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ISOLATED SCATTERING NUMBER OF SPLIT GRAPHS AND GRAPH PRODUCTS

FENGWEI LI✉[1], QINGFANG YE[1] and XIAOYAN ZHANG[2]

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Abstract

Computer or communication networks are so designed that they do not easily get disrupted under external attack. Moreover, they are easily reconstructed when they do get disrupted. These desirable properties of networks can be measured by various parameters, such as connectivity, toughness and scattering number. Among these parameters, the isolated scattering number is a comparatively better parameter to measure the vulnerability of networks. In this paper we first prove that for split graphs, this number can be computed in polynomial time. Then we determine the isolated scattering number of the Cartesian product and the Kronecker product of special graphs and special permutation graphs.

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1. Introduction

A communication network is composed of processors and communication links. Cuts, node interruptions, software errors or hardware failures, and transmission failures at various points can interrupt the service for long periods of time; this causes a loss of effectiveness. We speak of the *vulnerability* of communication networks: the vulnerability of a communication network measures the resistance of the network to the disruption of its operation after the failure of certain processors or communication links. Network designers give importance to the vulnerability of a network; they aim to design networks with less vulnerability or more reliability. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network.

[1]Department of Mathematics, Shaoxing University, Shaoxing, Zhejiang 312000, PR China;
e-mail: fengwei.li@hotmail.com, fqy-y@163.com.
[2]School of Mathematical Sciences, Nanjing Normal University, Nanjing, Jiangsu 210023, PR China;
e-mail: zhangxiaoyan@njnu.edu.cn.

The communication network often has as considerable an impact on a network's performance as the processors themselves. Performance measures for communication networks are essential to guide the designers in choosing an appropriate topology. In order to measure the performance, we are interested in the following performance metrics (there may be others): (1) the number of elements that are not functioning; (2) the number of remaining connected sub-networks; and (3) the size of a largest remaining group within which mutual communication can still occur.

The communication network can be represented as an undirected and unweighted graph, where a processor (station) is represented as a vertex, and a communication link between processors as an edge between corresponding vertices. If we use a graph to model a network, based on the above three quantities, a number of graph parameters, such as connectivity, toughness [7], scattering number [12], integrity [3], tenacity [8], rupture degree [16], isolated rupture degree [14, 15] and their edge analogues, have been proposed for measuring the vulnerability of networks.

Throughout this paper, we use Bondy and Murty's [4] terminology and notation, and only consider finite simple undirected graphs. The vertex set and edge set of a graph G are denoted by V and E, respectively. For $S \subseteq V(G)$, let $\omega(G-S)$ and $i(G-S)$ denote, respectively, the number of components and the number of components which are isolated vertices in $G-S$. Let u be a vertex in G; the open neighbourhood of u is defined as $N(u) = \{v \in V(G) \mid (u, v) \in E(G)\}$. Analogously, we define the open neighbourhood $N(S) = \bigcup_{u \in S} N(u)$ for any $S \subseteq V(G)$. Let $\kappa(G)$ denote the connectivity of graph G. A vertex set $S \subseteq V(G)$ is a *cut-set* of G, if either $G-S$ is disconnected or $G-S$ has only one vertex. Let $C(G)$ denote the set of all cut-sets of G.

One of the vulnerability parameters noted above is the scattering number, which takes into account the quantities (1) and (2). Introduced by Jung [12] in 1978, the *scattering number* of an incomplete connected graph G is defined as

$$s(G) = \max\{\omega(G-S) - |S| \mid S \in C(G), \omega(G-S) > 1\}.$$

Motivated by Jung's scattering number, by replacing $\omega(G-S)$ with $i(G-S)$ in the definition of $s(G)$, Wang et al. [20] introduced the *isolated scattering number*, $\mathrm{isc}(G)$, as a new parameter to measure the vulnerability of networks.

