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OSCILLATION CRITERIA FOR A CLASS OF PERTURBED SCHRÖDINGER EQUATIONS

BY
TAKAŜI KUSANO AND MANABU NAITO

1. **Introduction.** We are concerned with the oscillatory behavior of the second order elliptic equation

(1) $$\Delta u + c(x, u) = f(x), \qquad x \in E,$$

where Δ is the Laplace operator in n-dimensional Euclidean space R^n, E is an exterior domain in R^n, and $c: E \times R \to R$ and $f: E \to R$ are continuous functions.

A function $v: E \to R$ is called oscillatory in E if $v(x)$ has arbitrarily large zeros, that is, the set $\{x \in E : v(x) = 0\}$ is unbounded. For brevity, we say that equation (1) is oscillatory in E if every solution $u \in C^2(E)$ of (1) is oscillatory in E.

Our purpose here is to establish criteria for equation (1) with $f(x) \not\equiv 0$ to be oscillatory in E. This work has been motivated by the observation that there seems to be no literature except for Allegretto [1] dealing with oscillation theory of perturbed elliptic equations. We note that the unperturbed case of (1) ($f(x) \equiv 0$) has so far been studied in great detail; see e.g. Kreith [6] and Swanson [9].

In Section 2, with the help of the method of spherical means introduced by Noussair and Swanson [7], the problem under study is reduced to the problem of oscillation of ordinary differential inequalities of the form

(2) $$(t^{n-1}y')' + H(t)\Phi(y) \le F(t), \qquad F(t) \not\equiv 0.$$

Section 3 develops new oscillation criteria for a class of perturbed ordinary differential inequalities including (2). Thus Section 3 has independent interest. The desired oscillation criteria for equation (1) are derived in Section 4 by specializing the results of Section 3 to inequalities of the form (2).

2. **Reduction to a one-dimensional oscillation problem.** We employ the notation:

$$E_t = \{x \in R^n : |x| \ge t\}, \qquad S_t = \{x \in R^n : |x| = t\}, \quad t > 0,$$

where $|x|$ denotes the Euclidean length of x.

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THEOREM 1. *Suppose that* $c(x, u)$ *satisfies the following conditions:*

(i) $c(x,-u)=-c(x,u)$ *for* $x\in E$ *and* $u>0$;

(ii) $c(x,u)\geq H(|x|)\Phi(u)$ *for* $x\in E$ *and* $u>0$, *where* $H:[0,\infty)\to(0,\infty)$ *is continuous and* $\Phi:(0,\infty)\to(0,\infty)$ *is continuous and convex.*

Let $F(t)$ *be the spherical mean of* $f(x)$ *over* S_t, *i.e.*

$$F(t)=\frac{1}{\sigma_n t^{n-1}}\int_{S_t} f(x)\,dS, \tag{3}$$

where σ_n *is the area of the unit sphere* S_1.

Then equation (1) *is oscillatory in* E *if the ordinary differential inequalities*

$$(t^{n-1}y')'+t^{n-1}H(t)\Phi(y)\leq t^{n-1}F(t), \tag{4}$$

$$(t^{n-1}y')'+t^{n-1}H(t)\Phi(y)\leq -t^{n-1}F(t), \tag{5}$$

are oscillatory at $t=\infty$ *in the sense that neither* (4) *nor* (5) *has a solution which is positive on* $[t_0,\infty)$ *for any* $t_0>0$.

Proof. Suppose that equation (1) has a solution $u(x)$ which is positive in E_T for some $T>0$. Let $U(t)$ denote the spherical mean of $u(x)$ over S_t. Then, by a result of Noussair and Swanson [7], $U(t)$ satisfies

$$(t^{n-1}U'(t))'=\frac{1}{\sigma_n}\int_{S_t}\Delta u(x)\,dS, \qquad t\geq T,$$

from which, using (1), condition (ii) and the convexity of Φ, we see that

$$(t^{n-1}U'(t))'\leq -t^{n-1}H(t)\Phi(U(t))+t^{n-1}F(t), \qquad t\geq T.$$

This shows that $U(t)$ is a positive solution of inequality (4). Similarly, it can be shown that if $u(x)$ is a negative solution of (1) in E_T, then the function $-U(t)$ is a positive solution of inequality (5). This contradicts the hypothesis of the theorem, and the proof is complete.

