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REFLEXIVE TOPOLOGICAL SEMILATTICES

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The duality between compact 0-dimensional semilattices and discrete semilattices studied by K. H. Hofmann *et al.* [2] is here extended to larger categories of topological semilattices.

We regard topological semilattices as objects in the category *CvSl* of convergence semilattices, believing *CvSl* to be the appropriate setting for this study. The referee objected to this at first, which prompted us to set forth more carefully our reasons for doing so, as follows. Firstly, *CvSl* has an exponential law

$$[A\otimes B, C]\simeq[A,[B, C]]$$

relating internal hom-objects $[B, C]$ to tensor products (see L. D. Nel [5] for further details and motivation). The dual A' is the hom-object $[A, 2]$ where 2 denotes the discrete meet semilattice $\{0, 1\}$. In this setting the functor $[\ ,2]$ automatically has a right adjoint with adjunction $@_A : A \to A''=[[A, 2], 2]$ where $@(x)(h)=h(x)$. Reflexiveness of A means simply that $@_A$ is an isomorphism in *CvSl*. Several useful categorical facts become available (see [5]) which would disappear in the absence of an exponential law.

Secondly, *CvSl* is preferable to other categories of topological semilattices that likewise uphold an exponential law. It contains all desired objects (all topological semilattices) while the structure carried by its hom-objects $[A, B]$ (namely continuous convergence) is exactly the desired one, reducing as it does to the compact-open topology whenever A and B are topological with A locally compact. Thus the above canonical dual A' agrees with the notion of dual used by K. H. Hofmann *et al.* [2], but would differ from a dual $A^\wedge$ that carries the compact-open topology for arbitrary topological semilattices A (cf. example 3.5). We point out (Section 4) that when A is locally compact, its dual $A'(=A^\wedge)$ need no longer be locally compact and such A is reflexive in the sense that $A\simeq A^{\wedge\wedge}$ only if it is already reflexive in our canonical sense. In several other situations reflexiveness has been studied likewise to good effect in the presence of an exponential law (see [5] and papers cited there).

Our presentation emphasizes continuous convergence. This canonical function space structure reduces in special cases (as a pleasant surprise sometimes)

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to more familiar structures such as the compact-open topology, the Vietoris topology, the pointwise topology and others (see 1.8 and 2.5). However, none of the latter have a proper claim to serve as definition for a well-behaved dualizing functor.

A Pontryagin-type duality involves only reflexive objects. Moreover, a full subcategory of *CvSl* formed by reflexive objects is automatically dual to the full subcategory formed by all its dual objects. Thus, in order to extend a given duality, it is enough to find a larger class of reflexive objects. This is our main concern in this paper.

In Section 1 we show that among totally ordered convergence semilattices the reflexive objects are precisely those that carry a locally convex 0-dimensional topology. The proof proceeds through several different characterizations, each of which sheds further light on the nature of these semilattices. In Section 2 we prove reflexiveness of the semilattice $C(X)$ of all maps $X \to 2$ (equivalently, all open closed $W \subset C$) where X is locally compact and 0-dimensional. We then show (Section 3) that reflexiveness is preserved under formation of products and coproducts in *CvSl*, the dual of a product being the coproduct of the duals. While this is hardly a surprise, it is not obvious either. Our techniques for proving reflexiveness proceed as far as possible along categorical lines also applicable in analogous situations such as vector spaces or abelian groups.

Basic Definitions. For background on convergence spaces we refer to the first five pages of [1]. *Cv* will denote the category of convergence spaces and maps (= continuous functions). $C_c(X, Y)$ is the space of maps $X \to Y$ equipped with continuous convergence which gives the exponential law $C_c(W \times X, Y) \simeq C_c(W, C_c(X, Y))$ for *Cv*.

A convergence semilattice A is more precisely a triple (A, b, n) where $A \in Cv$, b is a continuous binary operation on A that is associative, commutative and idempotent and n is a nullary operation such that $b(x, n) = x$ for all x. We will usually write $(A, \wedge, 1)$ for (A, b, n) and speak of a meet semilattice, but where this gives an order opposite to a desired one we write $(A, \vee, 0)$ and speak of a join semilattice. The category *CvSl* is formed with such (A, b, n) as objects and with operation-preserving maps as morphisms in the usual way.