DEFINITION 1.1 [20]. The *isolated scattering number* of an incomplete connected graph G is defined as

$$\mathrm{isc}(G) = \max\{i(G-S) - |S| \mid S \in C(G)\},$$

where the maximum is taken over all the cut-sets of G; in particular, we define $\mathrm{isc}(K_n) = 2 - n$.

DEFINITION 1.2. A cut-set S of G is called an *isc-set* of G, if $\mathrm{isc}(G) = i(G-S) - |S|$.

This parameter is of particular interest, because it is considered to be a reasonable measure for the vulnerability of graphs. The scattering number and isolated scattering number differ in how they represent the vulnerability of networks. This can be shown

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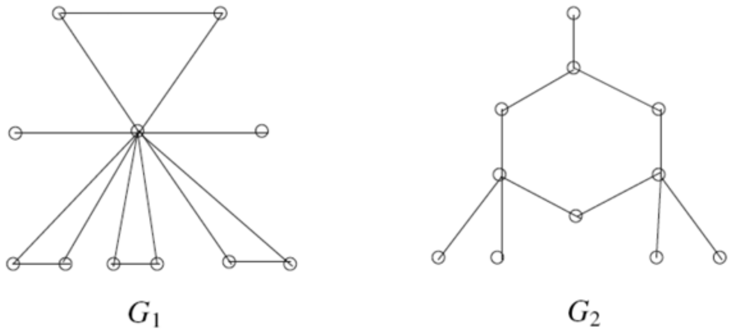

FIGURE 1. Two graphs with the same scattering number but with different isolated scattering numbers.

as follows. Consider the graphs G_1 and G_2 in Figure 1. It is not difficult to check that $s(G_1) = s(G_2) = 5$, but $\mathrm{isc}(G_1) = 1 \neq 5 = \mathrm{isc}(G_2)$.

Hence, the isolated scattering number is a reasonable parameter for distinguishing the vulnerability of these graphs. Note that the smaller the isolated scattering number of a network, the more stable it is considered to be. Wang et al. [20] gave formulas for the isolated scattering number of joint graphs and some bounds of the isolated scattering number, and they also developed a recursive algorithm for computing the isolated rupture degree of trees. In this paper, first we prove that for split graphs, this number can be computed in polynomial time. Then we give the exact values of the isolated scattering numbers of the Cartesian product and the Kronecker product of special graphs. Finally, we determine the isolated scattering number of special permutation graphs.

2. Isolated scattering number for split graphs

In this section we fix our attention on the isolated scattering number of split graphs.

DEFINITION 2.1 [9]. A graph $G = (V, E)$ is called a *split graph* if its vertex set V can be partitioned into a clique C and an independent set I.

Usually, the split graph G is denoted by $G = (C, I, E)$. To avoid trivialities, assume that G is connected, $C \neq \emptyset$, and $I \neq \emptyset$. If $N(I) \neq C$, then by choosing a vertex $v \in C - N(I)$, and replacing C by $C - \{v\}$ and I by $I \cup \{v\}$, G can be expressed as $G = (C - \{v\}, I \cup \{v\}, E)$, in which $N(I \cup \{v\}) = C - \{v\}$. Hence, in the following, we always assume that $N(I) = C$ for any split graph $G = (C, I, E)$.

By rewriting the problem of computing the toughness of a split graph as minimization of a submodular function, Woeginger [21] proved that the problem can be solved in polynomial time. Zhang et al. [23] showed that the problem of computing the scattering number of a split graph can be solved in polynomial time. Here, we prove that the isolated scattering number of a split graph can be computed in polynomial time by modifying their method [23].

Let I be a finite set. A function $f : 2^I \to \mathbb{R}$ is called *submodular* [11] on 2^I if

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

holds for all $X, Y \subseteq I$. The problem of finding the minimum of $f(U)$ has been studied by several authors. Schrijver [19] provided a strongly polynomial-time algorithm to solve this problem.

