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LIPSCHITZ STABILITY OF IMPULSIVE FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract

An initial value problem is considered for impulsive functional-differential equations. The impulses occur at fixed moments of time. Sufficient conditions are found for Lipschitz stability of the zero solution of these equations. An application in impulsive population dynamics is also discussed.

1. Introduction

Functional-differential equations are adequate mathematical models of various real processes and phenomena and hence the fundamental and qualitative theory of these equations has been the subject of intensive study in recent years ([4, 5, 7–9, 11, 12]).

Impulsive functional-differential equations are natural generalizations of functional-differential equations. In spite of the great possibilities for application, the mathematical theory of these equations is developing rather slowly due to technical and theoretical difficulties which are connected with the specific features of these equations ([1, 2]).

In the present paper we define the notion of Lipschitz stability of solutions of a system of impulsive functional-differential equations. Let us note that this notion has been introduced by Dannan and Elaydi [6] for ordinary differential equations without impulses.

By virtue of a comparison equation and differential inequalities for piecewise continuous functions, sufficient conditions are found for Lipschitz stability of the zero solution of a system of nonlinear impulsive functional-differential equations.

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2. Preliminary notes and definitions

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the norm $|\cdot|$, $\mathbb{R}_+ = [0, \infty)$, $t_0 \in \mathbb{R}$ and $\tau > 0$.

We consider the system of impulsive functional-differential equations

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x_t), && t > t_0,\ t \neq t_k, \\ \Delta x(t_k) &= x(t_k + 0) - x(t_k - 0) = I_k(x(t_k - 0)), && t_k > t_0,\ k = 1, 2, \dots, \end{aligned} \tag{1}$$

where $f : (t_0, \infty) \times \mathbb{R}^n \times D \to \mathbb{R}^n$; $D = \{\Phi : [-\tau, 0] \to \mathbb{R}^n,\ \Phi(t)$ is continuous everywhere except a finite number of points $\tilde{t}$ at which $\Phi(\tilde{t}-0)$ and $\Phi(\tilde{t}+0)$ exist and $\Phi(\tilde{t} - 0) = \Phi(\tilde{t})\}$; $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \dots$; $t_0 < t_1 < t_2 < \cdots$; $\lim_{k\to\infty} t_k = \infty$ and for $t > t_0$, $x_t \in D$ is defined by $x_t(s) = x(t+s)$, $-\tau \le s \le 0$.

Let $\varphi_0 \in D$. We denote by $x(t) = x(t; t_0, \varphi_0)$ the solution of the system (1), which satisfies the initial conditions

$$\begin{aligned} x(t; t_0, \varphi_0) &= \varphi_0(t - t_0), \quad t_0 - \tau \le t \le t_0, \\ x(t_0 + 0; t_0, \varphi_0) &= \varphi_0(0). \end{aligned} \tag{2}$$

The solution $x(t; t_0, \varphi_0) = x(t)$ of the initial value problem (1) and (2) is characterized by the following properties.

(a) For $t_0 - \tau \le t \le t_0$ the solution $x(t)$ satisfies the initial conditions (2).

(b) For $t_0 < t \le t_1$ the solution $x(t) = x(t; t_0, \varphi_0)$ of the initial value problem (1) and (2) coincides with the solution of the problem

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x_t), \quad t > t_0, \\ x_{t_0} &= \varphi_0. \end{aligned}$$

At the moment $t = t_1$ the mapping point $(t, x(t; t_0, \varphi_0))$ moves 'instantly' from the position $(t_1, x(t_1; t_0, \varphi_0))$ into the position $(t_1, x(t_1; t_0, \varphi_0) + I_1(x(t_1; t_0, \varphi_0)))$.