Order theoretic notation generally follows K. H. Hoffman *et al.* [2]. In particular, for $p, q, x \in A$ with $p < q$, $B \subset A$ we put

$$\uparrow x = \{y \mid x \wedge y = x\} \quad \text{and} \quad \downarrow x = \{w \mid x \wedge w = w\}$$

$$[p, q] = \uparrow p \cap \downarrow q = \{x \mid p \leq x \leq q\}; \; [p, q) = \uparrow p \setminus \uparrow q$$

$$\uparrow B = \bigcup_{x \in B} \uparrow x, \qquad \downarrow B = \bigcup_{x \in B} \downarrow x.$$

Members of A' will be called *characters* on A. We will say A has *enough characters* if for any $x, y \in A$ the equality $h(x) = h(y)$ for all characters $h : A \to 2$ implies $x = y$. A subset B of A is order convex, briefly *convex*, if $p, q \in B$ implies $[p, q] \subset B$. The *convexification* of a filter $\mathcal{F}$ is the filter generated by all convex members of $\mathcal{F}$ and when this coincides with $\mathcal{F}$ we call $\mathcal{F}$ a *convex filter.* To say $A \in CvSl$ is *locally convex* means that the convergence of $\mathcal{F}$ to x implies that the convexification of $\mathcal{F}$ also converges to x.

1. **Reflexive totally ordered semilattices.** A convergence semilattice A will be called *orderly* if it is totally ordered, locally convex and its convergence structure is a 0-dimensional topology. Our main purpose in this section is to show that among totally ordered convergence semilattices the reflexive ones are precisely those that are orderly.

1.1 Examples. The following are orderly semilattices.

(a) The discrete join semilattice N of non-negative integers.

(b) The meet semilattice N^* (= one·point compactification of N).

(c) The joint semilattice O of all countable ordinal numbers, with the usual order topology.

(d) The meet semilattice P of all non-negative irrational numbers together with ∞ as isolated point, where $P\backslash\{\infty\}$ has the usual topology.

(e) The join semilattice Q of all non-negative rationals, with the usual topology and order.

(f) The join semilattice R of non-negative real numbers with the usual order and the Sorgenfrey topology: basic neighbourhoods at x are the intervals $[x, y)$ where $y > x$.

For the remainder of this section we will suppose T to denote a totally ordered convergence semilattice.

1.2 Lemma. *If T is orderly, then T has enough characters.*

Proof. Given $w < x$ in T we can choose a convex U and an open closed V so that $x \in V \subset U$ and $w \notin U$. Then $w \notin \uparrow V$ and $\uparrow V$ is open. To show $\uparrow V$ is also closed, note that every point outside $\uparrow V$ is also outside V and thus has a convex neighbourhood disjoint from V, hence disjoint from $\uparrow V$. Now the characteristic function h of $\uparrow V$ is a character with $h(w) = 0$ and $h(x) = 1$.

1.3 Lemma. *If T is locally convex with enough characters, then each point x in T has a smallest filter convergent to it and this filter must be one of the following:*

the filter $[x]$ generated by $\{x\}$,
the filter $[x\ldots]$ generated by intervals $[x, y]$ with $x < y$,
the filter $[\ldots x]$ generated by intervals $[w, x]$ with $w < x$,
the filter $[\ldots x\ldots]$ generated by intervals $[w, y]$ with $w < x < y$.

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Proof. Take any x in T and any filter $\mathcal{G}$ convergent to x in T. Let $\mathcal{F}$ be the convexification of $\mathcal{G}\cap[x]$. Then $\mathcal{F}$ converges to x and since T has enough characters $\cap\mathcal{F}=\{x\}$. It is enough to show that $\mathcal{F}$ is one of the four filters mentioned, since the intersection of two or more of them is again one of them.

If $\mathcal{F}\neq[x]$, then every member of $\mathcal{F}$ contains x as well as at least one other point and precisely one of the following three cases can occur: Case 1: $\uparrow x\in\mathcal{F}$; Case 2: $\uparrow x\in\mathcal{F}$; Case 3: every member of $\mathcal{F}$ contains an interval $[w, y]$ with $w<x<y$. In this latter case we claim that, conversely, every such interval $[w, y]$ contains a member of $\mathcal{F}$. Indeed, there are convex members F_w and F_y of $\mathcal{F}$ such that $w\notin F_w$, $y\notin F_y$. By convexity, $F_w\subset\uparrow w$ and $F_y\subset\downarrow y$. Hence $[w, y]=\uparrow w\cap\downarrow y\supset F_w\cap F_y$ as claimed. Thus in Case 3, $\mathcal{F}=[\ldots x\ldots]$. It is proved similarly that in Case 1 $\mathcal{F}=[x\ldots]$ and in Case 2 $\mathcal{F}=[\ldots x]$.

The initial topology induced on a convergence semilattice A by its characters $h: A\to 2$ will be called the *character topology* of A.