A family $\mathfrak{R}$ of subsets of I is called a *crossing family* [11] if

$$X, Y \in \mathfrak{R}, X \cap Y \neq \emptyset, X \cup Y \neq I \Longrightarrow X \cap Y \in \mathfrak{R}, X \cup Y \in \mathfrak{R}.$$

It was shown that minimizing a submodular function on a crossing family can be easily reduced to $|I|(|I| - 1)$ problems of minimizing submodular functions on 2^I [11]. At the same time, it is easy to check that all the nonempty proper subsets of I form a crossing family. So if f is a submodular function on $2^I - \{I, \emptyset\}$, then $\min\{f(U) \mid U \neq \emptyset, U \subset I\}$ can also be computed in polynomial time.

Lemma 2.2 [20]. *Let G be an incomplete connected graph of order n. Then $3 - n \leq \mathrm{isc}(G) \leq n - 2$.*

Theorem 2.3. *The isolated scattering number of an incomplete connected split graph of order n can be computed in polynomial time.*

Proof. Let $G = (C, I, E)$ be an incomplete connected split graph of order n. For an integer s with $3 - n \leq s \leq n - 2$, our first goal is to decide whether there exists a cut-set S^* of G for which $i(G - S^*) - |S^*| > s$ holds. If such an S^* exists, we may assume without loss of generality that $S^* \subseteq C$; otherwise, replace S^* by $S^* \cap C$. This dose not increase $|S^*|$ and cannot decrease $i(G - S^*)$. We present two cases.

Case 1. Let $S^* = C$. Then $i(G - S^*) = |I|$ holds, and this case is trivial to check.

Case 2. If $S^* \subset C$, then $i(G - S^*)$ equals the number of vertices $v \in I$ with $N(v) \subseteq S^*$.

We claim that if $|I| - |C| \leq s$, then there exits a cut-set $S^* \subset C$ of G satisfying $i(G - S^*) - |S^*| > s$ if and only if there exists a $U^* \subset I$ satisfying $|N(U^*)| - |U^*| < -s$. Suppose that there exists a cut-set $S^* \subset C$ satisfying $i(G - S^*) - |S^*| > s$. Set $U^* = \{u \mid u \in I, N(u) \subseteq S^*\}$. Then $|N(U^*)| \leq |S^*|$ and $i(G - S^*) = |U^*|$ hold. So $|U^*| - |N(U^*)| > s$, that is, $|N(U^*)| - |U^*| < -s$.

Conversely, if there exists a $U^* \subset I$ satisfying $|N(U^*)| - |U^*| < -s$, set $S^* = N(U^*)$, then $i(G - S^*) \geq |U^*|$ holds. So we have $i(G - S^*) - |S^*| \geq |U^*| - |N(U^*)| > s$. Hence, our problem boils down to whether there exists $U^* \subset I$ such that $|N(U^*)| - |U^*| < -s$.

Let $f(U) = |N(U)| - |U|$ be a function defined on $2^I - \{I, \emptyset\}$. Notice that $f(U)$ is a submodular function on the crossing family $2^I - \{I, \emptyset\}$. Then from our analysis on submodular functions above, $\min\{f \mid f(U) \neq \emptyset, U \subset I\}$ can be computed in polynomial time by using the combinatorial algorithm presented by Grötschel et al. [11]. Consequently, we can decide whether there exists a cut-set $S^* \subset C$ such that $i(G - S^*) - |S^*| > s$ in polynomial time.

It follows from Lemma 2.2 that s is bounded by $3 - n$ and $n - 2$. Then we can enumerate all these s, and check whether $\mathrm{isc}(G) > s$ in polynomial time. This completes the proof. □

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3. Isolated scattering number of graph products

In this section we determine the isolated scattering number of the Cartesian product and the Kronecker product of special graphs and special permutation graphs.

DEFINITION 3.1 [3]. A subset S of V is called an *independent set* of G if no two vertices of S are adjacent in G. An independent set S is called a maximum independent set if G has no independent set S' with $|S'| > |S|$. The *independence number* of G, $\alpha(G)$, is the number of vertices in a maximum independent set of G.