(c) For $t_1 < t \le t_2$ the solution of (1) and (2) coincides with the solution of the problem

$$\begin{aligned} \dot{y}(t) &= f(t, y(t), y_t), \quad t > t_1, \\ y_{t_1} &= \varphi_1, \quad \varphi_1 \in D, \end{aligned}$$

where

$$\varphi_1(t - t_1) = \begin{cases} \varphi_0(t - t_1), & t \in [t_0 - \tau, t_0] \cap [t_1 - \tau, t_1], \\ x(t; t_0, \varphi_0), & t \in (t_0, t_1), \\ x(t; t_0, \varphi_0) + I_1(x(t; t_0, \varphi_0)), & t = t_1. \end{cases}$$

At the moment $t = t_2$ the mapping point $(t, x(t; t_0, \varphi_0))$ moves 'instantly' and so on.

The solution $x(t) = x(t; t_0, \varphi_0)$ of the initial value problem (1) and (2) is a piecewise continuous function for $t > t_0$ with points of discontinuity of the first kind t_k, $k = 1, 2, \dots$, where it is continuous from the left.

We also introduce the notation:

$$I = [t_0 - \tau, \infty), \quad I_0 = [t_0, \infty),$$

$$G_k = \{(t, x) \in I_0 \times \mathbb{R}^n : t_{k-1} < t < t_k\}, \quad k = 1, 2, \dots, \quad \text{and} \quad G = \bigcup_{k=1}^{\infty} G_k.$$

Let $J \subset \mathbb{R}$ be an interval. We define the following classes of functions:

$PC[J, \mathbb{R}^n] = \{\sigma : J \to \mathbb{R}^n : \sigma(t)$ is continuous everywhere except some points t_k at which $\sigma(t_k - 0)$ and $\sigma(t_k + 0)$ exist and $\sigma(t_k - 0) = \sigma(t_k)\}$;

$PC^1[J, \mathbb{R}^n] = \{\sigma \in PC[J, \mathbb{R}^n] : \sigma(t)$ is continuously differentiable everywhere except some points t_k at which $\dot\sigma(t_k - 0)$ and $\dot\sigma(t_k + 0)$ exist and $\dot\sigma(t_k - 0) = \dot\sigma(t_k)\}$;

$\mathscr{K} = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : a(u)$ is a strictly increasing function and $a(0) = 0\}$;

$\mathscr{V}_0 = \{V : I_0 \times \mathbb{R}^n \to \mathbb{R}_+ : V \in C[G, \mathbb{R}_+]$, V satisfies locally a Lipschitz condition with respect to $x \in \mathbb{R}^n$ on each of the sets G_k, $V(t_k - 0, x) = V(t_k, x)$ and $V(t_k + 0, x) = \lim_{\substack{t > t_k \\ t \to t_k}} V(t, x)$ exists$\}$;

$\Omega_1 = \{x \in PC[I_0, \mathbb{R}^n] : V(s, x(s)) \le V(t, x(t)), t - \tau \le s \le t, t \ge t_0, V \in \mathscr{V}_0\}$.

Let $V \in \mathscr{V}_0$, $x \in PC[I_0, \mathbb{R}^n]$ and $t \ne t_k$, $k = 1, 2, \dots$.

We introduce the function

$$D_- V(t, x(t)) = \liminf_{h \to 0^-} \frac{1}{h}[V(t + h, x(t) + hf(t, x(t), x_t)) - V(t, x(t))].$$

We consider the system (1) together with

$$\begin{aligned} \dot u &= g(t, u), & & t \ne t_k,\ t > t_0, \\ u(t_k + 0) &= B_k(u(t_k)), & & t_k > t_0,\ k = 1, 2, \dots, \end{aligned} \tag{3}$$

where $g : I_0 \times \mathbb{R}_+ \to \mathbb{R}_+$, $B_k : \mathbb{R}_+ \to \mathbb{R}_+$, $k = 1, 2, \dots$.

We denote by $u^+(t; t_0, u_0)$ the maximal solution of (3), for which $u^+(t_0 + 0; t_0, u_0) = u_0 \in \mathbb{R}_+$.