1.4 Lemma. *If T has enough characters and at each x in T one of $[x]$, $[x\ldots]$, $[\ldots x]$ or $[\ldots x\ldots]$ is the smallest filter convergent to x, then T carries the character topology as convergence structure.*

Proof. A routine verification shows that the assignment to each x in T of a filter $\mathcal{N}_x$ chosen from among $[x]$, $[x\ldots]$, $[\ldots x]$, or $[\ldots x\ldots]$ results in a neighbourhood system defining a topology for T. Thus the convergence structure of T is topological. If $\mathcal{N}_x=[x]$, then $\uparrow x$ and $\uparrow x\backslash\{x\}$ are readily seen to be open closed sets whose characteristic functions furnish characters h, k such that $\{x\}=h^{-1}(1)\cap k^{-1}(0)$ and this set is a neighbourhood of x in the character topology. In case $\mathcal{N}_x=[\ldots x\ldots]$, choose $w<x<y$ and characters h, k such that $h(w)=0$, $h(x)=1$, $k(x)=0$, $k(y)=1$. Then $[w, y]\supset h^{-1}(1)\cap k^{-1}(0)$, showing $\mathcal{N}_x$ to the contained in the neighbourhood filter of the character topology, hence equal to it. A similar conclusion holds in the other two cases.

If A is a topological semilattice, then its dual A' need not be topological (see 3.5) but will be so in special cases. One well known special case is that in which A is locally compact. The next proposition brings to light another special case.

1.5 Proposition. *T' is orderly. If T is also orderly, then T' carries the pointwise topology.*

Proof. If h and k are different characters on T, the fiters $h^{-1}(1)$, $k^{-1}(1)$ can differ only in such a way that one is strictly contained in the other. Thus one of h, k is smaller than the other and T' is totally ordered. The continuous convergence structure on T' is clearly locally convex and T' has enough character. It follows (1.3 and 1.4) that T' carries the character topology which is evidently 0-dimensional. Thus T' is orderly. Let T^* denote the dual of T equipped with the pointwise topology. Then the identity function $id: T'\to T^*$

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is continuous, since the convergence $\mathcal{H} \rightsquigarrow h$ in T' implies $\mathcal{H}[x] \rightsquigarrow h(x)$ in 2 ($x \in T$). For the continuity of $id: T^* \to T'$, consider $\mathcal{H} \rightsquigarrow h$ in T^* and suppose $\mathcal{F} \rightsquigarrow x$ in T, where $h(x)=1$ say (the case $h(x)=0$ is similar). We have $\mathcal{H}(x) \rightsquigarrow 1$, so there exists $H \in \mathcal{H}$ such that $H(x)=1$ and $H(y)=1$ for any $y > x$. Therefore, in case $\mathcal{F} = [x\ldots]$, we have $F=[x, y] \in \mathcal{F}$ such that $H(F)=1$ and we conclude $\mathcal{H}(\mathcal{F}) \rightsquigarrow h(x)$. In case $\mathcal{F} = [\ldots x]$ the continuity of h furnishes an interval $[w, x]$ such that $h([w, x])=1$. Now $\mathcal{H}(w) \rightsquigarrow h(w)=1$, whence $\mathcal{H}([w, x])=1$; again we conclude $\mathcal{H}(\mathcal{F}) \rightsquigarrow h(x)$. The case $\mathcal{F} = [\ldots x \ldots]$ is treated similarly and the case $\mathcal{F} = [x]$ is obvious.

1.6 Theorem. *For a totally ordered convergence semilattice T the following statements are equivalent:*

(a) *T is orderly;*

(b) *T has enough characters and is locally convex;*

(c) *T has enough characters and for each point x in T one from among $[x]$, $[x\ldots]$, $[\ldots x]$, $[\ldots x \ldots]$ is the smallest filter convergent to x;*

(d) *T has enough characters and carries the character topology;*

(e) *T is reflexive.*

Proof. (a) implies (b) by 1.2, (b) implies (c) by 1.3 and (c) implies (d) by 1.4. The character topology is clearly locally convex and when enough characters are present it is 0-dimensional; thus (d) implies (a). By 1.5, (e) implies (a). It remains to be shown that (a) implies (e). Since T has enough characters, the adjunction $@: T \to T''$ is $1-1$. Let $ev: T'' \times T' \to 2$ be the evaluation: $ev(u, h)=u(h)$. Then for each character $h: T \to 2$ we have $ev(\ , h) \circ @ = h$. Since T carries the initial topology induced by all such h, the map $@$ must be an embedding. Therefore we can complete the proof by showing $@$ to be onto. Take any $u \in T''$ and put $D = \bigcap \{h^{-1}(1) \mid h \in u^{-1}(1)\}$. We claim the closed filter D must have a smallest member. Suppose not. Then each d in D lies in an open set of the form $\uparrow x \setminus \{x\} \subset D$ for some x in D. Thus D is open closed and its characteristic function k (say) belongs to T'. We will force a contradiction by showing $k \in u^{-1}(0) \cap u^{-1}(1)$. Since T' carries the pointwise topology (1.5), this can be achieved by showing that for any $q_0 \in k^{-1}(0)$ and $q_1 \in k^{-1}(1)$ the set

