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THE MEET OPERATOR IN THE LATTICE OF GROUP TOPOLOGIES

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ABSTRACT. It is well known that the lattice of topologies on a set forms a complete complemented lattice. The set of topologies which make G into a topological group form a complete lattice $L(G)$ which is not a sublattice of the lattice of all topologies on G.

Let G be an infinite abelian group. No nontrivial Hausdorff topology in $L(G)$ has a complement in $L(G)$. If τ_1 and τ_2 are locally compact topologies then $\tau_1 \wedge \tau_2$ is also a locally compact group topology. The situation when G is nonabelian is also considered.

1. **Introduction**. It is well known that the lattice of topologies on a set is a complete complemented lattice. The first proof of this was done by Steiner [5]. Shorter versions of this result are due to Van Rooij [6] and Schnare [3]. In [1], Berri studied the complements of certain topological groups in the lattice of all topologies on the group.

One might wish to study the class of all topologies on a group G which make G into a topological group. We shall call such a topology a group topology. As is pointed out in [2], the intersection of two group topologies need not be a group topology. Thus the collection of group topologies on G is not a sublattice of the lattice of all topologies on G. However, if $\{\tau_\alpha\}_{\alpha \in \Delta}$ is any family of group topologies for G, then the join topology $\bigvee_{\alpha \in \Delta} \tau_\alpha$ created using $\bigcup_{\alpha \in \Delta} \tau_\alpha$ as a subbasis will also be a group topology. We note that the indiscrete topology is a group topology. Thus for any collection of group topologies $\{\tau_\alpha\}_{\alpha \in \Delta}$ we can find a nonempty collection of group topologies A such that $\tau \in A$ if and only if $\tau \subseteq \tau_\alpha$ for every $\alpha \in \Delta$. We define $\bigwedge_{\alpha \in \Delta} \tau_\alpha = \bigvee_{\tau \in A} \tau$. Hence the collection of all group topologies on G can be made into a complete lattice $L(G)$.

Obviously the meet operator in the lattice of group topologies is far from tractable. The purpose of this is paper to develop a better method of defining the meet operator in special situations and to study the structure of $L(G)$. In particular we shall show that if G is abelian then no nontrivial Hausdorff topology in $L(G)$ has a complement in $L(G)$. In the last section of this paper we shall consider the situation when G is nonabelian.

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Following the notation of [5] we shall let 0 denote the indiscrete topology on G and 1 denote the discrete topology on G. We shall assume as an additional hypothesis that G is both abelian and of infinite order. In section 4 we shall assume that G is nonabelian and of infinite order.

2. **The Meet Operator**. Let t_i be a group topology for G with G_i a copy of G endowed with the topology t_i for $i \in \mathbb{N}$. If we place the box topology on $\bigoplus_{i\in\mathbb{N}} G_i$ we have endowed $\bigoplus_{i\in\mathbb{N}} G_i$ with a group topology. Given an ω-tuple x contained in $\bigoplus_{i\in\mathbb{N}} G_i$, one can find an $n \in \mathbb{N}$ such that the k-th coordinate of x is the identity element e of G whenever $k > n$. We define $m: \bigoplus_{i\in\mathbb{N}} G_i \to G$ by $m(x) = g_1 \cdot g_2 \cdots g_n$. Obviously m is well defined and since G is abelian, m is a homomorphism. Let T_m be the quotient topology generated on G by m.

THEOREM 1. $T_m = \bigwedge_{i\in\mathbb{N}} t_i$.

PROOF: Let $e \in U$ and $U \in T_m$. We can find a basis element W containing the identity element of $\bigoplus_{i\in\mathbb{N}} G_i$ and such that $m(W) \subseteq U$. Let $U_i = \pi_i(W \cap Y_i)$ where Y_i is the i-th axis of $\bigoplus_{i\in\mathbb{N}} G_i$ and π_i is the projection map onto the i-th factor G_i. Certainly $U_i \in t_i$ and $m(W \cap Y_i) = U_i \subset U$. Therefore $T_m \subseteq t_i$.

Now suppose that T is a group topology for G with the property that $T \subseteq t_i$ for all $i \in \mathbb{N}$. Suppose that $e \in U$ and $U \in T$. We can find a sequence $V_1, V_2, V_3, \ldots$ of neighborhoods of e in T such that $V_1 = U$ and $V_{i+1}^2 \subseteq V_i$ for all $i \in \mathbb{N}$. Since $V_i \in t_i$ for all $i \in \mathbb{N}$ we have that $(\times_{i\in\mathbb{N}} V_i) \cap (\bigoplus_{i\in\mathbb{N}} G_i) = W$ is an open set in the box topology on $\bigoplus_{i\in\mathbb{N}} G_i$. If $x \in W$, a simple induction will show that $m(x) \in U$ and thus $m(W) \subseteq U$. Hence $T \subseteq T_m$.