Let $\rho > 0$. In what follows we will use the notation:

$$B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\};$$

$$\|\varphi_0\| = \sup_{s \in [-\tau, 0]} |\varphi_0(s)|, \quad \varphi_0 \in D; \quad \text{and}$$

$$\|A\|_1 = \sup_{|x|=1} |Ax| \text{ is the norm of the } n \times n\text{-matrix } A.$$

Now we shall define two kinds of Lipschitz stability for the zero solution of the system (1).

DEFINITION 1. The zero solution of the system (1) is said to be:

(a) *uniformly Lipschitz stable* if

$$(\exists M > 0)\ (\exists \delta > 0)\ (\forall \varphi_0 \in D\colon \|\varphi_0\| < \delta)(\forall t \in I_0)\colon\ |x(t; t_0, \varphi_0)| \le M\|\varphi_0\|;$$

(b) *globally uniformly Lipschitz stable* if

$$(\exists M > 0)\ (\forall \varphi_0 \in D)\ (\forall t \in I_0)\colon |x(t; t_0, \varphi_0)| \le M\|\varphi_0\|.$$

When studying Lipschitz stability of the zero solution of the system (1) we will use the following definitions of Lipschitz stability of the zero solution of (3) ([10]).

DEFINITION 2. The zero solution of (3) is said to be:

(a) *uniformly Lipschitz stable* if

$$(\exists M > 0)\ (\exists \delta > 0)\ (\forall u_0 \in \mathbb{R}_+\colon u_0 < \delta)(\forall t > t_0)\colon\ u^+(t; t_0, u_0) \le M u_0;$$

(b) *globally uniformly Lipschitz stable* if

$$(\exists M > 0)\ (\forall u_0 \in \mathbb{R}_+)\ (\forall t > t_0)\colon u^+(t; t_0, u_0) \le M u_0.$$

We introduce the following conditions.

H1. $t_0 < t_1 < t_2 < \cdots$.
H2. $\lim_{k\to\infty} t_k = \infty$.
H3. $f \in PC[I_0 \times \mathbb{R}^n \times D, \mathbb{R}^n]$.
H4. $f(t, 0, 0) = 0$, $t \in I_0$.
H5. $I_k \in C[\mathbb{R}^n, \mathbb{R}^n]$, $k = 1, 2, \ldots$.
H6. $I_k(0) = 0$, $k = 1, 2, \ldots$.
H7. $g \in PC[I_0 \times \mathbb{R}_+, \mathbb{R}]$.
H8. $g(t, 0) = 0$, $t \in I_0$.
H9. $B_k \in \mathscr{K}$ and $B_k\colon [0, \rho_0) \to [0, \rho)$, $k = 1, 2, \ldots$, $\rho_0 = \text{const} > 0$, $\rho = \text{const} > 0$.
H10. For $x \in B_\rho$ and for all $k = 1, 2, \ldots$ the inequalities $|x + I_k(x)| \le B_k(|x|)$ hold.
H11. The zero solution of (3) is uniformly Lipschitz stable (globally uniformly Lipschitz stable).

When proving the main results we shall use the following lemmas:

LEMMA 1. *Let the following conditions hold.*

1. *Conditions* H1–H10 *are met.*

2. *The solution* $x(t) = x(t; t_0, \varphi_0)$ *of the initial value problem* (1) *and* (2) *is such that* $x \in PC[I, B_\rho] \cap PC^1[I_0, B_\rho]$, $\rho = const > 0$.