$$W = \{h \in T' \mid h(q_0)=0 \quad \text{and} \quad h(q_1)=1\}$$

meets both of the closed sets $u^{-1}(0)$ and $u^{-1}(1)$. Now $q_1 \in D$ and we can choose $p \in D$ with $p < q_1$ and then $g \in T'$ with $g(p)=0$ and $g(q_1)=1$. Since $q_0 \in \downarrow D \setminus D$, we have $g \in W$. Since $g^{-1}(1)$ is strictly smaller than D we must have $u(g)=0$. Thus $g \in W \cap u^{-1}(0)$. On the other hand, since $q_0 \notin D$ we can choose $f \in T'$ so that $f(q_0)=0$, $u(f)=1$, $f(q_1)=1$. Thus $f \in W \cap u^{-1}(1)$. We are now able to conclude that D must have a smallest element x_u (say). We claim $u = @(x_u)$. If $u(h)=1$, then $@(x_u)(h)=h(x_u)=1$. And if $u(h)=0$ we conclude $h(x_u)=0$ by noting that there must exist some $y \in D$ for which $h(y)=0$.

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We thank the referee for pointing out that our weaker earlier version of 1.6 could be sharpened to give the equivalence of (b) and (e) for locally compact totally ordered topological semilattices. This stimulated the foregoing analysis which also showed local compactness to be dispensible.

1.7 Example. $Q' = P$ and therefore $P' = Q$ where P, Q are the objects defined in 1.1.

Proof. For any $p \in P$, put $h_p(x) = 1$ when $x < p$ and $h_p(x) = 0$ when $x > p$. Then h_p is a character on Q. Conversely, for any character h on Q, put $p = \sup h^{-1}(1)$ where the supremum is taken in the extended real line. Since $h^{-1}(1)$ is an open closed subset of Q, p must be ∞ or irrational. We have $h(x) = 1$ when $x < p$ so that $h = h_p$. This gives a bijective correspondence between Q' and P. It is easy to see that the pointwise topology of Q' (see 1.5) corresponds to the given topology of P under this correspondence.

We note that the duals of the other semilattices N, O, R of 1.1 can be calculated similarly.

1.8 Corollary. *On the dual T' of an orderly semilattice T the following structures coincide:* (a) *continuous convergence;* (b) *the compact-open topology;* (c) *the pointwise topology;* (d) *the character topology.*

Proof. In view of theorem 1.6 we need only show that (b) coincides with (c). Using the fact that $\uparrow x \backslash \{x\}$ is open, it is readily verified that every compact subset of T has a smallest and largest element. Thus a typical basic open set in the compact-open topology

$$W(P, Q) = \{h \in T' \mid h(P) = 0 \quad \text{and} \quad h(Q) = 1\}$$

where P, Q are compact in T, satisfies the relation $W(P, Q) = W(\{p\}, \{q\})$ where $p = \sup P$, $q = \inf Q$. Thus (b) and (c) have the same basic open sets.

2. **Reflexiveness of** $C(X)$. Throughout this section X will denote a locally compact 0-dimensional topological space. $C(X)$ is the semilattice of all continuous functions $X \to 2$, equipped with pointwise operations and continuous convergence, the latter reducing of course to the compact-open topology. In proving reflexiveness of $C(X)$, its dual will be represented as the join semilattice of compact subsets of X. It turns out that on the dual of $C(X)$ the canonical structure of continuous convergence coincides with five naturally arising topologies.

2.1 Lemma. *Each of the sets $K_0 = \{g \in C(X) \mid g^{-1}(0)$ is compact$\}$, $K_1 = \{g \in C(X) \mid g^{-1}(1)$ is compact$\}$ is dense in $C(X)$.*

Proof. Consider the non-empty basic open set

$$W_{PQ} = \{f \in C(X) \mid f(P) = 0 \quad \text{and} \quad f(Q) = 1\}$$

where P, Q are compact sets. Each x in P, hence P itself, is contained in an open compact set N disjoint from Q. The characteristic function f of $X\backslash N$ belongs to W_{PQ} and has $f^{-1}(0)=N$ compact. Thus K_0 and similarly K_1 is dense in $C(X)$.