3. **Oscillation of ordinary differential inequalities.** We now consider the ordinary differential inequality

$$(q(t)(p(t)y)')'+h(t,y)\leq r(t), \tag{6}$$

in generalization of (4) and (5). The objective of this section is to obtain conditions under which inequality (6) is oscillatory at $t=\infty$ in the sense that it has no eventually positive solution. The following conditions are always assumed to hold:

(a) $p,q:[a,\infty)\to(0,\infty)$ and $r:[a,\infty)\to R$ are continuous, and

$$\int_a^\infty \frac{dt}{q(t)}=\infty; \tag{7}$$

(b) $h:[a,\infty)\times(0,\infty)\to(0,\infty)$ is continuous and non-decreasing in the second variable.

We define

$$Q(t, s)=\int_s^t \frac{d\sigma}{q(\sigma)}, \qquad t, s \in [a, \infty). \tag{8}$$

THEOREM 2. *Suppose that*

$$\liminf_{t\to\infty} \frac{1}{Q(t, T)} \int_T^t Q(t, s)r(s)\, ds = -\infty \tag{9}$$

for all sufficiently large $T > a$.

Then inequality (6) *has no solution which is positive on* $[t_0, \infty)$ *for any* $t_0 > a$.

Proof. Let $y(t)$ be a positive solution of (6) defined on $[t_0, \infty)$. From (6) we have

$$(q(t)(p(t)y(t))')' = r(t) - h(t, y(t)) \leq r(t), \qquad t \geq t_0.$$

Integrating the above inequality twice, we get

$$\begin{aligned} p(t)y(t) &\leq c_1 + c_2\int_{t_0}^t \frac{ds}{q(s)} + \int_{t_0}^t \frac{1}{q(s)} \int_{t_0}^s r(\sigma)d\sigma\, ds \\ &= c_1 + c_2 Q(t, t_0) + \int_{t_0}^t Q(t, s)r(s)\, ds, \qquad t \geq t_0, \end{aligned}$$

where c_1 and c_2 are constant. We divide the above by $Q(t, t_0)$ and let $t \to \infty$. In view of (9) we then conclude that

$$\liminf_{t\to\infty} \frac{p(t)y(t)}{Q(t, t_0)} = -\infty,$$

which contradicts the assumption that $y(t)$ is positive on $[t_0, \infty)$. This completes the proof.

REMARK 1. In Theorem 2 the unperturbed inequality

$$(q(t)(p(t)y)')' + h(t, y) \leq 0 \tag{10}$$

may or may not have eventually positive solutions.

THEOREM 3. *Suppose that inequality* (10) *has no eventually positive solution. Suppose moreover that there exists a continuous function* $\rho : [a, \infty) \to R$ *which is oscillatory and satisfies*

$$(q(t)(p(t)\rho(t))')' = r(t), \qquad t \geq a, \tag{11}$$

and

$$\liminf_{t\to\infty} [p(t)\rho(t)] = 0. \tag{12}$$

Then inequality (6) *has no eventually positive solution.*

Proof. Suppose that (6) has a solution $y(t)$ which is positive on $[t_0, \infty)$. Put $z(t) = y(t) - \rho(t)$. Then, in view of (11), $z(t)$ satisfies

$$(q(t)(p(t)z(t))')' \le -h(t, y(t)) < 0, \qquad t \ge t_0. \tag{13}$$

It follows that $z(t)$ is eventually of constant sign, say for $t \ge t_1 > t_0$. If $z(t) < 0$, $t \ge t_1$, then $y(t) < \rho(t)$, $t \ge t_1$, which implies that $y(t)$ takes negative values in any neighborhood of infinity. Thus we must have $z(t) > 0$ for $t \ge t_1$. We note that $(p(t)z(t))' > 0$ for $t \ge t_1$.