The *Cartesian product* of two graphs G_1 and G_2, denoted by $G_1 \times G_2$, is defined as $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_1v_1 \in E(G_1)$ and $u_2 = v_2$. The Cartesian product of n graphs $G_1, G_2, \ldots, G_n$, denoted by $G_1 \times G_2 \times \cdots \times G_n$, is defined inductively as the Cartesian product of $G_1 \times G_2 \times \cdots \times G_{n-1}$ and G_n. In particular, the Cartesian product of k copies of K_2, denoted by Q_k, is called a *hypercube* of dimension k. The Cartesian product $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}$ is called a grid, where $n_1, n_2, \ldots, n_k$ are k integers not less than 2. It is clear that hypercubes are grids. The Cartesian product $C_{n_1} \times C_{n_2} \times \cdots \times C_{n_k}$ is called a torus.

It is well known that Cartesian products like hypercubes, grids and tori are highly recommended for the design of interconnection networks in multiprocessor computing systems. Hence, there is a lot of study of the vulnerability of these graphs in the literature [18, 22]. The aim of the following is to determine the isolated scattering number of grids, and that of the hypercubes as a special case.

The following lemmas will be used later.

LEMMA 3.2. *Let H be a spanning subgraph of an incomplete connected graph G. Then* $\mathrm{isc}(H) \geq \mathrm{isc}(G)$.

LEMMA 3.3 [20]. *Let G be a connected graph of order n. Then* $\mathrm{isc}(G) \geq 2\alpha(G) - n$. *The equality holds if G is a connected bipartite graph of order n.*

LEMMA 3.4 [20]. *The isolated scattering number of $K_{m,n} (m \geq n > 1)$ is* $\mathrm{isc}(K_{m,n}) = m - n$.

LEMMA 3.5 [20]. *Let P_n be be a path of order n. Then*

$$\mathrm{isc}(P_n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

THEOREM 3.6. *Suppose that $n_1, n_2, \ldots, n_k$ are k integers not less than 2. Then*

$$\mathrm{isc}(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) = \begin{cases} 1 & \text{if all } n_i \text{ are odd,} \\ 0 & \text{if some } n_i \text{ is even.} \end{cases}$$

Proof. It is well known that if G is a bipartite graph with bipartition $[A, B]$ and H is bipartite graph with bipartition $[C, D]$, then the Cartesian product of these two bipartite graphs G and H, $G \times H$, is a bipartite graph with bipartition $[(A \times C) \cup (B \times D), (A \times D) \cup (B \times C)]$. Hence, it follows that if all n_i are odd, then

$$P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k} \subseteq K_{(n_1 n_2 \cdots n_k - 1)/2, (n_1 n_2 \cdots n_k + 1)/2}.$$

By Lemmas 3.2 and 3.4, we know that

$$\operatorname{isc}(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) \geq \operatorname{isc}(K_{(n_1 n_2 \cdots n_k - 1)/2, (n_1 n_2 \cdots n_k + 1)/2}) = 1.$$

On the other hand, it is easy to see that if a graph G contains a Hamiltonian path, then so does graph $G \times P_n$. So the grid $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}$ has a Hamiltonian path $P_{n_1 n_2 \cdots n_k}$. It follows from Lemmas 3.2 and 3.5 that

$$\operatorname{isc}(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) \leq \operatorname{isc}(P_{n_1 n_2 \cdots n_k}) = 1.$$

This completes the proof of the case where all n_i are odd.