2.2 Proposition. *There is a bijective correspondence between the characters $h: C(X)\to 2$ and the compact subsets M of X, given by $h \to M_h = \bigcap\{f^{-1}(1) \mid h(f)=1\}$* and $M\to h_M$ where $h_M(f)=\inf f(M)$.

Proof. Given the character h, there exists $g\in h^{-1}(1)$ with $g^{-1}(1)$ compact (2.1). The intersection of all such $g^{-1}(1)$ is a compact set containing M_h as a closed subset. Hence M_h is compact. Given a compact $M\subset X$, h_M is clearly a semilattice homomorphism. To see that h_M is continuous, we note that $h_M^{-1}(1)=\{f \mid f(M)=1\}$ is open in the compact-open topology and the set $h_M^{-1}(0)=\{f \mid f(m)=0 \text{ for some } m\in M\}=\bigcup_{m\in M} G_m$, where $G_m=\{f \mid f(m)=0\}$, is also open. It is easy to see that $M\to h_M$ is an injective function. For surjectivity, take any character h and put $M=\bigcap\{f^{-1}(1) \mid h(f)=1\}$. We claim $h=h_M$. For $f\in h^{-1}(1)$ we clearly have $h(f)=h_M(f)=1$. To show $h(g)=h_M(g)$ for $g\in h^{-1}(0)$ it is enough, by 2.1 and the continuity of h_M, to consider the case where $g^{-1}(0)$ is compact. For such g, if $h_M(g)=1$ then $M\cap g^{-1}(0)=\emptyset$. By compactness we can find $f\in h^{-1}(1)$ such that $f(g^{-1}(0))=0$. But then $h(f)=h(f\wedge g)=h(g)\wedge h(f)=0$, a contradiction. Therefore $h_M(g)=0$.

2.3 Lemma. *The characters of the form h_F, with F a finite subset of X form a dense subset of $C(X)'$.*

Proof. Take any character $h: C(X)\to 2$ and let $M=\bigcap\{f^{-1}(1) \mid h(f)=1\}$. For each finite subset $F\subset M$ let $H_F=\{h_G \mid F\subset G\subset M$ and G is finite$\}$. These sets H_F generate a filter $\mathcal{H}$ on $C(X)'$. To complete the proof it is enough to verify that $\mathcal{H}$ converges to h in $C(X)'$. Therefore, given $f\in C(X)$, w must find a neighbourhood W_{PQ} of f in $C(X)$ and a member H_F of $\mathcal{H}$ such that $H_F(W_{PQ})=h(f)$. If $h(f)=0$, we have $\inf f(M)=0$. Choose $P=M\cap f^{-1}(0)\neq\emptyset$, Q any compact subset of $f^{-1}(1)$ and F any finite subset of P. Then $H_F(W_{PQ})=0=h(f)$. If $h(f)=1$, then $M\subset f^{-1}(1)$ and this time we can take P to be any compact subset of $f^{-1}(0)$, $Q=M$ and F any finite subset of M to get $H_F(W_{PQ})=h(f)$.

2.4 Theorem. *$C(X)$ is reflexive.*

Proof. Let $@: X\to C(X)'$ be the map $@(x)(f)=f(x)$ and let $u, v: C(X)'\to 2$ be $CvSl$-morphisms such that $u\circ @=v\circ @$. We will show that $u=v$ and thus by the categorical condition obtained in [5], conclude that $C(X)$ is reflexive. For any finite set $F\subset X$ we have

$$u(h_F)=u\left(\inf_{x\in F} @(x)\right)=\inf_{x\in F} u(@(x))=\inf_{x\in F} v(@(x))=v(h_F).$$

Thus by lemma 2.3 we have u and v agreeing on a dense subset of $C(X)'$ and so by continuity $u = v$.

The dual $C(X)'$ as space of functions $C(X) \to 2$ naturally carries continuous convergence. By 2.2, $C(X)'$ is representable as the join semilattice $K(X)$ of compact subsets of X, with union as operation. $K(X)$ is contained in the space 2^X of closed subsets of X for which the Vietoris (=exponential) topology has been extensively studied (see [4]). Thus $C(X)'$ can also carry the Vietoris topology.