3. *The function* $V \in \mathscr{V}_0$, $V: I_0 \times B_\rho \to \mathbb{R}_+$ *is such that*

$$\begin{aligned} &V(t_0 + 0, \varphi_0(0)) \le u_0, \\ &D_- V(t, x(t)) \le g(t, V(t, x(t))), \quad t \in I_0,\ t \ne t_k,\ x \in \Omega_1, \\ &V(t_k + 0, x(t_k) + I_k(x(t_k))) \le B_k(V(t_k, x(t_k))), \quad k = 1, 2, \dots. \end{aligned} \tag{4}$$

Then

$$V(t, x(t; t_0, \varphi_0)) \le u^+(t; t_0, u_0), \quad t \in I_0. \tag{5}$$

PROOF. For $t \in I_0$ the maximal solution $u^+(t; t_0, u_0)$ of (3) is defined by the equality

$$u^+(t; t_0, u_0) = \begin{cases} r_0(t; t_0, u_0^+), & t_0 < t \le t_1, \\ r_1(t; t_1, u_1^+), & t_1 < t \le t_2, \\ \dots\dots\dots\dots\dots\dots, \\ r_k(t; t_k, u_k^+), & t_k < t \le t_{k+1}, \\ \dots\dots\dots\dots\dots\dots, \end{cases}$$

where $r_k(t; t_k, u_k^+)$ is the maximal solution of the equation without impulses, $\dot{u} = g(t, u)$ in the interval $(t_k, t_{k+1}]$ for which $u_k^+ = B_k(r_{k-1}(t_k; t_{k-1}, u_{k-1}^+))$, $k = 1, 2, \dots$ and $u_0^+ = u_0$.

Let $t \in (t_0, t_1] \cap I_0$. Then the corresponding comparison lemma for the continuous case ([11]) implies

$$V(t, x(t; t_0, \varphi_0)) \le u^+(t; t_0, u_0),$$

that is, the inequality (5) is fulfilled for $t \in (t_0, t_1] \cap I_0$.

Suppose that (5) holds true for $t \in (t_{k-1}, t_k] \cap I_0$, $k > 1$. Then, using (4) and the fact that the function B_k is increasing, we obtain

$$\begin{aligned} V(t_k + 0, x(t_k + 0; t_0, \varphi_0)) &\le B_k(V(t_k, x(t_k; t_0, \varphi_0)) \\ &\le B_k(u^+(t_k; t_0, u_0)) = B_k(r_{k-1}(t_k; t_{k-1}, u_{k-1}^+)) = u_k^+. \end{aligned}$$

We apply again the corresponding comparison lemma for functional-differential equations without impulses [11] in the interval $(t_k, t_{k+1}] \cap I_0$ and obtain

$$V(t, x(t; t_0, \varphi_0)) \le r_k(t; t_k, u_k^+) = u^+(t; t_0, u_0),$$

that is, the inequality (5) is true for $t \in (t_k, t_{k+1}] \cap I_0$.

The proof is completed by induction.

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LEMMA 2 ([3]). *Let the following conditions hold*:

1. *The functions* $u,\ k \in PC[I_0, \mathbb{R}_+]$.
2. $c_0 = const > 0$, $\beta_k = const \geq 0$, $k = 1, 2, \ldots$.
3. *The function* $p \in C[\mathbb{R}_+, \mathbb{R}_+]$, *p is nondecreasing in* $\mathbb{R}_+$ *and positive in* $(0, \infty)$.
4. *For* $t \in I_0$ *the inequality*

$$u(t) \leq c_0 + \int_{t_0}^{t} k(s)p(u(s))\,ds + \sum_{t_0<t_k<t} \beta_k u(t_k)$$

holds true.

5. $B_k = \int_{c_k}^{u}(1/p(s))\,ds$, $k = 1, 2, \ldots$, *where* $c_k = (1+\beta_k)B_{k-1}^{-1}\left(\int_{t_{k-1}}^{t_k} k(s)\,ds\right)$.
6. $B_0 = \int_{u_0}^{u}(1/p(s))\,ds$, $u \geq u_0 > 0$.