An integration of (13) yields

$$q(t)(p(t)z(t))' \ge \int_t^\infty h(s, y(s))\, ds, \qquad t \ge t_1. \tag{14}$$

Dividing (14) by $q(t)$ and integrating from t_1 to t, we obtain

$$p(t)z(t) \ge c + \int_{t_1}^t \frac{1}{q(s)} \int_s^\infty h(\sigma, y(\sigma))\, d\sigma\, ds, \qquad t \ge t_1,$$

or

$$p(t)y(t) \ge c + p(t)\rho(t) + \int_{t_1}^t \frac{1}{q(s)} \int_s^\infty h(\sigma, y(\sigma))\, d\sigma\, ds, \qquad t \ge t_1,$$

where $c = p(t_1)z(t_1) > 0$. Using (12), we see that there is a $T > t_1$ such that

$$p(t)y(t) \ge \frac{c}{2} + \int_T^t \frac{1}{q(s)} \int_s^\infty h(\sigma, y(\sigma))\, d\sigma\, ds, \qquad t \ge T. \tag{15}$$

We denote by $w(t)$ the right hand side of (15) divided by $p(t)$. Then $w(t) \le y(t)$, $t \ge T$, and by differentiation

$$(q(t)(p(t)w(t))')' + h(t, y(t)) = 0, \qquad t \ge T. \tag{16}$$

Noting that $h(t, w(t)) \le h(t, y(t))$, $t \ge T$, we conclude that $w(t)$ is an eventually positive solution of the differential inequality (10). This contradiction proves the theorem.

REMARK 2. The idea of employing the function $\rho(t)$ satisfying (11) follows Kartsatos [3, 4]. It can be shown [2] that inequality (10) has an eventually positive solution if and only if so does the differential equation

$$(q(t)(p(t)y)')' + h(t, y) = 0. \tag{17}$$

4. **Oscillation of the partial differential equation (1).** Let us now turn to the original elliptic equation (1), for which effective oscillation criteria are derived on the basis of the results of the preceding sections.

According to Theorem 1 equation (1) is oscillatory in an exterior domain E if the ordinary differential inequalities (4) and (5) have no eventually positive solution. Inequality (4) [or (5)] is formally a special case of (6) in which

$p(t)=1$, $q(t)=t^{n-1}$, $h(t, y)=t^{n-1}H(t)\Phi(y)$, $r(t)=t^{n-1}F(t)$ [or $-t^{n-1}F(t)$], but condition (7) is violated in case $n\geq 3$. To avoid this difficulty we use the fact that

$$(t^{n-1}y')'=t^{n-2}(t^{3-n}(t^{n-2}y)')'$$

to replace (4) and (5) by the following equivalent inequalities:

$$(t^{3-n}(t^{n-2}y)')'+tH(t)\Phi(y)\leq tF(t), \tag{18}$$

$$(t^{3-n}(t^{n-2}y)')'+tH(t)\Phi(y)\leq -tF(t). \tag{19}$$

THEOREM 4. *Suppose that $c(x, u)$ satisfies the conditions of Theorem 1. Let $F(t)$ denote the spherical mean of $f(x)$ defined by* (3). *Suppose that if $n=2$, then*

$$\liminf_{t\to\infty}\int_T^t\left(1-\frac{\log s}{\log t}\right)sF(s)\,ds=-\infty, \tag{20}$$

$$\limsup_{t\to\infty}\int_T^t\left(1-\frac{\log s}{\log t}\right)sF(s)\,ds=\infty, \tag{21}$$

for all large T, and if $n\geq 3$, then

$$\liminf_{t\to\infty}\int_T^t\left[1-\left(\frac{s}{t}\right)^{n-2}\right]sF(s)\,ds=-\infty, \tag{22}$$

$$\limsup_{t\to\infty}\int_T^t\left[1-\left(\frac{s}{t}\right)^{n-2}\right]sF(s)\,ds=\infty, \tag{23}$$

for all large T. Then equation (1) *is oscillatory in E.*

Proof. Inequality (18) [or (19)] is a special case of (6) in which $p(t)=t^{n-2}$, $q(t)=t^{3-n}$, $h(t, y)=tH(t)\Phi(y)$, $r(t)=tF(t)$ [or $-tF(t)$]. It is clear that (7) is satisfied and the function $Q(t, s)$ defined by (8) becomes

$$Q(t, s)=\begin{cases}\log t-\log s & (n=2),\\ \dfrac{1}{n-2}(t^{n-2}-s^{n-2}) & (n\geq 3).\end{cases}$$

Hence we are able to apply Theorem 2 to see that neither (18) nor (19) has an eventually positive solution. The desired conclusion now follows from Theorem 1. This completes the proof.