2.5 THEOREM. *On* $[C(X), 2] = K(X)$ *the following function space structures coincide:* (cc) *continuous convergence;* (co) *the compact open topology;* (Vt) *the Vietoris topology;* (it) *the initial topology induced by the maps* $@(f) : [C(X), 2] \to 2$, *where* $f \in C(X)$ *and* $f^{-1}(1)$ *is compact;* (pt) *the pointwise topology;* (ch) *the character topology.*

Proof. It is evident from theorem 2.4 and the definitions that (cc) is finer than (pt), (co) is finer than (pt), (pt) is finer than (it) and (pt) coincides with (ch). When relativized to $K(X)$ the topology (Vt) can be described by taking as basic open sets those of the form

$$U(E_1, \ldots, E_k; D) = \{M \in K(X) \mid M \not\subset E_i (i = 1, \ldots, k) \text{ and } M \subset D\}$$

where E_i and D are open compact subsets of X. Let e_i resp. d denote the characteristic functions of E_i resp. D. Then $U(E_1, \ldots, E_k; D) = \{M \in K(X) \mid @(e_i)(h_M) = \inf e_i(M) = 0 \ (i = 1, \ldots, k) \text{ and } @(d)(h_M) = 1\}$. The latter set is clearly open in (it), which is therefore finer than (Vt). To show that (Vt) is finer than (co) it is enough to check [3] that (Vt) makes the evaluation function $ev : C(X) \times K(X) \to 2$ continuous. Suppose $ev(g, M) = \inf g(M) = 1$. Then M must have an open compact neighbourhood Q such that $g(Q) = 1$. Then $\{f \mid f(Q) = 1\} \times \{N \in K(X) \mid N \subset Q\}$ is a neighbourhood of (g, M) in $C(X) \times K(X)$ and ev carries this neighbourhood to 1. Suppose next that $ev(g, M) = 0$. Choose open compact sets L, P such that $M \cap g^{-1}(0) \subset L \subset g^{-1}(0)$, $M \backslash L \subset P$ and $L \cap P = \emptyset$. Then $\{f \mid f(L) = 0\} \times U(P; L \cup P)$ is a neighbourhood of (g, M) in $C(X) \times K(X)$ which is carried by ev to 0. Hence ev is continuous when $K(X)$ carries the Vietoris topology. Thus (Vt) is finer than (co) and by the same token (Vt) is also finer than (cc) (see [1], p. 5).

We thank the referee for pointing out the equivalence of (cc), (co) and (Vt) in theorem 2.5 along with the following remarks.

2.6 REMARKS. Since X is open in its one point compactification Y, $K(X)$ is open in the compact Vietoris space 2^Y. Therefore, $K(X)$ is locally compact, which implies that its (co)-dual $K(X)^\wedge$ coincides with $K(X)' = C(X)$. By 2.5 we also have $C(X)^\wedge = C(X)' = K(X)$. Thus the reflexiveness of $C(X)$ can be expressed and proved within the category of topological semilattices.

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Propositions 2.1 and 2.2 already pave the way for such a proof: it is mainly a matter of replacing the categorical technique used (2.4) with point set topological reasoning. In such an approach (generally better known and left to the interested reader) it would be more natural to view $C(X)$ as a meet semilattice of open closed subsets and to describe its topology in Vietoris-like manner.

3. **Permanence properties of reflexive semilattices.** The results of this section will show that reflexive objects in *CvSl* are closed under formation of products and coproducts and from [5] it is known that they are also closed under formation of suitable subsemilattices.

Thus by using the special semilattices of Sections 1 and 2 as building blocks, these permanence properties bring to light a vast collection of reflexive objects in *CvSl*.

Given a product $\prod_{i\in I} A_i$ in *CvSl* and $M \subset I$ we will make extensive use of the canonical projection $pr_M : \prod_I A_i \to \prod_M A_i$ and injection $in_M : \prod_M A_i \to \prod_I A_i$.

3.1 PROPOSITION. *For every character h on $\prod_I A_i$ there exists a finite set $J \subset I$ such that $h = h \circ in_J \circ pr_J$.*

Proof. Let $\mathcal{W}$ be the filter on $\prod_I A_i$ generated by the sets $W_K = \{x \mid x_i = 1_i$ for $i \in K\}$ where K varies through finite subsets of I. Then $\mathcal{W}$ converges to 1 in $\prod_I A_i$ and by continuity of h there exists $W_J \in \mathcal{W}$ such that $h(W_J) = 1$. For any $x \in \prod_I A_i$ we have $x = in_J \circ pr_J(x) \wedge in_{I\setminus J} \circ pr_{I\setminus J}(x)$. By applying h and noting that $in_{I\setminus J}$ takes values only in W_J, we see that $h = h \circ in_J \circ pr_J$.