One might hope to remove the restriction of having a countable index set in Theorem 1. Certainly if Δ is an uncountable index set and t_α is a group topology for every $\alpha \in \Delta$, then $\bigoplus_{\alpha\in\Delta} G_\alpha$ is a topological group when endowed with the box topology and the associated group topology $T_m \subseteq t_\alpha$ for every $\alpha \in \Delta$. However, in general $T_m \neq \bigwedge_{\alpha\in\Delta} t_\alpha$. To demonstrate this, let t be any connected topology for G and let $t = t_\alpha$ for every $\alpha \in \Delta$. Obviously, $\bigwedge_{\alpha\in\Delta} t_\alpha = t$. But $T_m = 0$.

The differences between the lattice of group topologies and the usual lattice of topologies are considerable. For example, any two Hausdorff topologies on a set contain the finite complement topology in their intersection. But if t_p is the p-adic topology on $\mathbb{Z}$ and t_q is the q-adic topology on $\mathbb{Z}$ with $p \neq q$ then $t_p \wedge t_q = 0$.

3. **Structure Theorems**. The very nature of the meet operator in $L(G)$ yields our first structure theorem. Certainly if t_1 and t_2 are locally compact topologies on G, then $t_1 \wedge t_2$ is also locally compact. On the other hand suppose that $G = \mathbb{R}^1$ and that t_1 is the usual topology for $\mathbb{R}^1$. Let t_2 be the group topology on $\mathbb{R}^1$ generated using $\{r\mathbb{Q} \mid r \in \mathbb{R}^1\}$ as a basis. Clearly both t_1 and t_2 are locally compact topologies and $t_1 \vee t_2$ is not locally compact. Therefore the strongest result we can state concerning the locally compact topologies of $L(G)$ is the following:

THEOREM 2. *The set of locally compact topologies forms a subsemilattice of $L(G)$.*

THEOREM 3. *If t is a nontrivial Hausdorff group topology then t has no complement in $L(G)$.*

PROOF: Let $t_1, t_2 \in L(G)$ with $t_1 \wedge t_2 = 0$. Let $\{V_\alpha\}$ be a symmetric basis for the topology t_2 at e and $U \in t_1$. If $U \neq \emptyset$ and gV_α is a typical basis element in t_2 with $V_\alpha \neq \emptyset$, then $U \cdot V_\alpha = G$. Hence we can find a $u \in U$ and a $v \in V_\alpha$ with $u \cdot v = g$. Thus $u = g \cdot v^{-1} \in gV_\alpha$ and hence U is dense in t_2. (In fact, it is easy to show that $t_1 \wedge t_2 = 0$ if and only if every nonempty open set in t_1 is dense in t_2.)

If t_2 is a Hausdorff topology and $t_1 \wedge t_2 = 0$, then every nonempty open set $U \in t_1$ and every nonempty open set $V \in t_2$ have infinite intersection. Thus it is impossible for $t_1 \vee t_2 = 1$.

Although the most interesting topologies of $L(G)$ fail to have a complement in $L(G)$, it is possible for two non-Hausdorff group topologies to be complementary. Let $G = \mathbb{R}^1$. We can use Zorn's lemma to find a maximal subgroup $N \subseteq \mathbb{R}^1$ with $N \cap \mathbb{Q} = \{0\}$. Let t_1 be the group topology generated using $\{r\mathbb{Q} \mid r \in \mathbb{R}^1\}$ as a basis and let t_2 be the group topology generated using $\{rN \mid r \in \mathbb{R}^1\}$ as a basis. Since every nonempty open set in t_1 is dense in t_2, we have that $t_1 \wedge t_2 = 0$. Obviously, $t_1 \vee t_2 = 1$.

4. **The Nonabelian Case**. When G is nonabelian the map m defined in Section 2 fails to be a homomorphism. Thus a different technique is needed. Let t_1 and t_2 be two group topologies and let $\{U_\alpha\}_{\alpha \in \Delta}$ and $\{V_\beta\}_{\beta \in \Gamma}$ be bases for these topologies at e respectively. Let $B_1 = \{U_\alpha V_\beta \mid \alpha \in \Delta \text{ and } \beta \in \Gamma\}$ and $B_2 = \{V_\beta U_\alpha \mid \alpha \in \Delta \text{ and } \beta \in \Gamma\}$.