If some n_i is even, then we have

$$P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k} \subseteq K_{(n_1 n_2 \cdots n_k)/2, (n_1 n_2 \cdots n_k)/2}.$$

By Lemmas 3.2 and 3.4, we know that

$$\operatorname{isc}(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) \geq \operatorname{isc}(K_{(n_1 n_2 \cdots n_k)/2, (n_1 n_2 \cdots n_k)/2}) = 0.$$

On the other hand, it is easy to see that the grid $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}$ has a Hamiltonian path $P_{n_1 n_2 \cdots n_k}$. It follows from Lemmas 3.2 and 3.5 that

$$\operatorname{isc}(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) \leq \operatorname{isc}(P_{n_1 n_2 \cdots n_k}) = 0.$$

This completes the proof of the case where some n_i is even. Thus, the proof is complete. □

Corollary 3.7. *The isolated scattering number of the hypercube Q_k is* $\operatorname{isc}(Q_k) = 0$.

The *Kronecker product* (also named *direct product*, *tensor product* and *cross product*) $G_1 \otimes G_2$ is defined as $V(G_1 \otimes G_2) = V(G_1) \otimes V(G_2)$, and $E(G_1 \otimes G_2) = \{(u_1, v_1)(u_2, v_2) \mid u_1 u_2 \in E(G_1), v_1 v_2 \in E(G_2)\}$. Clearly the Kronecker product of two nontrivial connected graphs is connected if and only if at least one of the factors is not bipartite. The Kronecker product of graphs has been extensively investigated with regard to graph colourings, graph recognition and decomposition, graph embeddings, matching theory and stability in graphs [1, 5]. This graph product has several applications; for instance, it can be used in modelling concurrency in multiprocessor system [13] and in automata theory [10]. Since the Kronecker product of graphs has been widely used to model some practical structures used in the design of certain optimal networks [2, 10], it is significant to consider the vulnerability parameters of this product of graphs.

We now determine the isolated scattering number for the Kronecker product of two complete graphs. The following lemma on the components after removing a cut-set from $K_m \otimes K_n$ plays a key role in the proof of our main result.

Lemma 3.8 [17]. *Let m and n be integers with $n \geq m \geq 2$ and $n \geq 3$, and let S be a cut-set of $G = K_m \otimes K_n$. Then the following hold.*

(a) *Suppose that $\omega(G - S) = 2$ and let C_1, C_2 be two components of $G - S$. Then either $|C_1| = |C_2| = 2$ or $\min\{|C_1|, |C_2|\} = 1$.*
(b) *If $\omega(G - S) \geq 3$, then every component of $G - S$ is an isolated vertex and $|S| \geq mn - n$.*

Lemma 3.9 [17]. *Let m and n be integers with $n \geq m \geq 2$, $n \geq 3$. Then $\alpha(K_m \otimes K_n) = n$, $\kappa(K_m \otimes K_n) = (m - 1)(n - 1)$.*

Theorem 3.10. *Let m and n be integers with $n \geq m \geq 2$ and $n \geq 3$. Then $\mathrm{isc}(K_m \otimes K_n) = 2n - mn$.*

Proof. For convenience, we use the abbreviation $G = K_m \otimes K_n$, and let S be an isc-set of G. If $\omega(G - S) = 2$, then by Lemmas 3.8 and 3.9, we have

$$i(G - S) - |S| \leq i(G - S) - \kappa(G) \leq 1 - (m - 1)(n - 1) = m + n - mn.$$

If $\omega(G - S) \geq 3$, then, by Lemma 3.8, we have $|S| \geq mn - n$, and consequently,

$$i(G - S) - |S| = (mn - |S|) - |S| = mn - 2|S| \leq 2n - mn.$$

The proof comprises the following two cases.

Case 1. If $m = n$, then $2n - mn = (m + n) - mn$, and therefore $\mathrm{isc}(G) \leq (m + n) - mn = 2n - mn$. On the other hand, in this case, by Lemma 3.9, we know that $\alpha(G) = n$. Hence, by Lemma 3.3, we get $\mathrm{isc}(G) \geq 2\alpha(G) - mn = 2n - mn$. Thus, $\mathrm{isc}(G) = 2n - mn$.