3.2 THEOREM. *A product $\prod_I A_i$ in CvSl of reflexive objects is reflexive.*

Proof. By a result for suitable abstract categories in [5] a sufficient condition for reflexiveness of $\prod_I A_i$ is that each A_i is reflexive and every character on $\prod A_i$ depends on finitely many coordinates in the sense made precise by prop. 3.1.

Turning now to coproducts, we remind that the coproduct $\prod_I A_i$ is algebraically the subsemilattice of $\prod_I A_i$ consisting of all x having $x_i = 1$ for $i \in I\setminus F$ where F is finite. Thus one has canonical injections $u_J : \prod_J A_i \to \prod_I A_i$ where J is a finite subset of I. By endowing $\prod_I A_i$ with the final convergence structure induced by these u_J, one obtains the coproduct in *CvSl*. Since *CvSl* has finite biproducts we can identify $(\prod_J A_j)'$ with $\prod_J A_j'$ for finite J (see [5]).

3.3 PROPOSITION. *In CvSl* $(\prod_I A_i)' \simeq \coprod_I A_i'$.

Proof. The functor $' = [\ ,2]$ carries each projection $pr_i : \prod_I A_i \to A_i$ to a morphism $pr_i' : A_i' \to (\coprod_I A_i)'$. Thus collectively the projections induce a unique morphism $f : \prod_I A_i' \to (\coprod_I A_i)'$ such that $f \circ u_i = pr_i'$ for all i. Each $k \in \coprod_I A_i'$ admits the representation $k = \inf_J u_j(k_j)$ for a smallest finite $J \subset I$. Now define the function $e : (\prod_I A_i)' \to \coprod_I A_i'$ as follows: $e(h) = \inf_J u_j(h_j)$, where J is the smallest finite set for which h factors through the projection pr_J (see 3.1). Then $f \circ e(h) = f(\inf_J u_j(h_j)) = \inf_J f(u_j(h_j)) = \inf_J pr'_j(h_j) = \inf_J(h_j \circ in_j) = h$, where $in_j : A_j \to \prod_I A_i$ is the canonical map. Also $e \circ f(h) = h$ and the only remaining problem is the continuity of e. Consider any convergent filter $\mathcal{H}$ convergent to h in $(\prod_I A_i)'$. Choose $H \in \mathcal{H}$ and a finite set $M \subset I$ such that h as well as all $k \in H$ factor through the projection pr_M. To see that such a choice exists, form the filter $\mathcal{W}$ with basic members $W_J = \{w \in \prod_I A_i \mid w_j = 1 \text{ for } j \in J\}$ where J is finite and note that $\mathcal{W}$ converges to 1 in $\prod_I A_i$. Then there exist $H \in \mathcal{H}$, $W_M \in \mathcal{W}$ with $H(W_M) = h(1) = 1$ and this gives the required H and M. Now $u_M(in'_M(\mathcal{H}))$ converges to $u_M(in'_M(h))$ i.e. $u_M(\mathcal{H} \circ in_M)$ converges to $u_M(h \circ in_M) = e(h)$ in $\coprod_I A'_i$. Thus in order to get $e(\mathcal{H})$ convergent to $e(h)$ it is enough to show $e(\mathcal{H}) \supset u_M(\mathcal{H} \circ in_M)$. But the choice of H and M ensures that for any $G \in \mathcal{H}$ we have $u_M(G \circ in_M) \supset u_M(G \cap H) \circ in_M = e(G \cap H)$, as required.

3.4 THEOREM. *In CvSl, every coproduct of reflexive objects is reflexive and the dual of a coproduct is the product of the duals.*

Proof. 3.2 and 3.3.

3.5 EXAMPLE. A reflexive semilattice (even the dual of a reflexive topological semilattice) need not be topological. To see this, choose $A_i = N^*$ (see 1.1) for $i \in I$ (infinite) and put $A = \prod_I A'_i$. Since each A_i is compact, A is a product of discrete semilattices, hence topological. But $A' = \coprod_I A_i$ has no smallest filter convergent to its unit and thus cannot be topological.

The class of reflexive objects in *CvSl* is also closed under formation of subsemilattices of the following kind by a result for abstract categories in [5]. Specifically, let B, C be reflexive, $u, v : B \to C$ and $e : A \to B$ such that $u \circ e = v \circ e$ is an equalizer diagram and every character h on A factors through e. Then A is reflexive. We will show how this fact can be applied to demonstrate that the usual closed real interval I $(0 \le x \le 1)$ does not make an interesting substitute for 2 as a dualizing object.

