian variety A for the result of the Corollary to hold.

BIBLIOGRAPHY

[1] Atiyah, M.F., Vector bundles over an elliptic curve, Proc. Lond. Math. Soc. (3) **7** (1957) 414–452.

[2] Gieseker, D.. p-ample bundles and their Chern classes, Nagoya Math. J. Vol. 43 (this volume).

[3] Griffiths, P.A., Hermitian differential geometry, Chern classes, and positive vector bundles, in "Global Analysis", vol. ded. to K. Kodaira (1969), Univ. of Tokyo Press.

[4] Hartshorne, R., Ample vector bundles. Publ. Math. IHES **29** (1966) 63–94.

[5] ——————, Cohomological dimension of algebraic varieties, Ann. of Math. **88** (1968) 403–450.

[6] ——————, Ample subvarieties of algebraic varieties, Springer Verlag, Lecture Notes in Mathematics **156** (1970).
[7] Hironaka, H. and Matsumura, H., Formal functions and formal embeddings, J. Math. Soc. Japan, **20** (1968), 52–82.
[8] Kleiman, S., Ample vector bundles on algebraic surfaces, Proc. Amer. Math. Soc. **21** (1969) 673–676.
[9] Narasimhan, M.S. and Seshadri, C.S., Stable and unitary vector bundles on a compact Riemann surface, Ann. of Math. **82** (1965) 540–567.
[10] Oda, T., Vector bundles on an elliptic curve, Nagoya Math. J. Vol. 43 (this volume).

Harvard University

https://doi.org/10.1017/S0027763000014379 Published online by Cambridge University Press