Then

$$u(t) \leq B_k^{-1}\left(\int_{t_k}^{t} k(s)\,ds\right), \quad t_{k-1} < t \leq t_k, \quad k = 1, 2, \ldots.$$

3. Main results

THEOREM 1. *Let the following conditions hold*:

1. *Conditions* H1–H11 *are valid.*
2. *For* $t \in I_0$, $x \in \Omega_1 \cap B_\rho$ *and for sufficiently small* $h > 0$ *the inequality*

$$|x(t) + hf(t, x(t), x_t)| \leq |x(t)| + hg(t, |x(t)|) + \varepsilon(h)$$

holds true, where $\varepsilon(h)/h \to 0$ *as* $h \to 0^-$, $t \neq t_k$, $k = 1, 2, \ldots$.

Then the zero solution of the system (1) *is uniformly Lipschitz stable* (*globally uniformly Lipschitz stable*).

PROOF. Let $\rho^* = \min(\rho, \rho_0)$. It follows from condition H11 that there exist constants $M > 0$ and $\delta > 0$ ($M\delta < \rho^*$) such that for $0 \leq u_0 < \delta$ and $t > t_0$ we have

$$u^+(t; t_0, u_0) \leq Mu_0. \tag{6}$$

We will prove that $|x(t; t_0, \varphi_0)| \leq M\|\varphi_0\|$ for $\|\varphi_0\| < \delta$ and $t > t_0$.

Let us suppose the opposite. Then there exists a solution $x(t) = x(t; t_0, \varphi_0)$ of the system (1), $\varphi_0 \in D$: $\|\varphi_0\| < \delta$ and $t^* \in (t_k, t_{k+1}]$ for some positive integer k, such that

$$|x(t^*)| > M\|\varphi_0\|, \qquad |x(t)| \leq M\|\varphi_0\|, \quad t_0 < t \leq t_k.$$

It follows from condition H10 that

$$|x(t_k+0)| = |x(t_k) + I_k(x(t_k))| \le B_k(|x(t_k)|)$$
$$\le B_k(M\|\varphi_0\|) \le B_k(M\delta) \le G_k(\rho^*) \le \rho.$$

The last inequality asserts that there exists t^0, $t_k < t^0 \le t^*$, such that

$$M\|\varphi_0\| < |x(t^0)| < \rho \quad \text{and} \quad |x(t)| < \rho, \quad t < t \le t^0. \tag{7}$$

Let us define $V(t, x(t)) = |x(t)|$ and $u_0 = \|\varphi_0\|$. Since condition 2 of Theorem 1 is fulfilled, then for $t \in (t_0, t^0]$, $t \ne t_j$, $j = 1, 2, \dots, k$, the inequalities

$$\begin{aligned} D_- V(t, x(t)) &= \liminf_{h\to 0^-} \frac{1}{h}\big[|x(t+h)| - |x(t)|\big] \\ &\le \lim_{h\to 0^-} \frac{1}{h}\big[|x(t+h)| + hg(t, |x(t)|) + \varepsilon(h) - |x(t) + hf(t, x(t), x_t)|\big] \\ &\le g(t, |x(t)|) + \lim_{h\to 0^-} \frac{\varepsilon(h)}{h} + \lim_{h\to 0^-} \left|\frac{1}{h}\big[x(t+h) - x(t)\big] - f(t, x(t), x_t)\right| \\ &= g(t, |x(t)|) = g(t, V(t, x(t))) \end{aligned}$$

hold true.

The inequalities

$$V(t_j+0, x(t_j+0)) = |x(t_j+0)| = |x(t_j) + I_j(x(t_j))|$$
$$\le B_j(|x(t_j)|) = B_j(V(t_j, x(t_j)))$$

follow from condition H10 for $j = 1, 2, \dots, k$.