3.6 PROPOSITION. $[2, I] \simeq I$ *and* I *is not* I*-reflexive* (*i.e.* $@ : I \to [[I, I], I]$ *is not an isomorphism*). *Moreover, no finite orderly semilattice can be* I*-reflexive.*

Proof. Define $h : I \to [2, I]$ by $h(r)(1) = 1$ and $h(r)(0) = r$ for each $r \in I$. Then clearly h is an algebraic isomorphism. For any $r \in I$, a basic neighbourhood of $h(r)$ is formed by $N = \{g \mid g(0) \in U,\ g(1) \in V\}$, where U is a neighbourhood of r and V is a neighbourhood of 1 in I. Then it is obvious that $h(U) \subset N$. Thus h is continuous, hence a homeomorphism. Suppose I is I-reflexive. Then for any embedding $e : 2 \to I$ one readily constructs $u, v : I \to I$ such that $u \circ e = v \circ e$ is an equalizer diagram to which the above mentioned result applies. This however gives the contradictory fact that 2 is I-reflexive. It can be seen in the same way that no finite orderly semilattice can be I-reflexive.

4. **Remarks on locally compact semilattices.** If A is locally compact, is A' likewise locally compact? This is well known to be true in the Pontryagin duality theory of abelian groups, but the examples to follow show that it fails for semilattices. In the present context, "locally compact" means more precisely "locally compact Hausdorff topological". We remind that for such A, the dual A' coincides with the compact-open dual $A^\wedge$. We conclude by showing that for locally compact A the concept of reflexiveness with respect to compact open duals is contained in reflexiveness in the canonical sense.

4.1 Example. If A is a locally compact semilattice, its dual $A^\wedge$ need not be locally compact, even when A is orderly. This is illustrated by the orderly join semilattice O of all countable ordinals (1.1). The dual $O^\wedge$ can be expressed as the meet semilattice of countable ordinals together with the first uncountable ordinal Ω as unit. In $O^\wedge$ all countable ordinals are isolated points while the sets $\uparrow x (x < \Omega)$ form a base of open closed neighbourhoods of Ω. However, none of these $\uparrow x$ can be compact, since $\{\{x\}, \{x+1\} \cdots \{x+w\}, \uparrow(x+w+1)\}$ is an open cover of $\uparrow x$ which has no finite subcover.

4.2 Example. Another locally compact A whose dual fails to be locally compact. Let B_n be an infinite discrete semilattice ($n \in N$) and put $A = \coprod_N B_n'$. Then A is locally compact but $A^\wedge = \prod_N B_n$ is not locally compact.

4.3 Proposition. *For a locally compact semilattice A one has $A \simeq A''$ whenever $A \simeq A^{\wedge\wedge}$.*

Proof. We have $A^\wedge = A'$ so that the underlying sets of $A^{\wedge\wedge}$ and A'' can be identified. Since $@ : A \to A^{\wedge\wedge}$ is an isomorphism, while $id : A^{\wedge\wedge} \to A''$ is at least continuous, it is enough to show that the identity function $id : A'' \to A^{\wedge\wedge}$ is continuous. Suppose $\mathcal{H}$ is a filter on A'' which converges to h in A'' and consider the basic neighbourhood $W_{PQ} = \{k \in A^{\wedge\wedge} \mid k(P) = 0 \text{ and } k(Q) = 1\}$ of h in $A^{\wedge\wedge}$. For each $p \in P$ we have $h(p) = 0$ so we can choose a neighbourhood N_p of p and a member H_p of $\mathcal{H}$ such that $H_p(N_p) = 0$. By compactness there is an open $N_P \supset P$ and $H_P \in \mathcal{H}$ such that $H_P(N_P) = 0$; similarly an open $N_Q \supset Q$ and $H_Q \in \mathcal{H}$ with $H_Q(N_Q) = 1$. Thus we arrive at a member $H_P \cap H_Q$ of $\mathcal{H}$ contained in W_{PQ} as required.

4.4 Remark. Let A be a reflexive topological semilattice and let A^p denote the dual equipped with the pointwise topology. Then $A \simeq A^{pp}$ iff A has enough characters and carries the character topology. The condition is clearly necessary. For the converse, it is obvious that $@ : A \to A^{pp}$ is an embedding. Since $id : A' \to A^p$ is continuous, $A^{pp} \subset A'' \simeq A$. Thus $@$ is an isomorphism.

4.5 Remark. Using 3.1, one can easily show that for any product A of discrete semilattices, $A^\wedge \simeq A^p$. Hence by 3.2 and 4.4, products of discrete semilattices are also reflexive in the sense of compact-open duals.

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