We note that $(U_{\alpha_1} \cap U_{\alpha_2}) \cdot (V_{\beta_1} \cap V_{\beta_2}) \subseteq U_{\alpha_1} V_{\beta_1} \cap U_{\alpha_2} V_{\beta_2}$. Since we can find a $U' \in \{U_\alpha\}_{\alpha \in \Delta}$ with $U' \subseteq U_{\alpha_1} \cap U_{\alpha_2}$ and a $V' \in \{V_\beta\}_{\beta \in \Gamma}$ with $V' \subseteq V_{\beta_1} \cap V_{\beta_2}$ we have $U'V' \in B_1$ with $U'V' \subseteq U_{\alpha_1} V_{\beta_1} \cap U_{\alpha_2} V_{\beta_2}$. Now suppose $a = uv \in UV$. We can find a $U' \in \{U_\alpha\}_{\alpha \in \Delta}$ such that $u \cdot U'V \subseteq UV$. We can also find a $U'' \in \{U_\alpha\}_{\alpha \in \Delta}$ such that $vU''V^{-1} \subseteq U'$. Thus $aU''V^{-1}V \subseteq UV$. Since $v \in V$ we can find a $V' \in \{V_\beta\}_{\beta \in \Gamma}$ such that $v \cdot V' \subseteq V$. Hence $aU''V' \subseteq UV$. Obviously similar arguments hold for B_2. Let T_1 be the topology generated on G using $\{g \cdot U_\alpha V_\beta \mid g \in G \text{ and } \alpha \in \Delta \text{ and } \beta \in \Gamma\}$ as a basis and T_2 the topology generated on G using $\{gV_\beta U_\alpha \mid g \in G \text{ and } \alpha \in \Delta \text{ and } \beta \in \Gamma\}$ as a basis.

Finally, we note that if $g \in G$, then for any $U \in \{U_\alpha\}_{\alpha \in \Delta}$ and for any $V \in \{V_\beta\}_{\beta \in \Gamma}$ we can find a $U' \in \{U_\alpha\}_{\alpha \in \Delta}$ and a $V' \in \{V_\beta\}_{\beta \in \Gamma}$ such that $gU'g^{-1} \subseteq U$ and $gV'g^{-1} \subseteq V$. Hence $gU'V'g^{-1} \subseteq UV$ and thus B_1 nearly forms a fundamental system of neighborhoods of the identity. Clearly, the same is true of'B_2.

THEOREM 4. *The following statements are equivalent:*

(a) $T_1 = T_2$.
(b) T_1 *is a group topology.*
(c) $T_1 = t_1 \wedge t_2$.

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PROOF: (a) implies (b): For any $UV \in B_1$ we can find a $U' \in \{U_\alpha\}_{\alpha \in \Delta}$ and a $V' \in \{V_\beta\}_{\beta \in \Gamma}$ such that $(U')^2(V')^2 \subseteq UV$. Since $U'V'$ is open in T_2, we can find a $V''U'' \in B_2$ such that $V''U'' \subseteq U'V'$. Thus $U'V''U''V' \in UV$. We can find a $U''' \subseteq (U' \cap U'')$ and a $V''' \subseteq (V' \cap V'')$ and hence $(U''' \cdot V''')^2 \subseteq UV$. So whenever $UV \in B_1$ we can find a $U_1V_1 \in B_1$ with $(U_1V_1)^2 \subseteq UV$. Now since U_1V_1 is open in T_2, we can find a $V_2U_2 \subseteq U_1V_1$ and thus $U_1V_1V_2U_2 \subseteq (U_1V_1)^2$. As before, we can find a $U_3 \subseteq U_1 \cap U_2$ and a $V_3 \subseteq V_1 \cap V_2$. So $U_3V_3^2U_3 \subseteq (U_1V_1)^2$. Since we may assume without loss of generality that $\{U_\alpha\}_{\alpha \in \Delta}$ and $\{V_\beta\}_{\beta \in \Gamma}$ are collections of symmetric sets we have $(U_3V_3)(U_3V_3)^{-1} \subseteq UV$ and that T_1 is a group topology.

(b) implies (c): Clearly $T_1 \subseteq t_1 \wedge t_2$. Let X be a neighborhood of e in $t_1 \wedge t_2$. We can find a $Y \in t_1 \wedge t_2$ such that $Y^2 \subseteq X$. But $Y_1^2 \in T_1$ and hence $T_1 = t_1 \wedge t_2$.

(c) implies (a): If $T_1 = t_1 \wedge t_2$, then $(UV)^{-1} = VU$ is open in T_1 since inversion is continuous. So $T_2 \subseteq T_1$. Let $UV \in T_1$. Then $VU = (UV)^{-1} \in T_1 \cap T_2$. Since VU is open in T_1 we can find a $U_1V_1 \subseteq VU$. Therefore, $(U_1V_1)^{-1} = V_1U_1 \subseteq UV = (VU)^{-1}$. Hence $T_1 = T_2$.

There are a number of topologies that can be defined on G which must satisfy $T_1 = T_2$. For example, any topology t_1 of the type described by Sharma in [4] will satisfy $T_1 = T_2$ no matter what topology we choose for t_2. In the same fashion if t_1 is any topology generated using normal subgroups as a fundamental system, then $T_1 = T_2$ no matter what topology we choose for t_2.

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