EXAMPLE 1. Consider the equation

$$\Delta u+\frac{1}{4|x|}u^3=|x|\sin|x| \tag{24}$$

in $E_1=\{x\in R^3 : |x|\geq 1\}$. The spherical mean over S_t of $f(x)=|x|\sin|x|$ is

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$F(t)=t \sin t$, and it is easy to verify that

$$\liminf_{t\to\infty} \int_T^t \left(1-\frac{s}{t}\right)s^2 \sin s\, ds = -\infty$$

and

$$\limsup_{t\to\infty} \int_T^t \left(1-\frac{s}{t}\right)s^2 \sin s\, ds = \infty$$

for all large $T>1$. It follows from Theorem 4 that equation (24) is oscillatory in $E_1 \subset R^3$. We note that the unperturbed equation

$$\Delta u + \frac{1}{4|x|} u^3 = 0$$

has a positive solution $u(x)=|x|^{-1/2}$.

Applying Theorem 3 we have the second oscillation theorem for equation (1).

THEOREM 5. *Suppose that $c(x, u)$ satisfies the conditions of Theorem 1. Let $F(t)$ denote the spherical mean of $f(x)$ defined by (3). Equation (1) is oscillatory in E if the differential inequality*

$$(t^{3-n}(t^{n-2}y)')' + tH(t)\Phi(y) \le 0 \tag{25}$$

has no eventually positive solution and if there exists a C^2 function $\rho:[1,\infty)\to R$ with the following properties:

(i) *$\rho(t)$ is oscillatory;*
(ii) $(t^{3-n}(t^{n-2}\rho(t))')' = tF(t)$, $t \ge 1$,
(iii) $\lim_{t\to\infty}[t^{n-2}\rho(t)] = 0$.

EXAMPLE 2. Consider the equation

$$\Delta u + \frac{2}{|x|^2} u = \frac{3}{|x|^4}[\sin(\log|x|) - \cos(\log|x|)] \tag{26}$$

in $E_1 = \{x \in R^3 : |x| \ge 1\}$. The associated ordinary differential inequality is

$$(ty)'' + \frac{2}{t} y \le 0,$$

which has no eventually positive solution. It is easily verified that the function

$$\rho(t) = \frac{3}{5t^2}[2\sin(\log t) + \cos(\log t)]$$

satisfies conditions (i)–(iii) of Theorem 5. Hence all solutions of equation (26) are oscillatory in $E_1 \subset R^3$. One such solution is $u(x) = \sin(\log|x|)/|x|^2$. Theorem

4 is not applicable to (26). Note that the homogeneous equation

$$\Delta u + \frac{2}{|x|^2} u = 0$$

is oscillatory in E_1; see Noussair and Swanson [7].

REMARK 3. Our results cannot cover an important class of sublinear equations of the form

$$\Delta u + c(x)\,|u|^\gamma \operatorname{sgn} u = f(x), \qquad 0<\gamma<1. \tag{27}$$

The unperturbed case of (27) has been studied by Kitamura and Kusano [5] and Noussair and Swanson [8], but their techniques fail when the non-vanishing perturbation $f(x)$ is present.

REFERENCES

1. W. Allegretto, *Oscillation criteria for quasilinear equations*, Canad. J. Math. **26** (1974), 931–947.

2. F. V. Atkinson, *On second-order differential inequalities*, Proc. Roy. Soc. Edinburgh, Sect. A, **72** (1974), 109–127.

3. A. G. Kartsatos, *On the maintenance of oscillations of nth order equations under the effect of a small forcing term*, J. Differential Equations **10** (1971), 355–363.

4. A. G. Kartsatos, *Maintenance of oscillations under the effect of a periodic forcing term*, Proc. Amer. Math. Soc. **33** (1972), 377–383.

5. Y. Kitamura and T. Kusano, *An oscillation theorem for a sublinear Schrödinger equation*, Utilitas Math. **14** (1978), 171–175.

6. K. Kreith, *Oscillation Theory*, Lecture Notes in Mathematics, Vol. 324, Springer Verlag, Berlin, 1973.

7. E. S. Noussair and C. A. Swanson, *Oscillation theory for semilinear Schrödinger equations and inequalities*, Proc. Roy. Soc. Edinburgh, Sect. A, **75** (1975/76), 67–81.

8. E. S. Noussair and C. A. Swanson, *Oscillation of semilinear elliptic inequalities by Riccati transformations*, Canad. J. Math., to appear.

9. C. A. Swanson, *Semilinear second order elliptic oscillation*, Canad. Math. Bull. **22** (1979), 139–157.

HIROSHIMA UNIVERSITY,
HIROSHIMA 730, JAPAN.