Since

$$V(t_0+0, \varphi_0(0)) = |\varphi_0(0)| \le \|\varphi_0\| = u_0,$$

it follows from Lemma 1 that

$$|x(t)| = V(t, x(t)) \le u^+(t; t_0, u_0), \qquad t_0 < t \le t^0. \tag{8}$$

Hence (6)–(8) imply the inequalities

$$M\|\varphi_0\| < |x(t^0)| = V(t^0, x(t^0)) \le u^+(t^0; t_0, u_0) \le Mu_0 = M\|\varphi_0\|.$$

The contradiction obtained shows that

$$x(t; t_0, \varphi_0)| \le M\|\varphi_0\| \text{ for } \|\varphi_0\| < \delta \text{ and } t > t_0.$$

THEOREM 2. *Let the following conditions hold*:

1. *Conditions* H1–H11 *are met.*
2. *For* $t \in I_0$ *and* $x \in \Omega_1 \cap B_\rho$ *the inequality*

$$\big[x(t), f(t, x(t), x_t)\big]_- \le g(t, |x(t)|), \quad t \neq t_k,\ k = 1, 2, \dots,$$

holds true, where

$$[x, y]_- = \liminf_{h \to 0^-} \frac{1}{h}\big[|x + hy| - |x|\big], \quad x,\ y \in \mathbb{R}^n.$$

Then the zero solution of the system (1) *is uniformly Lipschitz stable* (*globally uniformly Lipschitz stable*).

The proof of Theorem 2 is analogous to the proof of Theorem 1.

Let us introduce the following conditions:

H12. The function $p \in C[\mathbb{R}_+, \mathbb{R}_+]$, p is nondecreasing in $\mathbb{R}_+$, positive in $(0, \infty)$ and it is submultiplicative, that is, $p(\lambda u) \le p(\lambda)p(u)$ for $\lambda > 0$, $u > 0$.
H13. $p(\lambda u) \ge \mu(\lambda)p(u)$ for $\lambda > 0$, $u > 0$ where $\mu(\lambda) > 0$ for $\lambda > 0$.
H14. For $(t, x, x_t) \in I_0 \times \mathbb{R}^n \times D$, $|f(t, x, x_t)| \le m(t)p(|x|)$, where $m \in C[I_0, \mathbb{R}_+]$.
H15. For $x \in \mathbb{R}^n$ and $k = 1, 2, \dots$, $|I_k(x)| \le \beta_k|x|$, $\beta_k = \text{const} > 0$.

THEOREM 3. *Let the following conditions hold*:

1. *Conditions* H1–H6 *and* H12–H15 *are met.*
2. $B_k = \displaystyle\int_{c_k}^{u} \frac{ds}{p(s)}$, $\quad c_k = (1 + \beta_k)B_{k-1}^{-1}\left(\displaystyle\int_{t_{k-1}}^{t_k} m(s)\,ds\right)$, $\quad k = 1, 2, \dots$ *and*
$$B_0 = \int_c^u \frac{ds}{p(s)}, \quad u \ge c > 0.$$
3. $B_k(\infty) = \infty$, $k = 0, 1, 2, \dots$.
4. *For every* $k = 0, 1, 2, \dots$, $t \in (t_k, t_{k+1}]$ *and* $\varphi_0 \in D$,

$$B_k^{-1}\left(\frac{p(\|\varphi_0\|)}{\|\varphi_0\|}\int_{t_k}^{t} m(s)ds\right) \le M, \qquad 0 < M = const.$$

Then the zero solution of the system (1) *is globally uniformly Lipschitz stable.*

PROOF. For $t_k < t \le t_{k+1}$, $k = 0, 1, 2, \dots$, the function $x(t) = x(t; t_0, \varphi_0)$ satisfies the integral equation

$$x(t) = x(t_k) + I_k(x(t_k)) + \int_{t_k}^{t} f(s, x(s), x_s)ds.$$

We obtain by induction

$$x(t) = x(t_0+0) + \sum_{t_0<t_k<t} I_k(x(t_k)) + \int_{t_0}^{t} f(s, x(s), x_s)ds, \quad t > t_0.$$