Case 2. If $m \neq n$, then $m \leq n - 1$ and $(m + n) - mn < 2n - mn$. Hence in this case, we have $\mathrm{isc}(G) \leq 2n - mn$. On the other hand, we know that $\alpha(G) = n$, so by Lemma 3.3, we have $\mathrm{isc}(G) \geq 2\alpha(G) - mn = 2n - mn$. Hence, we get $\mathrm{isc}(G) = 2n - mn$. □

The concept of a permutation graph was introduced by Chartrand and Harary [6]. It is well known that permutation graphs have high connectivity properties. Since then many parameters on graphs of this kind have been determined, such as connectivity, chromatic number and crossing number.

Definition 3.11 [6]. Let G be a graph whose vertices are labelled as $v_1, v_2, \ldots, v_n$ and let a permutation $\alpha \in S_n$, where S_n is the symmetric group on $\{1, 2, \ldots, n\}$. Then the *permutation graph* $P_\alpha(G)$ is obtained by taking two copies of G, say G_x with vertex set $\{x_1, x_2, \ldots, x_n\}$ and G_y with vertex set $\{y_1, y_2, \ldots, y_n\}$, along with a set of permutation edges joining x_i of G_x and $y_{\alpha(i)}$ of G_y $(i = 1, 2, \ldots, n)$.

Lemma 3.12. *Let G be a bipartite, k-connected, k-regular graph on n vertices, $k \geq 2$. Then $\mathrm{isc}(G) = 0$.*

Proof. It is easy to see that $\alpha(G) = n/2$. Thus, by Lemma 3.3 we have $\mathrm{isc}(G) = 2\alpha(G) - n = 0$. This completes the proof. □

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THEOREM 3.13. *Let G_1 be a bipartite, n-regular and n-connected graph with p_1 vertices, and G_2 a bipartite, m-regular and m-connected graph with p_2 vertices. Then* $\mathrm{isc}(G_1 \times G_2) = 0$.

PROOF. It is obvious that the graph $G_1 \times G_2$ is an $(m+n)$-regular and $(m+n)$-connected bipartite graph with mn vertices. Then, by Lemma 3.12, we have $\mathrm{isc}(G_1 \times G_2) = 0$. The proof is complete. □

The following result can be directly derived from Theorem 3.13.

THEOREM 3.14. *Let m and n be two even positive integers. Then* $\mathrm{isc}(C_n \times C_m) = 0$ *and* $\mathrm{isc}(C_n \times K_2) = 0$.

THEOREM 3.15. *Let G be a bipartite, k-regular and k-connected graph with partition* $[M, N]$ *on n vertices. Then, for a permutation* $\alpha \in S_n$ *that satisfies*

$$\alpha : \begin{cases} M_x \to N_y, \\ M_y \to N_x, \end{cases}$$

we have $\mathrm{isc}(P_\alpha(G)) = 0$, *where* $[M_x, M_y]$ *is the partition of the first copy of G, and* $[N_x, N_y]$ *is the partition of the second copy of G.*

PROOF. It is easy to verify that the graph $P_\alpha(G)$ is a $(k+1)$-regular and $(k+1)$-connected bipartite graph with partition $[M_x \cup M_y, N_x \cup N_y]$. By Lemma 3.12, we know that $\mathrm{isc}(P_\alpha(G)) = 0$. □

4. Conclusion

Network vulnerability is an important issue in the area of distributed computing. Network designers often build a network configuration around specific processing, performance and cost requirements. They also identify the critical points of failure and modify the design to eliminate them. Most of the early work in this area takes a probabilistic approach to the problem. However, sometimes it is important to incorporate subjective vulnerability estimates into the measure. In this paper we discuss a comparatively better parameter, the isolated scattering number, which can be used to measure the vulnerability of networks. We first prove that for split graphs this number can be computed in polynomial time. Then we determine the isolated scattering number of the Cartesian product and the Kronecker product of special graphs and special permutation graphs.

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