It then follows from conditions H12–H15 that

$$\begin{aligned} |x(t;t_0,\varphi_0)| &\le |\varphi_0(0)| + \sum_{t_0<t_k<t} |I_k(x(t_k))| + \int_{t_0}^{t} |f(s, x(s), x_s)|\, ds \\ &\le \|\varphi_0\| + \sum_{t_0<t_k<t} \beta_k |x(t_k)| + \int_{t_0}^{t} m(s)p(|x(s)|)\, ds, \end{aligned}$$

whence we obtain the estimates, for $t > t_0$,

$$\begin{aligned} \frac{|x(t;t_0,\varphi_0)|}{\|\varphi_0\|} &\le 1 + \sum_{t_0<t_k<t} \beta_k \frac{|x(t_k;t_0,\varphi_0)|}{\|\varphi_0\|} + \int_{t_0}^{t} \frac{m(s)}{\|\varphi_0\|} p\left(\|\varphi_0\| \frac{|x(s;t_0,\varphi_0)|}{\|\varphi_0\|}\right) ds \\ &\le 1 + \int_{t_0}^{t} \frac{p(\|\varphi_0\|)}{\|\varphi_0\|} m(s) p\left(\frac{|x(s;t_0,\varphi_0)|}{\|\varphi_0\|}\right) ds + \sum_{t_0<t_k<t} \beta_k \frac{|x(t_k;t_0,\varphi_0)|}{\|\varphi_0\|}. \end{aligned}$$

We apply Lemma 2 to the last inequality and obtain

$$|x(t;t_0,\varphi_0)| \le \|\varphi_0\| B_k^{-1}\left(\frac{p(\|\varphi_0\|)}{\|\varphi_0\|} \int_{t_0}^{t} m(s)\, ds\right),$$

$t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \dots$.

The last inequality and condition 4 of Theorem 3 imply $|x(t;t_0,\varphi_0)| \le M\|\varphi_0\|$ for all $\varphi_0 \in D$ and $t > t_0$.

4. Applications in impulsive population dynamics

We consider the impulsive population model

$$\begin{aligned} \dot N(t) &= N(t)\big[a + bN(t-\tau) - cN^2(t-\tau)\big], \quad t \ne t_k, \\ \Delta N(t_k) &= I_k(N(t_k)), \quad t_k > 0,\ k = 1, 2, \dots, \end{aligned} \tag{9}$$

where the velocity $\dot N(t)$ of increase in population is a quadratic form of the density $N(t)$ of the same population; $\tau > 0$ characterizes the maturity of the population; a, $c \in (0, \infty)$, $b \in \mathbb{R}$ are constants, different for the concrete population; I_k, $k = 1, 2, \dots$, characterize the value of the increase or decrease of the population under the action of external perturbations (for example, human action) at the moments $0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$.

THEOREM 4. *Let the following conditions hold*:

1. $[N(t), N(t)(a + bN(t-\tau) - cN^2(t-\tau))]_- \leq p(t)F(N(t))$, *for* $t \geq 0$, $|N(t)| < \rho$, $|N(t-\tau)| < \rho$, $F \in \mathscr{K}$, $p \in C[I_0, \mathbb{R}_+]$.

2. $|N + I_k(N)| \leq B_k(|N|)$, $|N| < \rho$, $k = 1, 2, \ldots$, *where* $B_k \in \mathscr{K}$ *and* $B_k\colon [0, \rho_0) \to [0, \rho)$, $k = 1, 2, \ldots$.

3. *For each* $h \in (0, \rho_0)$,

$$\int_{t_k}^{t_{k+1}} p(s)\,ds + \int_h^{B_k(h)} \frac{ds}{F(s)} \leq 0, \quad k = 1, 2, \ldots .$$

Then the zero solution of the system (9) *is uniformly Lipschitz stable.*

PROOF. It follows from the conditions of Theorem 4 that the zero solution of the system

$$\dot{N}(t) = p(t)F(N(t)), \quad t \neq t_k, \quad t > 0,$$
$$N(t_k + 0) = B_k(N(t_k)), \quad k = 1, 2, \ldots ,$$

is uniformly Lipschitz stable ([10]).

Since all conditions of Theorem 2 are fulfilled, then the zero solution of the system (9) is uniformly Lipschitz stable.

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